The extent of partially resolving uncertainty in assessing coherent conditional plausibilities

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Abstract

Handling uncertainty and reasoning under partial knowledge are challenging tasks that require to deal with coherent assessments and their extensions. Plausibility theory is shown to rest upon the principle of partially resolving uncertainty due to Jaffray, together with a systematically optimistic behavior. This means that we allow situations in which the agent may only acquire the information that a non-impossible event occurs, without knowing which is the true state of the world. This leads to assume that a target event is plausibly true if it is compatible with the acquired piece of information. The aim of the paper is to provide coherence conditions for a conditional plausibility assessment (namely, Pl-coherence), by referring to a suitable axiomatic definition based on the Dempster's rule of conditioning. We provide different equivalent notions of Pl-coherence in terms of consistency, betting scheme, and penalization that, as a by-product, highlight different interpretations. We then specialize the Pl-coherence conditions to the subclasses of (finitely additive) conditional probabilities and (finitely maxitive) conditional possibilities.

Keywords: Conditional plausibility, coherence, betting scheme, penalization, conditional probability, conditional possibility

1. Introduction

Uncertainty is usually modelled through a probability measure, however, applications in different fields (e.g., in decision theory, economics, and arti-

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ficial intelligence) demand for more flexible models, that are able to manage partial information on arbitrary domains. Starting from de Finetti's work on coherent finitely additive probability assessments [20], several equivalent notions have been introduced in the literature to rule incomplete probabilistic evaluations. This topic is related to the longstanding debate concerning normative theories of rational behavior, and the justification of Bayesianism and finitely additive probabilities.

A first definition of probabilistic coherence, known as *consistency*, requires the existence of a finitely additive probability measure on an algebra, extending the given assessment. Two other well-known definitions of coherence for probabilities are that based on the *betting scheme* and that based on the *penalty criterion*. The betting scheme offers an interpretation based on hypothetical bets: coherence guarantees to rule out the possibility of *Dutch books* bringing to a loss whichever state of nature occurs. On the other hand, the penalty criterion favors candor in providing probabilistic forecasts since it requires the non-existence of a rival assessment resulting in a lower loss in all the states of nature.

Though the three quoted notions of coherence are proved to be equivalent, the last two have gathered particular attention since they offer a semantics of probabilistic evaluations either as prices or as forecasts. Despite that, the main criticism (see, e.g., [20, 41]) of the classical betting scheme and penalty criteria relies on the underlying assumption of a linear utility scale, which is reasonable only for small stakes [20, 51]. Indeed, the linear utility assumption and the protocols for betting or forecasting do not allow to incorporate any attitude of the agent towards risk. Furthermore, the betting scheme criterion is exposed to strategic aspects due to the roles of bettor and bank [20, 54], which are absent in the penalty criterion.

All the three notions of coherence have been extended to cope with conditioning. Also in this case, three approaches to coherence have been investigated: that in terms of consistency [6, 14, 15], that in terms of betting scheme [65, 52, 37, 38, 62, 15] and that in terms of penalty criterion [32, 33].

A key feature of probabilistic coherence is the possibility of extending a coherent assessment to a new set of events by preserving coherence. In general, we have a class of coherent extensions, determining lower and upper envelopes that can be taken as non-additive measures. Hence, the notion of probabilistic coherence acts like a "bridge" between finitely additive probability theory and other uncertainty calculi. For instance, in some particular cases, due to specific logical conditions, these envelopes can be belief and plausibility functions [8, 16, 36] or even necessity and possibility measures [9, 26].

The paradigm of coherence has been extended referring to a class of more general uncertainty measures giving up on additivity (see, e.g., [63, 61, 62, 64, 65, 15, 18, 4, 29, 43]). Actually, de Finetti's coherence gives a natural criterion of rationality under uncertainty but its formulation is not immune to descriptive violations [27, 42]: an example in insurance is presented in [23]. In order to comply with these apparently paradoxical situations, in [24, 61, 65, 64, 32, 13] it has been shown that coherence can be adapted in order to justify non-additive probabilities, hedging, ambiguity aversion, and pessimism and optimism of agents. Following this line, in this work we focus on Dempster-Shafer theory [21, 56] which is known to possess many important connections with finitely additive probability theory.

As is well-known, the issue of conditioning in Dempster-Shafer theory is still an open problem and the most popular conditioning rules are the Dempster's rule [21], the product (or geometric) rule [57, 59], and the Bayesian rule [40, 61] (see also [8, 63]). In the particular case of possibility theory, many other conditioning rules have been proposed, and their connection to Walley's theory of imprecise probabilities has been investigated in [64, 61].

Here, we refer to the axiomatic definition of conditional plausibility function introduced in [18] and studied in [3, 13], that extends the Dempster's rule of conditioning. This definition turns out to have as particular cases conditional probability [25, 15] and T_P-conditional possibility [2, 28, 19, 7], where T_P is the algebraic product triangular norm (simply called conditional possibility, in the sequel).

We provide five notions of coherence for a conditional plausibility assessment (namely, *Pl-coherence*) defined on an arbitrary (possibly infinite) set of conditional events: a global consistency notion, a local consistency notion, a betting scheme notion, a geometric notion, and a penalty criterion notion. All such notions are proved to be equivalent, moreover, the Pl-coherence of an assessment is shown to be a necessary and sufficient condition for its Pl-coherent extendibility to any larger set of conditional events.

As is well-known, in the literature, betting scheme notions of coherence have been introduced by Williams [65] and Walley [61, 62] to rule lower/upper conditional prevision (or probability) assessments. Our betting scheme notion of Pl-coherence does not coincide with those of Williams and Walley due to the conditioning rule adopted in this paper. This means that a Pl-coherent conditional assessment is generally not coherent in the sense of Williams and Walley, i.e., it is not the upper envelope of a class of coherent conditional probabilities defined on the same events (see, e.g., [8, 60, 61]). In turn, this is due to the fact that a conditional plausibility according to our definition is not necessarily the upper envelope of a class of conditional probabilities [25]. Nevertheless, in the unconditional case, our notion of Pl-coherence guarantees that the assessment is a coherent upper probability in the sense of Williams and Walley [65, 61, 62]. Furthermore, Pl-coherence assures extendibility as a plausibility function, i.e., as a completely alternating upper probability.

Both notions of Pl-coherence in terms of betting scheme and penalty criteria assume a linear utility scale: this means that agent's desire for money does not vary with changes in his/her fortune. Nevertheless, the bets and forecasts are carried out with different protocols that allow us to incorporate a systematic attitude of the agent towards uncertainty. It turns out that the coherent conditional plausibility theory we develop rests upon the principle of *partially resolving uncertainty* due to Jaffray [39] and the assumption that the agent is *systematically optimistic*.

In detail, by partially resolving uncertainty we mean that, the agent, at the moment uncertainty is resolved, may only acquire the information that an event $B \neq \emptyset$ occurs, without knowing which is the true state of the world $\omega \in B$. Further, by systematically optimistic behavior, due to the lack of information, we mean that a target event A is taken as plausibly true if it is compatible with the acquired piece of information $B \neq \emptyset$ and false otherwise. In economic terms, assuming a linear utility scale, the agent is optimistic since betting on A he/she thinks to receive 1 monetary unit whenever, knowing that $B \neq \emptyset$ occurs, $B \cap A \neq \emptyset$ and 0 only when $B \cap A = \emptyset$. Notice that, if the agent considers as target events A and A^c, and he/she acquires the information that $B \neq \emptyset$ occurs with $A \cap B \neq \emptyset \neq A^c \cap B$, then the lack of knowledge of the true state of the world does not allow him/her to completely resolve uncertainty on A and A^c. In this case, the truth of both A and A^c is plausible, therefore agent's optimism is expressed in assuming that he/she will receive 1 monetary unit from both bets. This optimistic approach is dual to the pessimistic approach adopted in [39] to justify belief functions. The protocols for betting and making forecasts are changed by considering a total gain or a total loss, respectively, defined on non-impossible events, rather than on states of the world, like in the probabilistic case.

