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Asymptotic stability of solutions for some classes of impulsive differential equations with distributed delay

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Abstract: In this paper we show the asymptotic stability of the solutions of some differential equations with delay and subject to impulses. After proving the existence of mild solutions on the half-line, we give a Gronwall-Bellman-type theorem. These results are prodromes of the theorem on the asymptotic stability of the mild solutions to a semilinear differential equation with functional delay and impulses in Banach spaces and of its application to a parametric differential equation driving a population dynamics model.

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1 Introduction

The aim of the paper is to study the asymptotic stability of the mild solutions of the parametric differential equation

$$\frac{\partial u}{\partial t}(t,x) = -b(t,x)u(t,x) + g\left(t, u(t,x), \int_{-\tau}^{0} u(t+\theta,x)d\theta\right),\ t \in [t_0, +\infty[\,,\ x \in [0,1] \qquad (1)$$

under the action of instantaneous impulses $\mathcal{I}_k$ at given times t_k, $k \in \mathbb{N}$, with $\{t_k\}_{k\in\mathbb{N}}$ sequence of nonnegative real numbers diverging to $+\infty$.

The interest to this problem arises from the population dynamics mathematical theory, where the above equation drives a model for the time evolution of a population in a given environment normalized to the interval $[0,1]$: $u(t,x)$ represents the density of the population at time t and place x; $-b(t,x)$ is the removal coefficient including the death rate and the displacement of the population; the nonlinearity g is the population development law involving a feedback term given by its integral argument. The feedback term $\int_{-\tau}^{0} u(t+\theta,x)d\theta$ means that the system has the memory of the whole population evolution for the fixed past time of given amplitude τ preceding the present instant t. On the other hand, the impulse functions are the modeling of external forces acting on the population at prescribed moments, as happens with the administration of antibiotics on a bacterial population or of pesticides on an insect infestation, and in Biology are called *regulation functions.* Actually, impulse functions are not only encountered in Biology, but appear in numerous models of real phenomena from Physics, Economy, Chemistry, Engineering. For this reason, the problems involving these functions have been studied in recent decades and are still of great interest today; we refer, e.g., to [4, 7, 14].

To achieve our goal, we will read equation (1) as a particular case of the semilinear differential equation with functional delay in a Banach spaces E

$$y'(t) = A(t)y(t) + f(t, y(t), y_t) \ , \ t \in [t_0, +\infty[, \tag{2}$$

subject to impulses I_k defined in a functions' space and taking values on E. Here $\{A(t)\}_{0\leq t}$ is a family of linear operators, f is a nonlinear function, and $y_t(\theta) = y(t+\theta)$, $\theta \in [-\tau, 0]$, for every $t \in [t_0, +\infty[$.

During the last years this kind of equations have been the object of a wide study in several setting, especially in the less general forms $y'(t) = Ay(t)+f(t, y(t))$ or $y'(t) = Ay(t)+f(t, y_t)$. We refer e.g. to [10, 13, 19] for studies on asymptotic behavior of solutions. Nevertheless, as far as we know, our theorem on the existence of impulsive mild solution on the half-line to (2) is new. We needed this result to give consistency to the definitions of local and global attractivity that we introduce in the Section 4 and to the proof of our main theorem. Given the relevance that the existence result has in itself, we proved it in a very general setting and we dedicated Section 3 to it.

The concept of attractivity we define herein is on the line of that described in [3] and then adapted to classical Cauchy problems driven by the equation $y'(t) = Ay(t)+f(t, y(t))$ in [16]. We extend their definitions to Cauchy problems with functional initial datum and provide for these problems the definition of (global) asymptotic stability.

Moreover, in order to show the asymptotic stability of our impulsive problem with functional delay, we provide a suitable Gronwall-Bellmann-type delayed impulsive integral inequality. Our Lemma 4.1 is - obviously and obligatorily - in the wake of that for impulsive differential equations (see, e.g. [11, Theorem 1.5.1] or [17, Lemma 1, Section 1.2]). We here provide an extension of the classical results to the delayed case. This topic arouses interest in the literature, because of its applications to delayed impulsive differential problems. We refer for example to the very recent paper [12], where the authors propose a Gronwall-Bellmann-type inequality for functions with delay and subject to impulses. We point out that our result is more general than theirs, since the class of functions involved in inequality is enlarged, as the presence of delay is allowed to the impulsive functions too.
Our Gronwall-Bellmann-type lemma is contained in Section 4 immediately before the main asymptotic stability theorem, namely Theorem 4.1.

We conclude the paper with the discussion of the population dynamics model governed by equation (1) with impulsive effects. We first transform the model as an impulsive Cauchy problem with delay in the space $E = L^2([0, 1])$ and then apply our abstract result.

2 Preliminaries and position of the problem

Let $(E, \|\cdot\|)$ be a real Banach space and $\mathcal{L}(E)$ the space of all bounded linear operators from E to E furnished with the strong operator topology. We recall that (see, e.g. [18]) a family $\mathcal{T} = \{T(t, s)\}_{t\geq s\geq 0}$ of bounded linear operators on E is a *(strongly continuous, exponentially bounded) evolution system on the half-line* if

1) $T(s, s) = I, \quad T(t, r)T(r, s) = T(t, s)$ for $t \geq s \geq 0$;

2) for every $x \in E$, the map $\xi_x : (t,s) \mapsto T(t,s)x$ is continuous;

3) there are constants $D \geq 0, \omega \in \mathbb{R}$ such that

$$\|T(t,s)\|_{\mathcal{L}(E)} \leq De^{\omega(t-s)} \text{ for } t \geq s \geq 0. \tag{3}$$

The number

$$\omega_* := \inf\{\omega \in \mathbb{R} : \exists D \geq 0 \text{ such that } \|T(t,s)\|_{\mathcal{L}(E)} \leq De^{\omega(t-s)},\ t \geq s \geq 0\} \tag{4}$$

is said to be *the growth bound of* $\mathcal{T}$.
If $\omega_* < 0$, the evolution system on the half-line is said to be *exponentially stable* (cf. [18, Definition 2.1]).

For $\tau > 0$, the symbol $\mathcal{PC}_\tau$ denotes the space of piecewise continuous functions $\boldsymbol{y} : [-\tau, 0] \to E$ with a finite number of jump discontinuity points, endowed with the norm

$$\|\boldsymbol{y}\|_{\mathcal{PC}_\tau} := \frac{1}{\tau}\int_{-\tau}^{0} \|\boldsymbol{y}(\theta)\|\, d\theta. \tag{5}$$

Let $\{t_k\}_{k\in\mathbb{N}}$ be a set of fixed real numbers with $0 \leq t_0 < t_1 < \cdots < t_k < ...$ and $t_k \to +\infty$. By $\mathcal{PC}([t_0, +\infty[, E)$ we denote the seminormed space of functions

$$\mathcal{PC}([t_0,+\infty[,E) := \left\{ \begin{array}{lll} y : [t_0, +\infty[\to E & : & y_{|\,]t_{k-1},t_k]} \text{ continuous, for all } k \in \mathbb{N}^+; \\ & & \exists\, y\big(t_k^+\big) = \lim\limits_{h\to 0^+} y(t_k + h) \in E, \text{ for all } k \in \mathbb{N} \end{array} \right\}$$

with $\|y\|_{\infty,n} = \sup_{t\in[t_0,t_0+n]} \|y(t)\|$, $n \in \mathbb{N}^+$.
Moreover, we put

$$\mathcal{S}([t_0 - \tau, +\infty[, E) = \{y : [t_0 - \tau, +\infty[\to E : y_{t_0} \in \mathcal{PC}_\tau \text{ and } y_{|\,[t_0,+\infty[} \in \mathcal{PC}([t_0, +\infty[, E).$$

For more details on this set, we refer to [9, Section 3.1], where the authors make some useful comments on $\mathcal{S}([t_0-\tau, +\infty[, E)$ and give some pictures illustrating the possible discontinuities of its functions. For the sake of clarity, we have to say that their functions are defined on a compact interval, but their comments can be still applied to functions defined on a half-line.