Then, we specialize the introduced notions of coherence and the ensuing results on equivalence and extendibility, to the sub-frameworks of conditional probability and conditional possibility. We show that, coherent conditional probability theory is based on the principle of *completely resolving uncertainty* according to which the agent will always acquire, at the moment uncertainty is resolved, the information that an element of the finest partition of the sure event spanned by the given events occurs. On the other hand, coherent conditional possibility theory is based on the principle of partially resolving uncertainty, the assumption that the agent is systematically optimistic, and the assumption of *consonance* [56]. The latter means that the agent will always acquire, at the moment uncertainty is resolved, the information that an element of a chain of events spanned by the given events occurs.

The paper is structured as follows. Section 2 introduces the axiomatic definition of conditional plausibility we refer to, showing its properties and its specialization to conditional probability and conditional possibility. In Section 3 we present the different notions of coherence for a conditional plausibility assessment and prove their equivalence. Then, in Section 4 we specialize the coherence notions and results of Section 3 to work inside the subclasses of conditional probabilities and conditional possibilities. Finally, Section 5 collects conclusions.

2. Axiomatically defined conditional plausibility

Let Ω be an arbitrary non-empty set of *states of the world* and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ an algebra of its subsets representing *events*, where $\mathcal{P}(\Omega)$ stands for the power set of Ω. Further, let $\mathcal{A}^0 = \mathcal{A} \setminus \{\emptyset\}$.

Every event $A \in \mathcal{P}(\Omega)$ is associated to the *indicator* $\mathbf{1}_A : \Omega \to \{0,1\}$, where $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise. Moreover, denoting $\mathcal{U} = \mathcal{A}^0$, the corresponding *upper generalized indicator* is the function $\mathbf{1}_A^{\mathbf{U}} : \mathcal{U} \to \{0,1\}$ defined, for every $B \in \mathcal{U}$, as

$$\mathbf{1}_A^{\mathbf{U}}(B) = \max_{\omega \in B} \mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } B \cap A \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In a finite setting, the notion of upper generalized indicator, though not explicitly introduced, can be traced back to [58]. Notice that, for $B \in \mathcal{U}$, the set function $\nu_B : \mathcal{A} \to \{0,1\}$ defined as $\nu_B(A) = \mathbf{1}_A^{\mathbf{U}}(B)$, for every $A \in \mathcal{A}$, is the dual of a *unanimity game on* B (see [30, 31]), that can be also regarded as a *vacuous plausibility function on* B (see [60]).

In case of a finite algebra $\mathcal{A}$, let $\mathbf{atoms}(\mathcal{A}) = \{C_1, \ldots, C_m\}$ be the set of its *atoms*, that is the finest partition of Ω contained in $\mathcal{A}$. Further, if $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ is an arbitrary non-empty collection, then $\mathbf{algebra}(\mathcal{E})$ denotes the minimal algebra of subsets of Ω containing $\mathcal{E}$ and $\mathbf{additive}(\mathcal{E})$ the minimal class of subsets of Ω closed under finite unions containing $\mathcal{E}$.

A *plausibility function* is a mapping $Pl : \mathcal{A} \to [0,1]$ satisfying:

(i) $Pl(\emptyset) = 0$ and $Pl(\Omega) = 1$;

(ii) for every $k \geq 2$ and for every $A_1, \ldots, A_k \in \mathcal{A}$,

$$Pl\left(\bigcap_{i=1}^{k} A_i\right) \leq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} Pl\left(\bigcup_{i\in I} A_i\right).$$

Condition *(ii)* above is usually termed *complete alternance*, moreover, together with condition *(i)* it implies *monotonicity* with respect to set inclusion, that is, $Pl(A) \leq Pl(B)$ whenever $A \subseteq B$, for $A, B \in \mathcal{A}$. In short, in this paper for plausibility function we mean a *normalized completely alternating capacity* [5]. Every plausibility function Pl is associated to a *dual* mapping $Bel : \mathcal{A} \to [0, 1]$, said *belief function*, and defined, for every $A \in \mathcal{A}$, as $Bel(A) = 1 - Pl(A^c)$.

In case of a finite algebra $\mathcal{A}$, Pl is completely characterized by the *Möbius inverse* of its dual belief function Bel, i.e., by the function $m : \mathcal{A} \to [0, 1]$ defined, for every $A \in \mathcal{A}$, as

$$m(A) = \sum_{\substack{B \subseteq A \\ B \in \mathcal{A}}} (-1)^{|C_r \in \mathbf{atoms}(\mathcal{A})\, :\, C_r \subseteq A \setminus B|} Bel(B). \tag{2}$$

Indeed, given m, for every $A \in \mathcal{A}$, we have that

$$Bel(A) = \sum_{\substack{B \subseteq A \\ B \in \mathcal{A}}} m(B) \quad \text{and} \quad Pl(A) = \sum_{\substack{B \cap A \neq \emptyset \\ B \in \mathcal{A}}} m(B). \tag{3}$$

In general, the Möbius inverse of a belief function on a finite $\mathcal{A}$ turns out to satisfy the following properties

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \in \mathcal{A}} m(A) = 1. \tag{4}$$

Hence, if we disregard $m(\emptyset) = 0$, the Möbius inverse m can be considered as a probability distribution over $\mathcal{U} = \mathcal{A}^0$, giving rise to a probability measure over $\mathcal{P}(\mathcal{U})$. In particular, the elements of $\mathcal{U}$ where m is strictly positive are called *focal elements*.

As is well-known (see, e.g., [35, 56]), *finitely additive probability measures* turn out to be particular plausibility functions, where condition *(ii)* is replaced by *finite additivity*

(ii)* $Pl(A \cup B) = Pl(A) + Pl(B)$, for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$.

In this case, the common custom is to use the symbol P in place of Pl. Moreover, in case of a finite algebra $\mathcal{A}$, probability measures are characterized by the fact that focal elements can be only in $\mathbf{atoms}(\mathcal{A})$ (see [35, 56]).

Analogously, *finitely maxitive possibility measures* turn out to be particular plausibility functions (see again [35, 56]), where condition *(ii)* is replaced by *finite maxitivity*

*(ii**)* $Pl(A \cup B) = \max\{Pl(A), Pl(B)\}$, for every $A, B \in \mathcal{A}$.

In this case, the common custom is to use the symbol Π in place of Pl. The dual of a finitely maxitive possibility measure is denoted by N and is said *finitely minitive necessity measure*. Moreover, in case of a finite algebra $\mathcal{A}$, possibility measures are characterized by the fact that focal elements are nested (see [35, 56]).

We consider the following axiomatic definition of conditional plausibility function introduced in [18] and studied in [3, 13].

Definition 1. *Let $\mathcal{H} \subseteq \mathcal{A}^0$ be an additive class, i.e., a non-empty family closed with respect to finite unions. A* **conditional plausibility function** *is a mapping $Pl : \mathcal{A} \times \mathcal{H} \to [0, 1]$ satisfying the following conditions:*

(i) $Pl(E|H) = Pl(E \cap H|H)$, for every $E \in \mathcal{A}$ and $H \in \mathcal{H}$;

(ii) $Pl(\cdot|H)$ is a plausibility function on $\mathcal{A}$, for every $H \in \mathcal{H}$;

(iii) $Pl(E \cap F|H) = Pl(E|H) \cdot Pl(F|E \cap H)$, for every $H, E \cap H \in \mathcal{H}$ and $E, F \in \mathcal{A}$.

Further we say that a conditional plausibility function is *full on $\mathcal{A}$* if $\mathcal{H} = \mathcal{A}^0$, i.e., if it is defined on the entire $\mathcal{A} \times \mathcal{A}^0$.

Condition *(ii)* of Definition 1 requires that, for every $H \in \mathcal{H}$, $Pl(\cdot|H)$ is a normalized completely alternating capacity on $\mathcal{A}$.

Remark 1. *In the above definition, a conditional event is simply regarded as a pair (E, H) of subsets of Ω, where $H \neq \emptyset$, that we denote as $E|H$. As is well-known, the set $\mathcal{A} \times \mathcal{H}$ can be endowed with the relation $\subseteq_{GN}$ due to Goodman and Nguyen [34] (see also [46]) that induces the equivalence relation $=_{GN}$. It is easily shown (see, e.g., [15, 48]) that a conditional plausibility Pl according to Definition 1 is monotonic with respect to $\subseteq_{GN}$. In turn, since $E|H =_{GN} E \cap H|H$, defining Pl on the quotient of $\mathcal{A} \times \mathcal{H}$ with respect to $=_{GN}$ makes axiom (i) superfluous. This last identification is in line with the interpretation of a conditional event as a three-valued logical entity [20, 15].*

The axiomatic definition of conditional plausibility function given above generalizes Dempster's rule of conditioning [21]. In particular, every conditional plausibility function $Pl(\cdot|\cdot)$, agreeing with Definition 1, has an associated *dual* mapping $Bel : \mathcal{A} \times \mathcal{H} \to [0,1]$ defined, for every $E|H \in \mathcal{A} \times \mathcal{H}$, as $Bel(E|H) = 1 - Pl(E^c|H)$. The function $Bel(\cdot|\cdot)$ is such that $Bel(\cdot|H)$ is a belief function on $\mathcal{A}$, for every $H \in \mathcal{H}$, thus is referred to as *conditional belief function.*

Remark 2. *This notion of conditional plausibility leads to conditional measures that are not necessarily coherent in the sense of Williams and Walley [65, 62]: this is due to the adopted notion of conditioning that generalizes Dempster's rule. Dempster's rule (and its generalization) has been justified in the literature [18, 21, 56] and a comparison with the Bayesian rule is carried out, e.g., in [8]. Notice that, a conditional plausibility according to Definition 1 is such that, locally on every* $H \in \mathcal{H}$, $Pl(\cdot|H)$ *is a coherent unconditional upper probability on* $\mathcal{A}$ *in the sense of Williams and Walley.*

The notion of conditional plausibility function given in Definition 1 has *(finitely additive) conditional probability* in the sense of Dubins [25] (see also [15]) as particular case, when condition *(ii)* in Definition 1 is replaced by

(ii)* $Pl(\cdot|H)$ is a finitely additive probability on $\mathcal{A}$, for every $H \in \mathcal{H}$.

Analogously to the unconditional case, in this case, the common custom is to use the symbol $P(\cdot|\cdot)$ in place of $Pl(\cdot|\cdot)$. Condition *(ii*)* above requires that, for every $H \in \mathcal{H}$, $Pl(\cdot|H)$ is a normalized finitely additive capacity on $\mathcal{A}$.

It turns out that Definition 1 embraces also the notion of *(finitely maxitive)* T_P*-conditional possibility* introduced in [2] and systematically studied in [28, 19, 7], where T_P is the algebraic product triangular norm. Here, we simply use the name *conditional possibility.* For that we have to substitute condition *(ii)* in Definition 1 with the condition

*(ii**)* $Pl(\cdot|H)$ is a finitely maxitive possibility on $\mathcal{A}$, for every $H \in \mathcal{H}$.

Analogously to the unconditional case, in this case, the common custom is to use the symbol $\Pi(\cdot|\cdot)$ in place of $Pl(\cdot|\cdot)$. Condition *(ii**)* above requires that, for every $H \in \mathcal{H}$, $Pl(\cdot|H)$ is a normalized finitely maxitive capacity on $\mathcal{A}$.

2.1. Layered representation for finite algebras

If $\mathcal{A}$ is a finite algebra, every conditional plausibility function $Pl(\cdot|\cdot)$ on $\mathcal{A} \times \mathcal{H}$ gives rise to a linearly ordered class $\{Pl_0, \ldots, Pl_k\}$ of plausibility functions on $\mathcal{A}$ said *$\mathcal{H}$-minimal agreeing class* (see [3]), associated to a class $\{H_0^0, \ldots, H_0^k\}$ of elements of $\mathcal{H}$ decreasingly ordered by set inclusion (this is a special case of results in [48]).

Definition 2. *Let $\mathcal{A}$ be a finite algebra and $\mathcal{H} \subseteq \mathcal{A}^0$ an additive class. A linearly ordered class of (unconditional) plausibility functions $\{Pl_0, \ldots, Pl_k\}$ on $\mathcal{A}$ is said a $\mathcal{H}$-**minimal agreeing class** on $\mathcal{A}$ if there exists a decreasing chain $H_0^0 \supset \cdots \supset H_0^k$ of elements of $\mathcal{H}$, whose enumeration agrees with the indices of plausibility functions, such that:*

(i) $H_0^0 = \bigcup_{H \in \mathcal{H}} H$;

(ii) $H_0^\alpha = \bigcup\{H \in \mathcal{H} : Pl_\beta(H) = 0, \beta = 0, \ldots, \alpha - 1\}$ *for* $\alpha = 1, \ldots, k$ *where* $H_0^{k+1} = \emptyset$;

(iii) $Pl_\alpha((H_0^\alpha)^c) = 0$.

Notice that for fixed $\mathcal{A}$ and $\mathcal{H}$, with $\mathcal{H} \subseteq \mathcal{A}^0$, the chain $H_0^0 \supset \cdots \supset H_0^k$ related to a $\mathcal{H}$-minimal agreeing class $\{Pl_0, \ldots, Pl_k\}$ on $\mathcal{A}$ is unique. Moreover, the plausibility functions in a $\mathcal{H}$-minimal agreeing class are such that, for $\alpha = 0, \ldots, k$, we have

$$Pl_\alpha(H_0^\alpha) = 1 \quad \text{and} \quad Pl_\alpha((H_0^\alpha)^c) = 0. \tag{5}$$

Theorem 1. *Let $\mathcal{A}$ be a finite algebra and $\mathcal{H} \subseteq \mathcal{A}^0$ an additive class. Conditional plausibility functions on $\mathcal{A} \times \mathcal{H}$ are in one-to-one correspondence with $\mathcal{H}$-minimal agreeing classes on $\mathcal{A}$.*

Proof. The proof is a straightforward adaptation of the proof of Theorem 1 in [48]. Given a conditional plausibility function $Pl(\cdot|\cdot)$ set:

- $Pl_0(\cdot) = Pl(\cdot|H_0^0)$ with $H_0^0 = \bigcup_{H \in \mathcal{H}} H$;
- for $\alpha > 0$, let $H_0^\alpha = \bigcup\{H \in \mathcal{H} : Pl_\beta(H) = 0, \beta = 0, \ldots, \alpha - 1\}$, if $H_0^\alpha \neq \emptyset$, then $Pl_\alpha(\cdot) = Pl(\cdot|H_0^\alpha)$, and the construction stops at index k such that $H_0^{k+1} = \emptyset$.