Remark 2.1 *We wish to recall that the norm defined in* (5) *is weaker than the Chebyshev sup-norm*

$$\|\boldsymbol{y}\|_{\infty\_0} = \sup_{-\tau\leq\theta\leq 0} \|y(\theta)\|. \tag{6}$$

In fact, for every $\boldsymbol{y} \in \mathcal{PC}_\tau$ *one trivially has*

$$\|\boldsymbol{y}\|_{\mathcal{PC}_\tau} \leq \frac{1}{\tau}\int_{-\tau}^{0} \sup_{-\tau\leq\theta\leq 0} \|y(\theta)\|\, d\theta = \|\boldsymbol{y}\|_{\infty\_0}; \tag{7}$$

but it is easy to see that it does not exist $c > 0$ *s.t.* $\|\boldsymbol{y}\|_{\infty\_0} \leq c\|\boldsymbol{y}\|_{\mathcal{PC}_\tau}$ *for every* $\boldsymbol{y} \in \mathcal{PC}_\tau$*. Furthermore, we notice that for* $y \in \mathcal{S}([t_0 - \tau, +\infty[, E)$ *the mapping* $t \mapsto y_t$ *is continuous if in the space* $\mathcal{PC}_\tau$ *we take the norm* $\|\cdot\|_{\mathcal{PC}_\tau}$*, but not if the norm is* $\|\cdot\|_{\infty\_0}$*.*

For $\boldsymbol{y} \in \mathcal{PC}_\tau$, we consider the impulsive Cauchy problem with functional delay on a half-line

$$(P)_{\boldsymbol{y}} \quad \begin{cases} y'(t) = A(t)y(t) + f\left(t, y(t), y_t\right) \ , \ t \in [t_0, +\infty[\ , \ t \neq t_k \ , \ k \in \mathbb{N}, \\ \\ y_{t_0} = \boldsymbol{y}, \\ \\ y(t_0^+) = \boldsymbol{y}(0) + I_0(\boldsymbol{y}), \\ \\ y(t_k^+) = y(t_k) + I_k(y_{t_k})\ , \ k \in \mathbb{N}^+; \end{cases}$$

where: $\{A(t)\}_{t\geq 0}$ is a family of linear operators; $f : [t_0, +\infty[\times E \times \mathcal{PC}_\tau \to E$; $I_k : \mathcal{PC}_\tau \to E$, $k \in \mathbb{N}$, are the impulse functions.

Suppose that the following property holds:

(A) $\{A(t)\}_{t\geq 0}$ is a family of linear operators, $A(t) : D(A) \subset E \to E$, $D(A)$ a dense subset of the Banach space E not depending on t, generating an evolution system on the half-line $\mathcal{T} = \{T(t, s)\}_{t\geq s\geq 0}$.

This means that $\mathcal{T}$ satisfies the further property

4) $\dfrac{\partial T(t,s)}{\partial t} = A(t)T(t, s)$ and $\dfrac{\partial T(t,s)}{\partial s} = -T(t, s)A(s),\ t \geq s \geq 0.$

Therefore we can state the following definition.

Definition 2.1 *A function* $y : [t_0-\tau, +\infty[\to E$ *is said to be a* delayed impulsive mild solution *(shortly, mild solution) of* $(P)_{\boldsymbol{y}}$ *if*

(i) $y(t) = T(t, t_0)[\boldsymbol{y}(0) + I_0(\boldsymbol{y})] + \sum_{t_0<t_i<t} T(t, t_i)I_i(y_{t_i}) + \int_{t_0}^{t} T(t, s)f(s, y(s), y_s)\, ds\ ,\ t > t_0;$

(ii) $y_{t_0} = \boldsymbol{y}$;

(iii) $y(t_0^+) = \boldsymbol{y}(0) + I_0(\boldsymbol{y})$;

(iv) $y(t_k^+) = y(t_k) + I_k(y_{t_k})\ ,\ k \in \mathbb{N}^+$,

with the agreement that $\sum_{t_0<t_k<t} T(t, t_k)I_k(y_{t_k}) = 0$ *if* $t \in [t_0, t_1]$.

Notice that if y is a mild solution of $(P)_{\boldsymbol{y}}$, then $y \in \mathcal{S}([t_0 - \tau, +\infty[, E)$.

3 Existence of solutions on a half-line

Let us fix a function φ in the space $\mathcal{PC}_\tau$ and consider the corresponding problem

$$(P)_\varphi \quad \begin{cases} y'(t) = A(t)y(t) + f\left(t, y(t), y_t\right) \ , \ t \in [t_0, +\infty[\ , \ t \neq t_k \ , \ k \in \mathbb{N}, \\ \\ y_{t_0} = \varphi, \\ \\ y(t_0^+) = \varphi(0) + I_0(\varphi), \\ \\ y(t_k^+) = y(t_k) + I_k(y_{t_k})\ , \ k \in \mathbb{N}^+. \end{cases}$$

On the nonlinearity $f : [t_0, +\infty] \times E \times \mathcal{PC}_\tau \to E$ we assume the following hypotheses:

(h1) $f(t,\cdot,\cdot)$ is continuous, for every $t \geq t_0$, and $f\left(\cdot, y(\cdot), y_{(\cdot)}\right)$ is measurable, for every $y \in \mathcal{S}([t_0-\tau, +\infty[, E)$;

(h2) there exists a locally integrable function $\alpha : [t_0, +\infty[\to \mathbb{R}^+$ such that

$$\|f(t, y, \boldsymbol{y})\| \leq \alpha(t)(1 + \|y\| + \|\boldsymbol{y}\|_{\mathcal{PC}_\tau}) \ , \text{ for a.e. } t \geq t_0 \text{ and all } y \in E, \boldsymbol{y} \in \mathcal{PC}_\tau \ ;$$

(h3) there exists a locally integrable function $h : [t_0, +\infty[\to \mathbb{R}^+$ such that

$$\chi(f(t, \Omega_1, \Omega_2)) \leq h(t) \left[\chi(\Omega_1) + \sup_{-\tau \leq \theta \leq 0} \chi(\Omega_2(\theta))\right] \ , \text{ for a.e. } t \geq t_0$$

for every bounded sets $\Omega_1 \subset E$, $\Omega_2 \subset \mathcal{PC}_\tau$, where χ is the Hausdorff measure of noncompactness in E.

We are now going to establish the existence result on the existence of mild solutions of problem $(P)_\varphi$ on the half-line.

Theorem 3.1 *Let E be a real Banach space and $\varphi \in \mathcal{PC}_\tau$, $\{t_k\}_{k\in\mathbb{N}}$ with $0 \leq t_0 < t_1 < \cdots < t_k < \ldots \to +\infty$, $I_k : \mathcal{PC}_\tau \to E$, $k \in \mathbb{N}$, be given. Assume that the family $\{A(t)\}_{t\geq 0}$ and the function $f : [t_0, +\infty] \times E \times \mathcal{PC}_\tau \to E$ respectively satisfy (A) and (h1)-(h3).*
Then problem $(P)_\varphi$ has at least one mild solution.

Proof. We proceed by iteration.
Consider the functional Cauchy problem

$$(P_1) \begin{cases} y'(t) = A(t)y(t) + f(t, y(t), y_t) \ , \quad t \in [t_0, t_1], \\ y_{t_0} = \varphi, \\ y(t_0^+) = \varphi(0) + I_0(\varphi). \end{cases}$$

Then, define the map $y^0 : [t_0 - \tau, t_0] \to E$ as

$$y^0(t) = \varphi(t - t_0), \ t_0 - \tau \leq t \leq t_0 \tag{8}$$

and, for any $y \in C([t_0, t_1], E)$, put

$$y[y^0](t) = \begin{cases} y(t), & t \in]t_0, t_1] \\ y^0(t), & t \in [t_0 - \tau, t_0]. \end{cases} \tag{9}$$

By (3) it follows that there exists $D_1 > 0$ such that $\|T(t,s)\|_{\mathcal{L}(E)} \leq D_1$ for every $t_0 \leq s \leq t \leq t_1$. Moreover, the functions α and h of assumptions (h2) and (h3) obviously belong to $L^1_+([t_0, t_1])$ when restricted to $[t_0, t_1]$. Hence we can use the same arguments of the proof of the Theorem 2.1 in [6], which acts on a compact interval $[0, T]$, and claim that the integral operator $\Gamma_1 : C([t_0, t_1], E) \to C([t_0, t_1], E)$ given by

$$\Gamma_1(y)(t) = T(t, t_0)\left[\varphi(0) + I_0(\varphi)\right] + \int_{t_0}^{t} T(t,s) f(s, y(s), y[y^0]_s)\, ds \ , \ t \in [t_0, t_1] \tag{10}$$

has a fixed point, say $y^1 \in C([t_0, t_1], E)$.
We show that the function $y^1[y^0] : [t_0 - \tau, t_1] \to E$ is a mild solution of (P_1).
First, by (9) and (10), for $t \in]t_0, t_1]$ one has

$$y^1[y^0](t) = y^1(t) = T(t, t_0)\,[\varphi(0) + I_0(\varphi)] + \int_{t_0}^{t} T(t,s) f(s, y^1(s), y^1[y^0]_s)\, ds;$$

since functions y^1 and $y^1[y^0]$ on the set $[t_0, t_1]$ are the same except in $s = t_0$, we get

$$\int_{t_0}^{t} T(t,s) f(s, y^1(s), y^1[y^0]_s)\, ds = \int_{t_0}^{t} T(t,s) f(s, y^1[y^0](s), y^1[y^0]_s)\, ds,$$

so

$$y^1[y^0](t) = T(t, t_0)\,[\varphi(0) + I_0(\varphi)] + \int_{t_0}^{t} T(t,s) f(s, y^1[y^0](s), y^1[y^0]_s)\, ds. \tag{11}$$

Further, by (9) and (8), for $\theta \in [-\tau, 0]$ it is $y^1[y^0]_{t_0}(\theta) = y^1[y^0](\theta + t_0) = y^0(\theta + t_0) = \varphi(\theta)$, that is

$$y^1[y^0]_{t_0} = \varphi.$$

Clearly, by means of (11) it also true that

$$y^1[y^0](t_0^+) = \varphi(0) + I_0(\varphi).$$

Now we can iterate the process.
First of all notice that, by (8), the map φ can be read as

$$\varphi^0 := \varphi = y^0_{t_0};$$

further we can define

$$\varphi^1 := y^1[y^0]_{t_1},$$

that, of course, belongs to $\mathcal{PC}_\tau$.
So, at the k-step ($k \geq 2$) one gets:

- the map φ^{k-1} in $\mathcal{PC}_\tau$ defined by

$$\varphi^{k-1} = y^{k-1}[...[y^0]...]_{t_{k-1}} \tag{12}$$

  where $y^{k-1}[...[y^0]...] : [t_0 - \tau, t_{k-1}] \to E$ is the prolonging function built by means of the concatenation of the fixed points of the integral operators at the preceding steps; i.e. for $i = 1, ..., k-1$ it is defined $\Gamma_i : C([t_{i-1}, t_i], E) \to C([t_{i-1}, t_i], E)$

$$\begin{aligned} \Gamma_i(y)(t) &= T(t, t_{i-1})\left[\varphi^{i-1}(0) + I_{i-1}(\varphi^{i-1})\right] \\ &\quad + \int_{t_{i-1}}^{t} T(t,s) f(s, y(s), y[y^{i-1}[...[y^0]...]]_s)\, ds \end{aligned}$$

having a fixed point $y^i \in C([t_{i-1}, t_i], E)$ (by (3) there exists $D_i > 0$ s. t. $\|T(t,s)\|_{\mathcal{L}(E)} \leq D_i$ for every $t_{i-1} \leq s \leq t \leq t_i$; then also in this case the proof of the existence of the fixed point retraces the one of [6, Theorem 2.1]), leading to the desired map

$$y^{k-1}[...[y^0]...](t) = \begin{cases} y^{k-1}(t), & t \in]t_{k-2}, t_{k-1}] \\ ... \\ y^1(t), & t \in]t_0, t_1] \\ y^0(t), & t \in [t_0 - \tau, t_0]; \end{cases}$$

- the functional Cauchy problem

$$(P_k) \begin{cases} y'(t) = A(t)y(t) + f(t, y(t), y_t) \ , \quad t \in [t_{k-1}, t_k], \\ y_{t_{k-1}} = \varphi^{k-1}, \\ y(t_{k-1}^+) = \varphi^{k-1}(0) + I_{k-1}(\varphi^{k-1}); \end{cases}$$

- the integral operator $\Gamma_k : C([t_{k-1}, t_k], E) \to C([t_{k-1}, t_k], E)$

$$\begin{aligned} \Gamma_k(y)(t) \quad &= \quad T(t, t_{k-1}) \left[\varphi^{k-1}(0) + I_{k-1}(\varphi^{k-1})\right] \\ &\quad + \int_{t_{k-1}}^{t} T(t,s) f(s, y(s), y[y^{k-1}[...[y^0]...]]_s) \, ds; \end{aligned} \tag{13}$$

- the fixed point $y^k \in C([t_{k-1}, t_k], E)$ of Γ_k;
- the function $y^k[...[y^0]...] : [t_0 - \tau, t_k] \to E$,

$$y^k[...[y^0]...](t) = \begin{cases} y^k(t), & t \in]t_{k-1}, t_k] \\ y^{k-1}[...[y^0]...](t), & t \in [t_0 - \tau, t_{k-1}], \end{cases}$$

  mild solution of (P_k), i.e.

$$\begin{aligned} &y^k[...[y^0]...](t) = T(t, t_{k-1}) \left[\varphi^{k-1}(0) + I_{k-1}(\varphi^{k-1})\right] \\ &\qquad + \textstyle\int_{t_{k-1}}^{t} T(t,s) f(s, y^k[...[y^0]...](s), y^k[...[y^0]...]_s) \, ds, \ t \in]t_{k-1}, t_k]; \\ &y^k[...[y^0]...]_{t_{k-1}} = \varphi^{k-1}; \\ &y^k[...[y^0]...](t_{k-1}^+) = \varphi^{k-1}(0) + I_{k-1}(\varphi^{k-1}). \end{aligned}$$

Let us define now the function $y^\infty : [t_0 - \tau, +\infty[\to E$ by

$$y^\infty(t) = \begin{cases} y^k(t), & t \in]t_{k-1}, t_k], \ k \in \mathbb{N}^+ \\ y^0(t), & t \in [t_0 - \tau, t_0]. \end{cases} \tag{14}$$

We are going to show that y^∞ is a mild solution of $(P)_\varphi$.

Fix $t > t_0$; then the exists $k \in \mathbb{N}$ such that $t \in]t_{k-1}, t_k]$. So, (14), (13), (12) yield

$$\begin{aligned}
y^\infty(t) &= y^k(t) = T(t, t_{k-1}) \left[\varphi^{k-1}(0) + I_{k-1}(\varphi^{k-1})\right] \\
&\quad + \int_{t_{k-1}}^t T(t,s) f(s, y^k(s), y^k[y^{k-1}[...[y^0]...]]_s)\, ds \\
&= ... \\
&= T(t, t_0)[\varphi(0) + I_0(\varphi)] + \sum_{t_0 < t_i < t} T(t, t_i) I_i(\varphi^i) \\
&\quad + \int_{t_0}^t T(t,s) f(s, y^k(s), y^k[y^{k-1}[...[y^0]...]]_s)\, ds \\
&= T(t, t_0)[\varphi(0) + I_0(\varphi)] + \sum_{t_0 < t_i < t} T(t, t_i) I_i({y^\infty}_{t_i}) \qquad (15) \\
&\quad + \int_{t_0}^t T(t,s) f(s, y^\infty(s), {y^\infty}_s)\, ds.
\end{aligned}$$

So *(i)* of Definition 2.1 holds.