Vice versa, given a $\mathcal{H}$-minimal agreeing class $\{Pl_0, \ldots, Pl_k\}$ of plausibility functions on $\mathcal{A}$, for every $E|H \in \mathcal{A} \times \mathcal{H}$, denoting by α_H the minimum index in $\{0, \ldots, k\}$ such that $Pl_{\alpha_H}(H) > 0$, it holds that the function defined as

$$Pl(E|H) = \frac{Pl_{\alpha_H}(E \cap H)}{Pl_{\alpha_H}(H)}, \tag{6}$$

is a conditional plausibility function on $\mathcal{A} \times \mathcal{H}$. □

As shown in [48], given two distinct additive classes $\mathcal{H}, \mathcal{H}' \subseteq \mathcal{A}^0$, the same linearly ordered class $\{Pl_0, \ldots, Pl_k\}$ of plausibility functions on $\mathcal{A}$ can be both a $\mathcal{H}$-minimal agreeing class and a $\mathcal{H}'$-minimal agreeing class on $\mathcal{A}$. This means that $\{Pl_0, \ldots, Pl_k\}$ represents through (6) a plausibility function on $\mathcal{A} \times \mathcal{H}$, when seen as a $\mathcal{H}$-minimal agreeing class on $\mathcal{A}$, and a plausibility function on $\mathcal{A} \times \mathcal{H}'$, when seen as a $\mathcal{H}'$-minimal agreeing class on $\mathcal{A}$.

Every $\mathcal{H}$-minimal agreeing class of plausibility functions $\{Pl_0, \ldots, Pl_k\}$ on $\mathcal{A}$ gives rise to a linearly ordered class $\{m_0, \ldots, m_k\}$ of Möbius inverses of the dual belief functions. In particular, since every Pl_α satisfies (5), every Pl_α has focal elements in the set $\mathcal{U}_0^\alpha = \{B \in \mathcal{U} : B \subseteq H_0^\alpha\}$. Thus, for every $A \in \mathcal{A}$, it holds that

$$Pl_\alpha(A) = \sum_{B \in \mathcal{U}} \mathbf{1}_A^{\mathbf{U}}(B) \cdot m_\alpha(B) = \sum_{B \in \mathcal{U}_0^\alpha} \mathbf{1}_A^{\mathbf{U}}(B) \cdot m_\alpha(B), \tag{7}$$

where the first equality above has been established in [58] (see also [30]).

The following theorem states that, in case of a finite algebra $\mathcal{A}$, every conditional plausibility function can be extended to a full conditional plausibility function on a finite super-algebra $\mathcal{B}$.

Theorem 2 (Theorem 2.4 in [3]). *Let $\mathcal{A}, \mathcal{B}$ be finite algebras with $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{H} \subseteq \mathcal{A}^0$ an additive class. If $Pl : \mathcal{A} \times \mathcal{H} \to [0,1]$ is a conditional plausibility function, then there exists a full conditional plausibility function Pl' on $\mathcal{B}$ such that $Pl'_{|\mathcal{A}\times\mathcal{H}} = Pl$.*

By virtue of Theorem 1, full conditional plausibility functions on $\mathcal{A}$ are in one-to-one correspondence with $\mathcal{A}^0$-minimal agreeing classes of plausibility functions on $\mathcal{A}$. Moreover, Theorem 2 implies that every conditional plausibility function on $\mathcal{A} \times \mathcal{H}$ can be extended, generally not in a unique way, to a full conditional plausibility on $\mathcal{A}$ which, in turn, corresponds to a unique $\mathcal{A}^0$-minimal agreeing class on $\mathcal{A}$. Every conditional plausibility function $Pl(\cdot|\cdot)$ on $\mathcal{A} \times \mathcal{H}$ corresponds to possibly many $\mathcal{A}^0$-minimal agreeing classes, each representing an extension to $\mathcal{A} \times \mathcal{A}^0$: plainly, we have a unique $\mathcal{A}^0$-minimal agreeing class if $Pl(\cdot|\cdot)$ is full on $\mathcal{A}$ (see [3]).

In the particular case of a conditional probability $P(\cdot|\cdot)$ on $\mathcal{A} \times \mathcal{H}$, the notion of $\mathcal{H}$-minimal agreeing class reduces to that of agreeing class of probabilities $\{P_0, \ldots, P_k\}$ on $\mathcal{A}$ (and the ensuing notion of *zero-layers*) [15, 44]. In this case, since every P_α satisfies (5), every P_α has focal elements in the set $\mathcal{C}_0^\alpha = \{C \in \mathbf{atoms}(\mathcal{A}) : C \subseteq H_0^\alpha\}$. Thus, for every $A \in \mathcal{A}$, it holds that

$$P_\alpha(A) = \sum_{B \in \mathcal{U}} \mathbf{1}_A^{\mathbf{U}}(B) \cdot m_\alpha(B) = \sum_{C \in \mathcal{C}_0^\alpha} \mathbf{1}_A^{\mathbf{U}}(C) \cdot m_\alpha(C). \tag{8}$$

Analogously, in the particular case of a conditional possibility $\Pi(\cdot|\cdot)$ on $\mathcal{A}\times\mathcal{H}$, we get a $\mathcal{H}$-minimal agreeing class of possibility measures $\{\Pi_0,\dots,\Pi_k\}$ on $\mathcal{A}$ (see [28]).

Consider $H \in \mathcal{U} = \mathcal{A}^0$ and let $\mathbf{atoms}(\mathcal{A}) = \{C_1,\dots,C_m\}$. If $H = C_{i_1} \cup \dots \cup C_{i_h}$, denote by $\mathbf{chains}(\mathcal{U}, H)$ the collection of subfamilies of $\mathcal{U}$ such that $\mathcal{D} = \{D_1,\dots,D_h\} \in \mathbf{chains}(\mathcal{U}, H)$ if and only if $D_1 = C_{i_{\sigma(1)}}$, $D_2 = C_{i_{\sigma(1)}} \cup C_{i_{\sigma(2)}}, \dots, D_h = C_{i_{\sigma(1)}} \cup \dots \cup C_{i_{\sigma(h)}} = H$, where σ is a permutation of $\{1,\dots,h\}$. Notice that the elements of $\mathbf{chains}(\mathcal{U}, H)$ are in one-to-one correspondence with permutations of atoms contained in H. In this case, since every Π_α satisfies (5), every Π_α has all focal elements in a set $\mathcal{D}_0^\alpha \in \mathbf{chains}(\mathcal{U}, H_0^\alpha)$. Thus, for every $A \in \mathcal{A}$, it holds that

$$\Pi_\alpha(A) = \sum_{B\in\mathcal{U}} \mathbf{1}_A^{\mathbf{U}}(B) \cdot m_\alpha(B) = \sum_{D\in\mathcal{D}_0^\alpha} \mathbf{1}_A^{\mathbf{U}}(D) \cdot m_\alpha(D). \tag{9}$$

Remark 3. *The notion of $\mathcal{H}$-minimal agreeing class of possibility measures introduced in [28] does not coincide, in general, with that of $\mathcal{H}$-reduced T-nested class of possibility measures introduced in [7, 17, 19]. The essential difference in between the two concepts is that more possibility measures are needed to represent a T-conditional possibility $\Pi(\cdot|\cdot)$ if T is not a strictly monotone triangular norm. This happens since the equation $\Pi_\alpha(E \cap H) = T(\Pi(E|H), \Pi_\alpha(H))$ may not have a unique solution, even if $\Pi_\alpha(H) > 0$.*

3. Conditional plausibility assessments

We consider a non-empty set $\mathcal{G} \subseteq \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0$ of conditional events together with a conditional plausibility assessment $Pl : \mathcal{G} \to [0,1]$.