Then, property *(ii)* is satisfied since (14) and (8) allow to

$${y^\infty}_{t_0} = {y^0}_{t_0} = \varphi.$$

Further, by (15), (14), the fact that y^1 is a fixed point of (10), and (12), we have

$$\begin{aligned}
y^\infty(t_1^+) &= T(t_1, t_0)[\varphi(0) + I_0(\varphi)] + T(t_1, t_1) I_1(\varphi^1) + \int_{t_0}^{t_1} T(t_1, s) f(s, y^1(s), y^1[y^0]_s)\, ds \\
&= y^1(t_1) + I_1(\varphi^1) = \varphi^1(0) + I_1(\varphi^1),
\end{aligned}$$

so *(iii)* is satisfied.

Finally, by iteratively repeating calculations analogous to the previous ones, it can be proved that *(iv)* holds too, concluding the proof. □

4 Asymptotic stability

Let us suppose that the family $\{A(t)\}_{t \geq 0}$ satisfies property

(A)$'$ $\{A(t)\}_{t \geq 0}$ is a family of linear operators, $A(t) : D(A) \subset E \to E$, $D(A)$ a dense subset of the Banach space E not depending on t, generating an *exponentially stable* evolution system on the half-line $\mathcal{T} = \{T(t,s)\}_{t \geq s \geq 0}$.

and the function $f : [t_0, +\infty[\times E \times \mathcal{PC}_\tau \to E$ satisfies conditions

(h4) $f\left(\cdot, y, \boldsymbol{y}\right)$ is measurable, for every $y \in E$ and $\boldsymbol{y} \in \mathcal{PC}_\tau$;

(h5) there exist $c_1, c_2 > 0$ such that

$$\|f(t, z, \boldsymbol{z}) - f(t, y, \boldsymbol{y})\| \leq c_1 \|z - y\| + c_2 \|\boldsymbol{z} - \boldsymbol{y}\|_{\mathcal{PC}_\tau}, \text{ for all } t \geq t_0,\ z, y \in E,\ \boldsymbol{z}, \boldsymbol{y} \in \mathcal{PC}_\tau.$$

Remark 4.1 *If the Banach space E is separable, then properties (h4) and (h5) imply property (h1). In fact, we first note that the norm $\|\cdot\|_{\mathcal{PC}_\tau}$ in $\mathcal{PC}_\tau$ is trivially equivalent to the usual L^1-norm. So we can see $\mathcal{PC}_\tau$ as a subset of the space $L^1([-\tau,0],E)$, which is separable being E separable and metric being normed. Therefore, $\mathcal{PC}_\tau$ is separable in the L^1-norm and $(\mathcal{PC}_\tau, \|\cdot\|_{\mathcal{PC}_\tau})$ is separable as well. Moreover, for every $y \in \mathcal{S}([t_0-\tau,+\infty[,E)$ the map $t \mapsto y_t$ is continuous when it takes values in $(\mathcal{PC}_\tau, \|\cdot\|_{\mathcal{PC}_\tau})$ (see Remark 2.1). Hence, we can apply [8, Corollary 2.5.24] which leads (h1) to be true.*

Further, if (h5) is assumed, then properties (h2) immediately follows.

Let us show that (h5) implies (h3) as well. Indeed, for any $\boldsymbol{y} \in \mathcal{PC}_\tau$ one has (see (7) *and* (6)*)*

$$\|\boldsymbol{y}\|_{\mathcal{PC}_\tau} \leq \sup_{-\tau\leq\theta\leq 0} \|\boldsymbol{y}(\theta)\|. \tag{16}$$

Hence for all $t \geq t_0$, $z, y \in E$, $\boldsymbol{z}, \boldsymbol{y} \in \mathcal{PC}_\tau$ we get

$$\begin{aligned} \|f(t,z,\boldsymbol{z}) - f(t,y,\boldsymbol{y})\| &\leq c_1\|z-y\| + c_2\|\boldsymbol{z}-\boldsymbol{y}\|_{\mathcal{PC}_\tau} \\ &\leq c\left[\|z-y\| + \sup_{-\tau\leq\theta\leq 0}\|\boldsymbol{z}(\theta)-\boldsymbol{y}(\theta)\|\right] \end{aligned}$$

where $c = \max\{c_1, c_2\}$. Let us define a metric in the space $E\times\mathcal{PC}_\tau$ by putting $d\left((z,\boldsymbol{z}),(y,\boldsymbol{y})\right) = \|z-y\| + \sup_{-\tau\leq\theta\leq 0}\|\boldsymbol{z}(\theta)-\boldsymbol{y}(\theta)\|$; we can therefore apply the comparison between Lipschitz conditions and measures of noncompactness as exposed in [1, Section 2] and obtain property (h3) to be true with $h(t) = c/2$ for a.e. $t \geq t_0$.

On the impulse functions $I_k : \mathcal{PC}_\tau \to E$, $k \in \mathbb{N}$ we suppose that

(h6) for every $k \in \mathbb{N}$, there exists $a_k > 0$ such that

$$\|I_k(\boldsymbol{z}) - I_k(\boldsymbol{y})\| \leq a_k\|\boldsymbol{z}-\boldsymbol{y}\|_{\mathcal{PC}_\tau}, \text{ for all } \boldsymbol{z},\boldsymbol{y} \in \mathcal{PC}_\tau \tag{17}$$

and

$$\sum_{k=0}^{\infty} a_k \text{ converges.} \tag{18}$$

In the wake of the definitions described in [2, Section 1.8] and in [16, Section 4], we introduce some concepts of attractivity of mild solutions.

Definition 4.1 *We say that a mild solution y of $(P)_{\boldsymbol{y}}$, $\boldsymbol{y} \in \mathcal{PC}_\tau$, is* locally attractive *if there exists $r > 0$ such that for every z mild solution of $(P)_{\boldsymbol{z}}$ with $\boldsymbol{z} \in B_{\mathcal{PC}_\tau}(\boldsymbol{y}, r)$ we have that*

$$\lim_{t\to+\infty} \|z(t) - y(t)\| = 0. \tag{19}$$

Definition 4.2 *We say that a mild solution y of $(P)_{\boldsymbol{y}}$, $\boldsymbol{y} \in \mathcal{PC}_\tau$, is* uniformly locally attractive *if there exists $r > 0$ such that the limit* (19) *is uniform with respect to the ball $B_{\mathcal{PC}_\tau}(\boldsymbol{y}, r)$; i.e.*

there exists $r > 0$ such that for every $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that for every $t \geq T(\varepsilon)$ then $\|z(t) - y(t)\| \leq \varepsilon$, for every z mild solution of $(P)_{\boldsymbol{z}}$ with $\boldsymbol{z} \in B_{\mathcal{PC}_\tau}(\boldsymbol{y}, r)$.

Analogously, we give the next definitions of global type.

Definition 4.3 *We say that a mild solution y of $(P)_{\mathbf{y}}$, $\mathbf{y} \in \mathcal{PC}_\tau$, is* globally attractive *if for every z mild solution of $(P)_{\mathbf{z}}$ with $\mathbf{z} \in \mathcal{PC}_\tau$, we have that (19) holds.*

Definition 4.4 *We say that a mild solution y of $(P)_{\mathbf{y}}$, $\mathbf{y} \in \mathcal{PC}_\tau$, is* uniformly globally attractive *if the limit (19) is uniform overall the space $\mathcal{PC}_\tau$; i.e.*

for every $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that for every $t \geq T(\varepsilon)$ then $\|z(t) - y(t)\| \leq \varepsilon$, for every z mild solution of $(P)_{\mathbf{z}}$ with $\mathbf{z} \in \mathcal{PC}_\tau$.

Remark 4.2 *We first wish to note that the above definitions are well-posed, since from Remark 4.1 and Theorem 3.1 we can deduce that for every $\mathbf{y} \in \mathcal{PC}_\tau$ the corresponding problem $(P)_{\mathbf{y}}$ has at least one mild solution in the sense of Definition 2.1.*

Moreover, from Definition 4.2 we can say that the mild solutions of the differential equation (2) subject to the impulses I_k at times t_k, $k \in \mathbb{N}$, (briefly, impulsive mild solutions of (2)) are uniformly locally attractive if

there exists $r > 0$ such that for every $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that for every $t \geq T(\varepsilon)$ then $\|z(t) - y(t)\| \leq \varepsilon$, for every z solution of $(P)_{\mathbf{z}}$ and y solution of $(P)_{\mathbf{y}}$ with $\|\mathbf{z} - \mathbf{y}\|_{\mathcal{PC}_\tau} \leq r$;

in this case the mild solutions are said to be asymptotically stable.

Analogously, from Definition 4.4 we can say that the impulsive mild solutions of (2) are uniformly globally attractive if

for every $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that for every $t \geq T(\varepsilon)$ it is $\|z(t) - y(t)\| \leq \varepsilon$, for every z solution of $(P)_{\mathbf{z}}$ and y solution of $(P)_{\mathbf{y}}$ with $\mathbf{z}, \mathbf{y} \in \mathcal{PC}_\tau$;

in this case the mild solutions are said to be globally asymptotically stable.

Finally, the above definitions are then linked together according to the following scheme

uniform global attractivity $\Longleftrightarrow$ *global asymptotic stability*	$\Longrightarrow$	*global attractivity*
$\Downarrow$		$\Downarrow$
uniform local attractivity $\Longleftrightarrow$ *asymptotic stability*	$\Longrightarrow$	*local attractivity*

To achieve our main result, we need the following Lemma on a Gronwall-Bellman-type impulsive inequality with delay.

Let $\mathcal{PC}([t_0 - \tau, t_0], \mathbb{R})$ be the space of the piecewise continuous functions from $[t_0 - \tau, t_0]$ to $\mathbb{R}$ with finite points of discontinuity where functions are left continuous and have the right limits, endowed with the sup-norm $\|m\|_{\infty\_t_0} = \sup_{s\in[t_0-\tau,t_0]} |m(s)|$. Observe that, since $\sup_{s\in[t_0-\tau,t_0]} |m(s)| = \sup_{\theta\in[-\tau,0]} |m(t_0 + \theta)| = \|m(t_0 + \cdot)\|_{\infty\_0}$, then the t_0-left-translation of $\mathcal{PC}([t_0 - \tau, t_0], \mathbb{R})$ is a subset of $(\mathcal{PC}_\tau, \|\cdot\|_{\infty\_0})$.

Lemma 4.1 *Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence with $0 \le t_0 < t_1 < \cdots < t_k < \ldots \to +\infty$ and $m : [t_0 - \tau, +\infty[\to \mathbb{R}$ be a nonnegative piecewise continuous function with left-continuous jump-discontinuities at t_k and at a finite number of points $s_i \in [t_0 - \tau, t_0]$.*
Suppose that there exist constants $c \ge 0$, $\beta_k \ge 0$, $k \in \mathbb{N}^+$, and two continuous function $p, q : [t_0, +\infty[\to \mathbb{R}^+$ such that

$$m(t) \le c\,m(t_0) + \sum_{t_0<t_i<t} \beta_i \sup_{-\tau\le\theta\le 0} m(t_i+\theta) + \int_{t_0}^{t} p(s)m(s)ds + \int_{t_0}^{t} q(s) \sup_{-\tau\le\theta\le 0} m(s+\theta)ds,\ t \ge t_0. \tag{20}$$

Then,

$$m(t) \le \tilde{c}\,\|m\|_{\infty\text{-}t_0} \prod_{t_0<t_i<t} (1+\beta_i)\, e^{\int_{t_0}^{t}[p(s)+q(s)]ds},\ t \ge t_0, \tag{21}$$

where $\tilde{c} = \max\{1, c\}$.

Proof. Let us define the function $M : [t_0 - \tau, +\infty[\to E$ as

$$M(t) = \sup_{s\in[t_0-\tau,t]} m(s),\ t \ge t_0 - \tau. \tag{22}$$

By the assumptions it is easy to see that M is well-posed.
First of all, note that by (26) and (20) for every $t \ge t_0$ it is

$$\begin{aligned} M(t) &\le \max\Big\{\sup_{s\in[t_0-\tau,t_0]} m(s), \sup_{s\in[t_0,t]} m(s)\Big\} \\ &\le \max\Big\{\|m\|_{\infty\text{-}t_0}, c\,m(t_0) + \sup_{s\in[t_0,t]}\Big[\sum_{t_0<t_i<s}\beta_i \sup_{-\tau\le\theta\le 0} m(t_i+\theta) \\ &\qquad + \int_{t_0}^{s} p(\xi)m(\xi)d\xi + \int_{t_0}^{s} q(\xi)\sup_{-\tau\le\theta\le 0} m(\xi+\theta)d\xi\Big]\Big\} \\ &\le \tilde{c}\,\|m\|_{\infty\text{-}t_0} + \sup_{s\in[t_0,t]}\Big[\sum_{t_0<t_i<s}\beta_i \sup_{-\tau\le\theta\le 0} m(t_i+\theta) \\ &\qquad + \int_{t_0}^{s} p(\xi)m(\xi)d\xi + \int_{t_0}^{s} q(\xi)\sup_{-\tau\le\theta\le 0} m(\xi+\theta)d\xi\Big]. \end{aligned} \tag{23}$$

Now, observe that

$$\begin{aligned} \sup_{-\tau\le\theta\le 0} m(t_i+\theta) &= \sup_{t_i-\tau\le w\le t_i} m(w) \le M(t_i); \\ m(\xi) &\le \sup_{t_0-\tau\le w\le \xi} m(w) = M(\xi); \\ \sup_{-\tau\le\theta\le 0} m(\xi+\theta) &= \sup_{\xi-\tau\le w\le \xi} m(w) \le M(\xi); \end{aligned}$$

hence (23) yields

$$\begin{aligned} M(t) &\le \tilde{c}\,\|m\|_{\infty\text{-}t_0} + \sup_{s\in[t_0,t]}\Big[\sum_{t_0<t_i<s}\beta_i M(t_i) + \int_{t_0}^{s} p(\xi)M(\xi)d\xi + \int_{t_0}^{s} q(\xi)M(\xi)d\xi\Big] \\ &\le \tilde{c}\,\|m\|_{\infty\text{-}t_0} + \sum_{t_0<t_i<t}\beta_i M(t_i) + \int_{t_0}^{t}[p(s)+q(s)]M(s)ds. \end{aligned}$$

We can now apply [3, Lemma 1] and get

$$M(t) \le \tilde{c}\, \|m\|_{\infty\_t_0} \prod_{t_0<t_i<t} (1+\beta_i)\, e^{\int_{t_0}^{t} [p(s)+q(s)]ds}.$$

So, taking into account that $m(t) \le M(t)$, we achieve the inequality (21). □

The main result of the paper is contained in the following theorem.