For every non-empty $\mathcal{K} \subseteq \mathcal{G}$ we denote by $\mathcal{O}(\mathcal{K}) = \{E, H \in \mathcal{P}(\Omega) : E|H \in \mathcal{K}\}$ and $\mathcal{E}(\mathcal{K}) = \{H \in \mathcal{P}(\Omega)^0 : E|H \in \mathcal{K}\}$. Moreover, let $\mathcal{A}_\mathcal{K} = \mathbf{algebra}(\mathcal{O}(\mathcal{K}))$, $\mathcal{H}_\mathcal{K} = \mathbf{additive}(\mathcal{E}(\mathcal{K}))$ and $\mathcal{U}_\mathcal{K} = \mathcal{A}_\mathcal{K} \setminus \{\emptyset\}$.

The first notion of coherence we present is a global consistency notion: it expresses the coherence of an assessment Pl in terms of its extendibility to a conditional plausibility function. We stress that this condition is merely syntactic and does not shed light on the meaning of a conditional plausibility assessment nor it provides an operational tool.

Definition 3. *An assessment $Pl : \mathcal{G} \to [0,1]$ is said to be* **Pl-coherent-1** *if there exists a conditional plausibility $Pl' : \mathcal{A}_\mathcal{G} \times \mathcal{H}_\mathcal{G} \to [0,1]$ such that $Pl'_{|\mathcal{G}} = Pl$.*

The second notion of coherence we introduce is a local consistency notion in terms of solvability of a linear system for every finite subset of conditional events. Though neither this condition helps in attaching an interpretation to a conditional plausibility assessment, it supplies an operational tool to check the consistency of an assessment.

Definition 4. *An assessment $Pl : \mathcal{G} \to [0,1]$ is said to be* **Pl-coherent-2** *if for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, denoting by $\mathcal{U}_0 = \{B \in \mathcal{U}_\mathcal{F} \ : \ B \subseteq H_0\}$ with $H_0 = \bigcup_{i=1}^n H_i$, the following system with unknowns x_B for all $B \in \mathcal{U}_0$*

$$\mathcal{S}_\mathcal{F} : \begin{cases} \sum\limits_{\substack{B \cap E_i \cap H_i \neq \emptyset \\ B \in \mathcal{U}_0}} x_B = Pl(E_i|H_i) \cdot \sum\limits_{\substack{B \cap H_i \neq \emptyset \\ B \in \mathcal{U}_0}} x_B, & \text{for } i = 1, \ldots, n, \\ \sum\limits_{B \in \mathcal{U}_0} x_B = 1, & \\ x_B \geq 0, & \text{for all } B \in \mathcal{U}_0, \end{cases} \tag{10}$$

is compatible.

The following result shows the equivalence of Pl-coherence-1 and Pl-coherence-2 conditions when the set of conditional events $\mathcal{G}$ is finite.

Lemma 1. *Let $\mathcal{G}$ be finite and $Pl : \mathcal{G} \to [0,1]$. Then the following statements are equivalent:*

(i) Pl is Pl-coherent-1;

(ii) Pl is Pl-coherent-2.

Proof. (i) $\Longrightarrow$ (ii). If the assessment Pl is Pl-coherent-1, there exists a conditional plausibility function $Pl' : \mathcal{A}_\mathcal{G} \times \mathcal{H}_\mathcal{G} \to [0,1]$, extending Pl. For every finite subset $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, denote by Pl'' the restriction to $\mathcal{A}_\mathcal{F}$ of the plausibility function $Pl'(\cdot|H_0)$. Such plausibility function is such that, for $i = 1, \ldots, n$,

$$Pl''(E_i \cap H_i) = Pl(E_i|H_i) \cdot Pl''(H_i),$$

and $Pl''(H_0) = 1$ and $Pl''(H_0^c) = 0$. Let $m'' : \mathcal{A}_\mathcal{F} \to [0,1]$ be the Möbius inverse of the dual belief function of Pl''. We have that m'' has focal elements contained in $\mathcal{U}_0$, thus setting $x_B = m''(B)$, for every $B \in \mathcal{U}_0$, we get a solution of system $\mathcal{S}_\mathcal{F}$. This implies that Pl is Pl-coherent-2.

(ii) $\Longrightarrow$ *(i)*. Suppose Pl is Pl-coherent-2. Let $\mathcal{F}_0 = \mathcal{G} = \{E_1|H_1,\ldots,E_n|H_n\}$ and denote $H_0^0 = \bigcup_{i=1}^n H_i$ and $\mathcal{U}_0^0 = \{B \in \mathcal{U}_{\mathcal{F}_0} \,:\, B \subseteq H_0^0\}$. Since Pl is Pl-coherent-2, then the corresponding system $\mathcal{S}_{\mathcal{F}_0}$ has a solution $\mathbf{x}^0$ with components x_B^0, for all $B \in \mathcal{U}_0^0$. Define $m_0 : \mathcal{A}_\mathcal{G} \to [0,1]$ setting $m_0(B) = x_B^0$, for all $B \in \mathcal{U}_0^0$, and 0 otherwise. The function m_0 is the Möbius inverse of a belief function whose dual plausibility function Pl_0 on $\mathcal{A}_\mathcal{G}$ is such that, for $i = 1,\ldots,n$,

$$Pl_0(E_i \cap H_i) = Pl(E_i|H_i) \cdot Pl_0(H_i),$$

and $Pl_0(H_0^0) = 1$ and $Pl_0((H_0^0)^c) = 0$.

For $\alpha > 0$, let $I_\alpha = \{i \in \{1,\ldots,n\} \,:\, Pl_\beta(H_i) = 0, \beta = 0,\ldots,\alpha-1\}$. If $I_\alpha = \emptyset$ the construction stops, otherwise denote $\mathcal{F}_\alpha = \{E_i|H_i\}_{i\in I_\alpha}$, $H_0^\alpha = \bigcup_{i\in I_\alpha} H_i$ and $\mathcal{U}_0^\alpha = \{B \in \mathcal{U}_{\mathcal{F}_\alpha} \,:\, B \subseteq H_0^\alpha\}$. Since Pl is Pl-coherent-2, then the corresponding system $\mathcal{S}_{\mathcal{F}_\alpha}$ has a solution $\mathbf{x}^\alpha$ with components x_B^α, for all $B \in \mathcal{U}_0^\alpha$. Define $m_\alpha : \mathcal{A}_\mathcal{G} \to [0,1]$ setting $m_\alpha(B) = x_B^\alpha$, for all $B \in \mathcal{U}_0^\alpha$, and 0 otherwise. The function m_α is the Möbius inverse of a belief function whose dual plausibility function Pl_α on $\mathcal{A}_\mathcal{G}$ is such that, for $i \in I_\alpha$,

$$Pl_\alpha(E_i \cap H_i) = Pl(E_i|H_i) \cdot Pl_\alpha(H_i),$$

and $Pl_\alpha(H_0^\alpha) = 1$ and $Pl_\alpha((H_0^\alpha)^c) = 0$.