Theorem 4.1 *Let E be a separable real Banach space and $\{t_k\}_{k\in\mathbb{N}}$ with $0 \le t_0 < t_1 < \cdots < t_k < ... \to +\infty$ be given. Assume on the family$\{A(t)\}_{t\ge0}$, the function $f : [t_0,+\infty] \times E \times \mathcal{PC}_\tau \to E$, and the impulse functions $I_k : \mathcal{PC}_\tau \to E$, $k \in \mathbb{N}$, the hypotheses (A)$'$, (h4), (h5), (h6) and*

(h7) $c_1 + c_2 e^{-\omega_*\tau} < |\omega_*|/D$.

Then, the mild solutions of (2) are globally asymptotically stable.

Proof. Let $\boldsymbol{z}, \boldsymbol{y} \in \mathcal{PC}_\tau$ be fixed. We consider mild solutions z and y to problems $(P)_{\boldsymbol{z}}$ and $(P)_{\boldsymbol{y}}$ respectively, whose existence is ensured by Remark 4.1 and Theorem 3.1.

Let us fix any $t \ge t_0$. By using *(i)* of Definition 2.1, (A)$'$ (recalling (3) and (4)), (h5), and (h6) we get the following estimate:

$$\begin{aligned}
\|z(t)-y(t)\| \;\le\;& \big\|T(t,t_0)[\boldsymbol{z}(0)-\boldsymbol{y}(0)+I_0(\boldsymbol{z})-I_0(\boldsymbol{y})]\big\| \\
&+ \sum_{t_0<t_i<t} \|T(t,t_i)[I_i(z_{t_i})-I_i(y_{t_i})]\| \\
&+ \int_{t_0}^{t} \|T(t,s)[f(s,z(s),z_s)-f(s,y(s),y_s)]\|\, ds \\
\le\;& De^{\omega_*(t-t_0)}\, \big\|\boldsymbol{z}(0)-\boldsymbol{y}(0)+I_0(\boldsymbol{z})-I_0(\boldsymbol{y})\big\| \\
&+ \sum_{t_0<t_i<t} De^{\omega_*(t-t_i)}\, \|I_i(z_{t_i})-I_i(y_{t_i})\| \\
&+ \int_{t_0}^{t} De^{\omega_*(t-s)}\, \|[f(s,z(s),z_s)-f(s,y(s),y_s)]\|\, ds \\
\le\;& De^{\omega_*(t-t_0)}\, \big\|\boldsymbol{z}(0)-\boldsymbol{y}(0)+I_0(\boldsymbol{z})-I_0(\boldsymbol{y})\big\| \\
&+ \sum_{t_0<t_i<t} De^{\omega_*(t-t_i)} a_i\, \|z_{t_i}-y_{t_i}\|_{\mathcal{PC}_\tau} \\
&+ \int_{t_0}^{t} De^{\omega_*(t-s)}\, [c_1\|z(s)-y(s)\| + c_2\|z_s-y_s\|_{\mathcal{PC}_\tau}]\, ds \\
=\;& e^{\omega_* t}\Big\{ De^{-\omega_* t_0}\, \big\|\boldsymbol{z}(0)-\boldsymbol{y}(0)+I_0(\boldsymbol{z})-I_0(\boldsymbol{y})\big\| \\
&+ \sum_{t_0<t_i<t} De^{-\omega_* t_i} a_i\, \|z_{t_i}-y_{t_i}\|_{\mathcal{PC}_\tau} \\
&+ \int_{t_0}^{t} De^{-\omega_* s}\, [c_1\|z(s)-y(s)\| + c_2\|z_s-y_s\|_{\mathcal{PC}_\tau}]\, ds \Big\},
\end{aligned}$$

from which we have

$$
\begin{aligned}
e^{-\omega_* t}\|z(t)-y(t)\| \quad \leq \quad & De^{-\omega_* t_0}\left\|\boldsymbol{z}(0)-\boldsymbol{y}(0)+I_0(\boldsymbol{z})-I_0(\boldsymbol{y})\right\| \\
& + \sum_{t_0<t_i<t} De^{-\omega_* t_i} a_i \left\|z_{t_i}-y_{t_i}\right\|_{\mathcal{PC}_\tau} \\
& + \int_{t_0}^{t} De^{-\omega_* s}\left[c_1\|z(s)-y(s)\| + c_2\|z_s-y_s\|_{\mathcal{PC}_\tau}\right] ds.
\end{aligned} \tag{24}
$$

Being $\omega^* < 0$ (cf. (A)$'$ and (4)), for any $\theta \in [-\tau, 0]$ we get $e^{\omega_* \theta} \leq e^{-\omega_* \tau}$; hence, for every $\xi \in [t_0, t]$ it holds that

$$
\begin{aligned}
e^{-\omega_* \xi} \sup_{-\tau\leq\theta\leq 0} \|z(\xi+\theta)-y(\xi+\theta)\| \quad &= \sup_{-\tau\leq\theta\leq 0} e^{-\omega_* \xi}\|z(\xi+\theta)-y(\xi+\theta)\| \\
&= \sup_{-\tau\leq\theta\leq 0} e^{\omega_* \theta} e^{-\omega_*(\xi+\theta)}\|z(\xi+\theta)-y(\xi+\theta)\| \\
&\leq \sup_{-\tau\leq\theta\leq 0} e^{-\omega_* \tau} e^{-\omega_*(\xi+\theta)}\|z(\xi+\theta)-y(\xi+\theta)\| \\
&= e^{-\omega_* \tau} \sup_{-\tau\leq\theta\leq 0} e^{-\omega_*(\xi+\theta)}\|z(\xi+\theta)-y(\xi+\theta)\|
\end{aligned} \tag{25}
$$

Therefore, by (24), (16), and (25) we obtain

$$
\begin{aligned}
e^{-\omega_* t}\|z(t)-y(t)\| \quad \leq \quad & De^{-\omega_* t_0}\left\|\boldsymbol{z}(0)-\boldsymbol{y}(0)+I_0(\boldsymbol{z})-I_0(\boldsymbol{y})\right\| \\
& + \sum_{t_0<t_i<t} a_i De^{-\omega_* t_i} \sup_{-\tau\leq\theta\leq 0} \|z(t_i+\theta)-y(t_i+\theta)\| \\
& + \int_{t_0}^{t} c_1 De^{-\omega_* s}\|z(s)-y(s)\| ds \\
& + \int_{t_0}^{t} c_2 De^{-\omega_* s} \sup_{-\tau\leq\theta\leq 0} \|z(s+\theta)-y(s+\theta)\| ds \\
\leq \quad & De^{-\omega_* t_0}\left\|\boldsymbol{z}(0)-\boldsymbol{y}(0)+I_0(\boldsymbol{z})-I_0(\boldsymbol{y})\right\| \\
& + \sum_{t_0<t_i<t} a_i De^{-\omega_* \tau} \sup_{-\tau\leq\theta\leq 0} e^{-\omega_*(t_i+\theta)} \|z(t_i+\theta)-y(t_i+\theta)\| \\
& + \int_{t_0}^{t} c_1 De^{-\omega_* s}\|z(s)-y(s)\| ds \\
& + \int_{t_0}^{t} c_2 De^{-\omega_* \tau} \sup_{-\tau\leq\theta\leq 0} e^{-\omega_*(s+\theta)}\|z(s+\theta)-y(s+\theta)\| ds.
\end{aligned}
$$

Now, we consider the function $m : [t_0-\tau, +\infty[\to \mathbb{R}^+$ defined by

$$
m(t) = \begin{cases}
e^{-\omega_* t}\|z(t)-y(t)\|, & t > t_0 \\
e^{-\omega_* t_0}\left\|\boldsymbol{z}(0)-\boldsymbol{y}(0)+I_0(\boldsymbol{z})-I_0(\boldsymbol{y})\right\|, & t = t_0 \\
e^{-\omega_* t}\|\boldsymbol{z}(t-t_0)-\boldsymbol{y}(t-t_0)\|, & t_0-\tau \leq t < t_0.
\end{cases} \tag{26}
$$

Hence the above inequality reads as

$$\begin{aligned} m(t) &\leq Dm(t_0) + \sum_{t_0<t_i<t} a_i D e^{-\omega_* \tau} \sup_{-\tau\leq\theta\leq 0} m(t_i+\theta) \\ &\quad + \int_{t_0}^{t} c_1 D m(s) ds + \int_{t_0}^{t} c_2 D e^{-\omega_*\tau} \sup_{-\tau\leq\theta\leq 0} m(s+\theta) ds. \end{aligned}$$

So we can apply Lemma 4.1 and obtain

$$m(t) \leq \tilde{D} \|m\|_{\infty\_t_0} \prod_{t_0<t_i<t} \left(1 + a_i D e^{-\omega_*\tau}\right) e^{D\left(c_1 + c_2 e^{-\omega_*\tau}\right)(t-t_0)}, \tag{27}$$

where $\tilde{D} = \max\{1, D\}$.
By (26) and recalling that $\theta \leq 0$, we have

$$\begin{aligned} \tilde{D} \|m\|_{\infty\_t_0} &= \tilde{D} \sup_{t_0-\tau\leq s\leq t_0} e^{-\omega_* s} \|z(s-t_0) - y(s-t_0)\| \\ &= \tilde{D} \sup_{-\tau\leq\theta\leq 0} e^{-\omega_*(t_0+\theta)} \|z(\theta) - y(\theta)\| \\ &\leq \tilde{D} e^{-\omega_* t_0} \sup_{-\tau\leq\theta\leq 0} \|z(\theta) - y(\theta)\|; \end{aligned}$$

hence, put

$$C(z, y) := \tilde{D} e^{-\omega_* t_0} \sup_{-\tau\leq\theta\leq 0} \|z(\theta) - y(\theta)\|,$$

by (27) and (26), for every $t > t_0$ the following estimate holds

$$e^{-\omega_* t} \|z(t) - y(t)\| \leq C(z, y) \prod_{t_0<t_i<t} \left(1 + a_i D e^{-\omega_*\tau}\right) e^{D\left(c_1 + c_2 e^{-\omega_*\tau}\right)(t-t_0)}.$$

Then, we get

$$\begin{aligned} \|z(t) - y(t)\| &\leq C(z, y) e^{\omega_* t} e^{\sum_{t_0<t_i<t} \log\left(1 + a_i D e^{-\omega_*\tau}\right)} e^{D\left(c_1 + c_2 e^{-\omega_*\tau}\right)(t-t_0)} \\ &\leq C(z, y) e^{\left[\omega_* + D\left(c_1 + c_2 e^{-\omega_*\tau}\right)\right](t-t_0)} e^{\sum_{t_0<t_i<t} \log\left(1 + a_i D e^{-\omega_*\tau}\right)}, \text{ for every } t > t_0. \end{aligned}$$

Notice that for fixed $t > t_0$ the term $\sum_{t_0<t_i<t} \log\left(1 + a_i D e^{-\omega_*\tau}\right)$ is a finite sum leading to the series $\sum_{k=1}^{\infty} \log\left(1 + a_k D e^{-\omega_*\tau}\right)$. This series has the same behavior of $\sum_{k=1}^{\infty} a_k$, which converges by (18) of hypothesis (h6).
Further, by (h4) and (h7) we have that

$$\omega_* + D\left(c_1 + c_2 e^{-\omega_*\tau}\right) < 0.$$

Therefore, we deduce that

$$\|z(t) - y(t)\| \leq C(z, y) e^{\left[\omega_* + D\left(c_1 + c_2 e^{-\omega_*\tau}\right)\right](t-t_0)} e^{\sum_{t_0<t_i<t} \log\left(1 + a_i D e^{-\omega_*\tau}\right)} \longrightarrow_{t\to+\infty} 0.$$

Note that this limit is uniform with respect to $z, y \in \mathcal{PC}_\tau$, concluding the proof. □

5 Application to the population dynamics model

Let us consider the population dynamics model driven by the parametric differential equation

$$\frac{\partial u}{\partial t}(t,x) = -b(t,x)u(t,x) + g\left(t, u(t,x), \int_{-\tau}^{0} u(t+\theta,x)d\theta\right),\ t \in [t_0,+\infty[\,,\ x \in [0,1]$$

with given initial functional datum given by a function ψ, i.e.

$$u(t_0+\theta,x) = \psi(\theta,x),\ \text{ for every } (\theta,x) \in [-\tau,0]\times[0,1].$$

The map $\psi(\cdot,x)$ is piecewise continuous on $[-\tau,0]$ in a finite number of discontinuity points not depending on $x \in [0,1]$.
The function u represents the density of a population in an environment normalized to the interval $[0,1]$ with death rate and displacement given by the removal coefficient $-b(t,x)$. The nonlinearity g represents the population development law; the dependence from the past state of the system is contained in this term by means of the integral $\int_{-\tau}^{0} u(t+\theta,x)d\theta$, generating a feedback control on the evolution of the population. Further, the external action on the system of instantaneous forces is provided by given functions $\mathcal{I}_k$, $k \in \mathbb{N}$. In the study of a pest, these functions represent the action of the pesticide used to regulate the ecosystem forcing the population to remain under some prescribed thresholds at fixed times.

The asymptotic stability of the model can be studied as an application of Theorem 4.1, as we show in what follows.
First of all, consider the family of linear functions $A(t) : L^2([0,1]) \to L^2([0,1])$, $t \geq 0$, defined by

$$A(t)y(x) = -b(t,x)y(x),\ \ x \in [0,1]. \tag{28}$$

Suppose on the function $b : [t_0,+\infty[\times[0,1] \to \mathbb{R}^+$ the assumptions

(b.1) b is measurable;

(b.2) there exist $\beta > 0$ and $s \in L^1_{loc}([t_0,+\infty[)$ such that

$$\beta < b(t,x) \leq s(t)\,,\ \text{ for every } t \geq t_0,\ \text{ a.e. } x \in [0,1];$$

(b.3) for every $x \in [0,1]$, the function $b(\cdot,x) : [t_0,+\infty[\to \mathbb{R}^+$ is continuous.

We have the following result.

Proposition 5.1 *Under assumptions (b.1)-(b.3) the family* $\{A(t)\}_{t\geq 0}$ *defined by* (28) *generates the noncompact evolution system*

$$[T(t,s)y](x) = e^{\int_s^t -b(\sigma,x)d\sigma}y(x),\ \ y \in L^2([0,1]),\ x \in [0,1],\ t \geq s \geq 0.$$

Further, the evolution system is exponentially stable.

Proof. The first part of the thesis is a consequence of [15, Section 3.1] and [5, Proposition 3.2 and Remark 3.1].