Let k be the first index such that $I_{k+1} = \emptyset$. Then $\{Pl_0,\ldots,Pl_k\}$ is by construction a $\mathcal{H}_\mathcal{G}$-minimal agreeing class of plausibility functions on $\mathcal{A}_\mathcal{G}$ corresponding (see Subsection 2.1) to a conditional plausibility function $Pl' : \mathcal{A}_\mathcal{G} \times \mathcal{H}_\mathcal{G} \to [0,1]$ that extends Pl. This implies that Pl is Pl-coherent-1. □

Remark 4. *In case of a finite $\mathcal{G}$, though condition Pl-coherence-2 requires the solvability of the linear system $\mathcal{S}_\mathcal{F}$ for every finite subset $\mathcal{F} \subseteq \mathcal{G}$, the proof of Lemma 1 shows that Pl-coherence-2 can be reformulated in terms of solvability of a sequence of systems $\mathcal{S}_{\mathcal{F}_0},\ldots,\mathcal{S}_{\mathcal{F}_k}$ with progressively less constraints and unknowns. We point out that $\mathcal{F}_0,\ldots,\mathcal{F}_k$ are decreasingly ordered by set inclusion, with $\mathcal{F}_0 = \mathcal{G}$ and $\mathcal{F}_{k+1} = \emptyset$. This implies that also $\mathcal{U}_{\mathcal{F}_0},\ldots,\mathcal{U}_{\mathcal{F}_k}$ are decreasingly ordered by set inclusion. Further, the structure of systems depends on the sequence of solutions $\mathbf{x}^0,\ldots,\mathbf{x}^k$, that actually are restrictions of Möbius inverses defined on $\mathcal{A}_\mathcal{G}$ corresponding to a $\mathcal{H}_\mathcal{G}$-minimal agreeing class $\{Pl_0,\ldots,Pl_k\}$.*

For a finite $\mathcal{G}$, it is easy to show that in systems $\mathcal{S}_{\mathcal{F}_0},\ldots,\mathcal{S}_{\mathcal{F}_k}$ it is equivalent to take, for $\alpha = 0,\ldots,k$, $\mathcal{U}_0^\alpha = \{B \in \mathcal{U}_\mathcal{G} \,:\, B \subseteq H_0^\alpha\}$. This translates in considering more unknowns without affecting solvability and allows to obtain all $\mathcal{H}_\mathcal{G}$-minimal agreeing classes of plausibility functions on $\mathcal{A}_\mathcal{G}$ compatible with the given assessment.

The notion of coherence in terms of solvability of a sequence of systems with the structure above has been introduced in [3].

The third notion we introduce has a betting scheme interpretation, analogous to that proposed for conditional probabilities [65, 37, 38, 52] by generalizing [20], but working under partially resolving uncertainty and a systematically optimistic behavior.

Given a finite subfamily $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, a *bet* on $E_i|H_i$ with stake $\lambda_i \in \mathbb{R}$ produces a gain defined, for every $B \in \mathcal{U}_0$, as

$$\lambda_i \cdot \left[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) - Pl(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(B)\right],$$

where $Pl(E_i|H_i)$ can be interpreted as the amount paid to participate to the bet. Notice that $\mathcal{U}_0$ collects the possible partial information we may acquire when uncertainty is resolved, which are the elements of $\mathcal{U}_{\mathcal{F}}$ contained in $H_0 = \bigcup_{i=1}^n H_i$. As motivated in [39], in situations depending, for instance, on contractual clauses, the occurrence of some events could remain undetermined to the observer, because of a lack of information, at the dates at which the clauses take effect. In other terms, in such situations, that we qualify as *partially resolving uncertainty*, we can acquire the information that an event $B \in \mathcal{U}_0$ has occurred but we may not be able to identify the true state of the world.

Acquiring the information that $B \in \mathcal{U}_0$ is true, the agent receives $\lambda_i(1 - Pl(E_i|H_i))$ if $B \cap E_i \cap H_i \neq \emptyset$ and pays $-\lambda_i Pl(E_i|H_i)$ if $B \cap E_i \cap H_i = \emptyset \neq B \cap H_i$, while the bet is called off if $B \cap H_i = \emptyset$. The bets on conditional events in $\mathcal{F}$ can be combined giving rise to the gain $G_{\mathcal{F}}$ defined on $\mathcal{U}_0$, obtained summing up the gains in the single bets. The combination of bets is coherent if the gain $G_{\mathcal{F}}$ is not uniformly negative over $\mathcal{U}_0$, otherwise we will have a so-called *Dutch book* under partially resolving uncertainty and systematically optimistic behavior.

We point out that, assuming a linear utility scale, the indicator $\mathbf{1}_A$ of an event A can be seen as a gamble paying 1 monetary unit on every state of the world $\omega \in A$ and 0 otherwise. Working under partially resolving uncertainty, that is considering a gain defined on events $B \in \mathcal{U}_0$, the agent is optimistic since betting on A he/she thinks to receive 1 monetary unit whenever $B \cap A \neq \emptyset$ and 0 only when $B \cap A = \emptyset$, i.e., in his/her mental speculations he/she refers to the generalized upper indicator $\mathbf{1}^{\mathbf{U}}_A$.

Definition 5. *An assessment $Pl : \mathcal{G} \to [0, 1]$ is said to be* **Pl-coherent-3** *if for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, denoting by $\mathcal{U}_0 = \{B \in \mathcal{U}_{\mathcal{F}} \ : \ B \subseteq H_0\}$ with $H_0 = \bigcup_{i=1}^n H_i$, for every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$,* *the function* $G_{\mathcal{F}} : \mathcal{U}_0 \to \mathbb{R}$ *defined, for every* $B \in \mathcal{U}_0$*, as*

$$G_{\mathcal{F}}(B) = \sum_{i=1}^{n} \lambda_i \cdot \left[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) - Pl(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(B)\right] \tag{11}$$

satisfies $\max_{B \in \mathcal{U}_0} G_{\mathcal{F}}(B) \geq 0$*.*

The above condition Pl-coherence-3 turns out to be a particular case of a more general condition introduced in [13] in the framework of conditional completely alternating Choquet expectations.

By identifying every unconditional event E with the conditional event $E|\Omega$, the condition Pl-coherence-3 for an unconditional assessment $Pl : \mathcal{G} \to [0,1]$ with $\mathcal{G} \subseteq \mathcal{P}(\Omega)$ reduces to: for every finite $\mathcal{F} = \{E_1, \ldots, E_n\} \subseteq \mathcal{G}$, for every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, the function $G_{\mathcal{F}} : \mathcal{U}_{\mathcal{F}} \to \mathbb{R}$ defined, for every $B \in \mathcal{U}_{\mathcal{F}}$, as

$$G_{\mathcal{F}}(B) = \sum_{i=1}^{n} \lambda_i \cdot \left[\mathbf{1}^{\mathbf{U}}_{E_i}(B) - Pl(E_i)\right] \tag{12}$$

satisfies $\max_{B \in \mathcal{U}_{\mathcal{F}}} G_{\mathcal{F}}(B) \geq 0$. We stress that, in case of a finite set of unconditional events $\mathcal{G}$, it is sufficient to require condition Pl-coherence-3 to hold only on $\mathcal{F} = \mathcal{G}$.

Limiting to the unconditional case, a dual form of condition Pl-coherence-3 has been introduced in [47] in the framework of belief functions. Such condition is a generalization to an arbitrary set of events of conditions given in [39, 53]. In case of a finite Ω and $\mathcal{G} = \mathcal{P}(\Omega)$, it is possible to show that such a dual form of our condition Pl-coherence-3 can be expressed in terms of the notion of B-consistency for a coherent betting function $R : \mathbb{R}^{\Omega} \to \{0, 1\}$ introduced in [43]. We also have that a betting condition for a belief assessment on many-valued events has been considered in [29]. Finally, condition Pl-coherence-3 is the quantitative counterpart of the qualitative notions for gambles given in [11, 12].