Let us show the last part. By (b.2), for every $t \geq s \geq 0$ we have

$$\begin{aligned} \|T(t,s)\|_{\mathcal{L}(L^2([0,1]))} &= \sup_{\|y\|_{L^2([0,1])}=1} \left\| e^{\int_s^t -b(\sigma,\cdot)d\sigma} y(\cdot) \right\|_{L^2([0,1])} \\ &\leq \sup_{\|y\|_{L^2([0,1])}=1} \sqrt{\int_0^1 \left[e^{-\beta(t-s)} y(x) \right]^2 dx} = e^{-\beta(t-s)}. \end{aligned}$$

Therefore, recalling the definition of ω^* (see (4)) we obtain

$$\omega^* \leq -\beta < 0.$$

□

As an immediate consequence of the previous proposition we have the following

Corollary 5.1 *Under assumptions (b.1)-(b.3), the family* $\{A(t)\}_{t\geq 0}$ *defined by* (28) *satisfies property (A)′.*

Now, let us take functions $u : [t_0 - \tau, +\infty[\times[0,1] \to \mathbb{R}$ with $u(t,\cdot) \in L^2([0,1])$ for every $t \geq t_0 - \tau$. In this way, the maps $y : [t_0 - \tau, +\infty[\to L^2([0,1])$ defined by

$$y(t)(x) = u(t,x),\ x \in [0,1],$$

are well posed. In the same way, the map $\psi : [-\tau, 0] \times [0,1] \to \mathbb{R}$ is such that $\psi(\theta,\cdot) \in L^2([0,1])$ for every $\theta \in [-\tau, 0]$. Further, the function $g : [t_0, +\infty[\times\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is such that $g\left(t, y(\cdot), \int_{-\tau}^0 \boldsymbol{y}(\theta)(\cdot)d\theta\right) \in L^2([0,1])$, for every $t \geq t_0$, $y \in L^2([0,1])$, $\boldsymbol{y} \in \mathcal{PC}_\tau{}^{L^2([0,1])}$; hence the mapping $f : [t_0, +\infty[\times L^2([0,1]) \times \mathcal{PC}_\tau{}^{L^2([0,1])} \to L^2([0,1])$ defined by

$$f(t, y, \boldsymbol{y})(x) = g\left(t, y(x), \int_{-\tau}^0 \boldsymbol{y}(\theta)(x)d\theta\right),\ x \in [0,1], \tag{29}$$

is well posed. Here the symbol $\mathcal{PC}_\tau{}^{L^2([0,1])}$ denotes the space $\mathcal{PC}_\tau$ of functions taking values in the particular $E = L^2([0,1])$.
Assume that $g : [t_0, +\infty[\times\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following properties.

(g.1) for every $y \in L^2([0,1])$, $\boldsymbol{y} \in \mathcal{PC}_\tau{}^{L^2([0,1])}$, the map $t \mapsto g\left(t, y(\cdot), \int_{-\tau}^0 \boldsymbol{y}(\theta)(\cdot)d\theta\right)$ is measurable;

(g.2) there exists $c_1, c_2 > 0$ such that

$$\left| g\left(t, p, \int_{-\tau}^0 \boldsymbol{y}(\theta)(\cdot)d\theta\right) - g\left(t, q, \int_{-\tau}^0 \boldsymbol{z}(\theta)(\cdot)d\theta\right) \right| \leq c_1|p-q| + c_2\|\boldsymbol{y} - \boldsymbol{z}\|_{\mathcal{PC}_\tau{}^{L^2([0,1])}},$$

for a.e. $t \geq t_0$ and every $p, q \in \mathbb{R}$, $\boldsymbol{y}, \boldsymbol{z} \in \mathcal{PC}_\tau{}^{L^2([0,1])}$.

The following statement holds.

Proposition 5.2 *Under assumptions (g.1) and (g.2), the function f defined by* (29) *satisfies properties (h4) and (h5).*

Proof. By (g.1) and (29) we directly have the function f satisfies (h4).

Let us fix $t \geq t_0$, $y, z \in L^2([0,1])$, $\mathcal{y}, \mathcal{z} \in \mathcal{PC}_\tau{}^{L^2([0,1])}$. By (29) and (g.2) we have that there exist $c_1, c_2 > 0$ such that

$$\begin{aligned}
\|f(t,y,\mathcal{y}) - f(t,z,\mathcal{z})\|^2_{L^2([0,1])} &= \int_0^1 \left[g\left(t, y(x), \int_{-\tau}^0 \mathcal{y}(\theta)(x)d\theta\right) \right. \\
&\qquad \left. -g\left(t, z(x), \int_{-\tau}^0 \mathcal{z}(\theta)(x)d\theta\right)\right]^2 dx \\
&\leq \int_0^1 \left[c_1|y(x) - z(x)| + c_2\|\mathcal{y} - \mathcal{z}\|_{\mathcal{PC}_\tau{}^{L^2([0,1])}}\right]^2 dx \\
&= \left[c_1\|y - z\|_{L^2([0,1])} + c_2\|\mathcal{y} - \mathcal{z}\|_{\mathcal{PC}_\tau{}^{L^2([0,1])}}\right]^2
\end{aligned}$$

so (h5) is satisfied. □

Finally, by the impulse maps acting on the system $\mathcal{I}_k : \mathbb{R} \to \mathbb{R}$, for every $k \in \mathbb{N}$ we define $I_k : \mathcal{PC}_\tau{}^{L^2([0,1])} \to L^2([0,1])$ as

$$I_i(\mathcal{y})(x) = \mathcal{I}_i\left(\int_{-\tau}^0 \mathcal{y}(\theta)(x)\, d\theta\right),\ x \in [0,1]. \tag{30}$$

On the maps $\mathcal{I}_k$ we assume the property

(I.1) for every $k \in \mathbb{N}$ there exists $a_k > 0$ such that for every $\mathcal{y}, \mathcal{z} \in \mathcal{PC}_\tau{}^{L^2([0,1])}$

$$\left|\mathcal{I}_k\left(\int_{-\tau}^0 \mathcal{y}(\theta)(\cdot)d\theta\right) - \mathcal{I}_k\left(\int_{-\tau}^0 \mathcal{z}(\theta)(\cdot)d\theta\right)\right| \leq a_k\|\mathcal{y} - \mathcal{z}\|_{\mathcal{PC}_\tau{}^{L^2([0,1])}},$$

with $\sum_{k=0}^{+\infty} a_k < +\infty$.

Proposition 5.3 *Under assumption (I.1), the functions I_k, $k \in \mathbb{N}$, defined by* (30) *satisfy property (h6).*

Proof. Let us fix $\mathcal{y}, \mathcal{z} \in \mathcal{PC}_\tau{}^{L^2([0,1])}$ and $k \in \mathbb{N}$. By (30) and (I.1) we have that there exists $a_k > 0$ such that

$$\begin{aligned}
\|I_k(\mathcal{y}) - I_k(\mathcal{z})\|_{L^2([0,1])} &= \sqrt{\int_0^1 \left[\mathcal{I}_k\left(\int_{-\tau}^0 \mathcal{y}(\theta)(x)d\theta\right) - \mathcal{I}_k\left(\int_{-\tau}^0 \mathcal{z}(\theta)(x)d\theta\right)\right]^2 dx} \\
&\leq \sqrt{\int_0^1 \left[a_k\|\mathcal{y} - \mathcal{z}\|_{\mathcal{PC}_\tau{}^{L^2([0,1])}}\right]^2 dx} = a_k\|\mathcal{y} - \mathcal{z}\|_{\mathcal{PC}_\tau{}^{L^2([0,1])}}.
\end{aligned}$$

Property (18) holds by assumption. □

From all the above propositions it is therefore evident that we can apply Theorem 4.1 so that the following theorem holds.

Theorem 5.1 *Under assumptions (b.1)-(b.3), (g.1), (g.2), (I.1) and*

(h7)' $c_1 + c_2 e^{-\omega_* \tau} < |\omega_*|$

the mild solutions of the system

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = -b(t,x)u(t,x) + g\left(t, u(t,x), \displaystyle\int_{-\tau}^{0} u(t+\theta,x)d\theta\right), \; t \geq t_0, \, t \neq t_k, \, k \in \mathbb{N}, \, x \in [0,1], \\ \\ u(t_0+\theta,x) = \psi(\theta,x), \, (\theta,x) \in [-\tau,0] \times [0,1], \\ \\ u(t_0^+,x) = \psi(0,x) + \mathcal{I}_0\left(\int_{-\tau}^{0} \psi(\theta,x)d\theta\right), \, x \in [0,1], \\ \\ u(t_k^+,x) = u(t_k,x) + \mathcal{I}_k\left(\int_{-\tau}^{0} u(t+\theta,x)d\theta\right), \, x \in [0,1] \;\;, \, k \in \mathbb{N}^+, \end{cases}$$

are globally asymptotically stable.

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