The next coherence condition has a geometric interpretation. It relies, for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, denoting by $\mathcal{U}_0 = \{B \in \mathcal{U}_{\mathcal{F}} \ : \ B \subseteq H_0\}$ with $H_0 = \bigcup_{i=1}^{n} H_i$, on the functions $Q_i : \mathcal{U}_0 \to [0,1]$, for $i = 1, \ldots, n$, defined, for every $B \in \mathcal{U}_0$, as

$$Q_i(B) = \begin{cases} 1 & \text{if } \mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) = 1 \text{ and } \mathbf{1}^{\mathbf{U}}_{H_i}(B) = 1, \\ 0 & \text{if } \mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) = 0 \text{ and } \mathbf{1}^{\mathbf{U}}_{H_i}(B) = 1, \\ Pl(E_i|H_i) & \text{otherwise.} \end{cases} \tag{13}$$

Moreover, given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s \in \mathbb{R}^n$, we denote by $\mathbf{conv}(\{\mathbf{v}_1, \ldots, \mathbf{v}_s\})$ their convex hull. While, for vectors $\mathbf{u} = (u_1, \ldots, u_n)^T, \mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$, denote by

$$d_E(\mathbf{u}, \mathbf{v}) = \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2},$$

their Euclidean distance. In the condition below, the coherence of the assessment is expressed requiring that, for every finite subset of events $\mathcal{F}$, the vector formed by the values of the assessment on $\mathcal{F}$ belongs to the convex hull of a finite collection of vectors built through the functions Q_i's.

Definition 6. *An assessment $Pl : \mathcal{G} \to [0,1]$ is said to be* **Pl-coherent-4** *if for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, denoting by $\mathcal{U}_0 = \{B \in \mathcal{U}_{\mathcal{F}} \,:\, B \subseteq H_0\}$ with $H_0 = \bigcup_{i=1}^{n} H_i$, the vectors in $\mathbb{R}^n$*

$$\begin{aligned} \mathbf{d} &= (Pl(E_1|H_1), \ldots, Pl(E_n|H_n))^T, && (14) \\ \mathbf{q}_B &= (Q_1(B), \ldots, Q_n(B))^T, \quad \textit{for all } B \in \mathcal{U}_0, && (15) \end{aligned}$$

are such that $\mathbf{d} \in \mathbf{conv}(\{\mathbf{q}_B \,:\, B \in \mathcal{U}_0\})$.

The last notion of coherence we propose is the generalization of the penalty criterion introduced in [20] for a probability assessment. Given a finite subfamily $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, the assessed $Pl(E_i|H_i)$ on $E_i|H_i$ causes to the agent a *penalty* defined, for every $B \in \mathcal{U}_0$, as

$$\left[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) - Pl(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(B)\right]^2.$$

Also in this case, we assume partially resolving uncertainty and a systematically optimistic behavior: $\mathcal{U}_0$ collects the possible partial information we may acquire when uncertainty is resolved and a target event is assumed to be plausibly true if it is compatible with the acquired piece of information.

Acquiring the information that $B \in \mathcal{U}_0$ is true, the agent incurs in a penalty of $(1 - Pl(E_i|H_i))^2$ if $B \cap E_i \cap H_i \neq \emptyset$ and $(-Pl(E_i|H_i))^2$ if $B \cap E_i \cap H_i = \emptyset \neq B \cap H_i$, while no penalty is assigned if $B \cap H_i = \emptyset$. Hence, the restriction of the assessment Pl to $\mathcal{F}$ will result in a global loss $L_{\mathcal{F}}$ defined on $\mathcal{U}_0$, obtained summing up the single penalties. The assessment is coherent if there is no distinct assessment $Pl^* : \mathcal{F} \to [0,1]$ such that the corresponding global loss $L^*_{\mathcal{F}}$ is uniformly lower than $L_{\mathcal{F}}$ over $\mathcal{U}_0$.

Definition 7. *An assessment $Pl : \mathcal{G} \to [0,1]$ is said to be* **Pl-coherent-5** *if for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, denoting* *by* $\mathcal{U}_0 = \{B \in \mathcal{U}_{\mathcal{F}} \ : \ B \subseteq H_0\}$ *with* $H_0 = \bigcup_{i=1}^n H_i$*, there is no distinct assessment* $Pl^* : \mathcal{F} \to [0,1]$ *such that the functions* $L_{\mathcal{F}}, L^*_{\mathcal{F}} : \mathcal{U}_0 \to \mathbb{R}$ *defined, for every* $B \in \mathcal{U}_0$*, as*

$$L_{\mathcal{F}}(B) = \sum_{i=1}^n \left[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) - Pl(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(B)\right]^2, \tag{16}$$

$$L^*_{\mathcal{F}}(B) = \sum_{i=1}^n \left[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) - Pl^*(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(B)\right]^2, \tag{17}$$

are such that $L^*_{\mathcal{F}}(B) < L_{\mathcal{F}}(B)$ *for every* $B \in \mathcal{U}_0$*.*

Again, by identifying every unconditional event E with the conditional event $E|\Omega$, the condition Pl-coherence-5 for an unconditional assessment $Pl : \mathcal{G} \to [0,1]$ with $\mathcal{G} \subseteq \mathcal{P}(\Omega)$ reduces to: for every finite $\mathcal{F} = \{E_1, \ldots, E_n\} \subseteq \mathcal{G}$, there is no distinct assessment $Pl^* : \mathcal{F} \to [0,1]$ such that the functions $L_{\mathcal{F}}, L^*_{\mathcal{F}} : \mathcal{U}_{\mathcal{F}} \to \mathbb{R}$ defined, for every $B \in \mathcal{U}_{\mathcal{F}}$, as

$$L_{\mathcal{F}}(B) = \sum_{i=1}^n \left[\mathbf{1}^{\mathbf{U}}_{E_i}(B) - Pl(E_i)\right]^2, \tag{18}$$

$$L^*_{\mathcal{F}}(B) = \sum_{i=1}^n \left[\mathbf{1}^{\mathbf{U}}_{E_i}(B) - Pl^*(E_i)\right]^2, \tag{19}$$

are such that $L^*_{\mathcal{F}}(B) < L_{\mathcal{F}}(B)$ for every $B \in \mathcal{U}_{\mathcal{F}}$. Also in this case, we stress that, in case of a finite set of unconditional events $\mathcal{G}$, it is sufficient to require condition Pl-coherence-5 to hold only on $\mathcal{F} = \mathcal{G}$.

Limiting to the unconditional case, a dual form of condition Pl-coherence-5 has been introduced in [47] in the framework of belief functions.

The following theorem shows that all the notions of coherence introduced so far are actually equivalent.

Theorem 3. *For a conditional plausibility assessment* $Pl : \mathcal{G} \to [0,1]$*, the following statements are equivalent:*

(i) Pl is Pl-coherent-1;

(ii) Pl is Pl-coherent-2;

(iii) Pl is Pl-coherent-3;

(iv) Pl is Pl-coherent-4;

(v) Pl is Pl-coherent-5.

Proof. (i) $\iff$ (ii). By Lemma 1, if the assessment Pl is Pl-coherent-1, then it is Pl-coherent-2. Moreover, if Pl is Pl-coherent-2, Lemma 1 implies that, for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, there exists a conditional plausibility function Pl'' on $\mathcal{A}_\mathcal{F} \times \mathcal{H}_\mathcal{F}$ extending $Pl_{|\mathcal{F}}$. Notice that every conditional plausibility function Pl'' on $\mathcal{A}_\mathcal{F} \times \mathcal{H}_\mathcal{F}$ extending $Pl_{|\mathcal{F}}$ is obtained through a sequence of solutions of systems $\mathcal{S}_{\mathcal{F}_0}, \ldots, \mathcal{S}_{\mathcal{F}_k}$ as highlighted in Remark 4. Denote by $\mathbf{P}_\mathcal{F}$ the set of mappings from $\mathcal{A}_\mathcal{G} \times \mathcal{H}_\mathcal{G}$ to $[0,1]$ whose restriction to $\mathcal{A}_\mathcal{F} \times \mathcal{H}_\mathcal{F}$ is a conditional plausibility function extending $Pl_{|\mathcal{F}}$, where the restriction is determined by a sequence of solutions of systems $\mathcal{S}_{\mathcal{F}_0}, \ldots, \mathcal{S}_{\mathcal{F}_k}$. The set $\mathbf{P}_\mathcal{F}$ is a non-empty closed subset of the compact and Hausdorff space $[0,1]^{\mathcal{A}_\mathcal{G} \times \mathcal{H}_\mathcal{G}}$ endowed with the product topology. Indeed, for every net $\{\nu_\lambda\}_{\lambda \in \Lambda}$ in $\mathbf{P}_\mathcal{F}$ converging pointwise to ν, a simple application of properties of limits of real nets shows that ν is a $[0,1]$-valued function whose restriction to $\mathcal{A}_\mathcal{F} \times \mathcal{H}_\mathcal{F}$ is a conditional plausibility and $\nu(E|H) = Pl(E|H)$ for every $E|H \in \mathcal{F}$. Thus, $\nu \in \mathbf{P}_\mathcal{F}$. It is easily seen that the family

$$\{\mathbf{P}_\mathcal{F} \,:\, \mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}, n \in \mathbb{N}\},$$

possesses the finite intersection property, thus it holds that

$$\bigcap \{\mathbf{P}_\mathcal{F} \,:\, \mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}, n \in \mathbb{N}\} \neq \emptyset$$

and so there exists $Pl' \in \bigcap \{\mathbf{P}_\mathcal{F} \,:\, \mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}, n \in \mathbb{N}\}$ which is a conditional plausibility function on $\mathcal{A}_\mathcal{G} \times \mathcal{H}_\mathcal{G}$ extending Pl.

(ii) $\iff$ (iii) Fix the enumeration of $\mathcal{U}_0 = \{A_1, \ldots, A_h\}$ and consider the matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{(n+1) \times h}$ with

$$\begin{aligned} a_{ij} &= \mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(A_j) - Pl(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(A_j), \\ a_{(n+1)j} &= \mathbf{1}^{\mathbf{U}}_{H_0}(A_j) = 1, \end{aligned}$$

and the vector $\mathbf{b} = (0, \ldots, 0, 1)^T \in \mathbb{R}^{(n+1)}$. The system $\mathcal{S}_\mathcal{F}$ in condition Pl-coherence-2 can be written in matrix form as

$$\mathcal{S}_\mathcal{F} : \begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}, \end{cases}$$

where $\mathbf{x} = (x_{A_1}, \ldots, x_{A_h})^T \in \mathbb{R}^h$ is an unknown column vector

By Farkas' lemma [45], the above system $\mathcal{S}_\mathcal{F}$ has solution if and only if the following system has no solution

$$\mathcal{S}^*_\mathcal{F} : \begin{cases} \mathbf{A}^T\mathbf{y} \leq \mathbf{0}, \\ \mathbf{b}^T\mathbf{y} > 0, \end{cases}$$

where $\mathbf{y} = (\lambda_1, \ldots, \lambda_n, y_{n+1}) \in \mathbb{R}^{(n+1)}$ and $\mathbf{b}^T\mathbf{y} = y_{n+1}$. It holds that $\mathbf{A}^T\mathbf{y} \in \mathbb{R}^h$ and, for $j = 1, \ldots, h$, the jth component of constraint $\mathbf{A}^T\mathbf{y} \leq \mathbf{0}$ is

$$\sum_{i=1}^{n} \lambda_i \cdot [\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(A_j) - Pl(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(A_j)] + y_{n+1} \leq 0.$$

Therefore, for every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $y_{n+1} > 0$ there must exist at least an index $j \in \{1, \ldots, h\}$ such that $(\mathbf{A}^T\mathbf{y})_j > 0$. Hence, the non-solvability of $\mathcal{S}^*_{\mathcal{F}}$ is equivalent, for every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, to

$$\max_{j \in \{1,\ldots,h\}} G_{\mathcal{F}}(A_j) \geq 0,$$

where we set

$$G_{\mathcal{F}}(A_j) = \sum_{i=1}^{n} \lambda_i \cdot [\mathbf{1}^{\mathbf{U}}_{E_1 \cap H_i}(A_j) - Pl(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(A_j)].$$

(ii) $\Longleftrightarrow$ *(iv)*. Fix the enumeration of $\mathcal{U}_0 = \{A_1, \ldots, A_h\}$ and consider the matrix $\mathbf{Q} = (\mathbf{q}_{A_1}, \ldots, \mathbf{q}_{A_h}) \in \mathbb{R}^{n \times h}$ and the column vector $\mathbf{d} = (Pl(E_1|H_1), \ldots, Pl(E_n|H_n))^T \in \mathbb{R}^n$. System $\mathcal{S}_{\mathcal{F}}$ in condition Pl-coherence-2 can be written as

$$\mathcal{S}_{\mathcal{F}} : \begin{cases} \mathbf{Q}\mathbf{x} = \mathbf{d}, \\ \sum_{j=1}^{h} x_{A_j} = 1, \\ \mathbf{x} \geq \mathbf{0}. \end{cases}$$

To see this, for $i = 1, \ldots, n$, notice that the first equation in (10) can be rewritten as

$$\begin{aligned} & \sum_{j=1}^{h} \mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(A_j) x_{A_j} = Pl(E_i|H_i) \cdot \sum_{j=1}^{h} \mathbf{1}^{\mathbf{U}}_{H_i}(A_j) x_{A_j} \\ & \Longleftrightarrow \sum_{j=1}^{h} \left[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(A_j) - Pl(E_i|H_i) \mathbf{1}^{\mathbf{U}}_{H_i}(A_j)\right] x_{A_j} = 0 \\ & \Longleftrightarrow \sum_{j=1}^{h} \left[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(A_j) + Pl(E_i|H_i) \left(\mathbf{1}^{\mathbf{U}}_{H_0}(A_j) - \mathbf{1}^{\mathbf{U}}_{H_i}(A_j)\right)\right] x_{A_j} = Pl(E_i|H_i) \end{aligned}$$

where the last equivalence follows since $\mathbf{1}^{\mathbf{U}}_{H_0}$ is constantly equal to 1 on $\mathcal{U}_0$ and $\sum_{j=1}^{h} x_{A_j} = 1$. Finally, by the definition of Q_i in (13), the last equation can be rewritten as

$$\sum_{j=1}^{h} Q_i(A_j) x_{A_j} = Pl(E_i|H_i).$$

Hence, $\mathbf{x} = (x_{A_1}, \ldots, x_{A_h})^T \in \mathbb{R}^h$ is a solution of system $\mathcal{S}_{\mathcal{F}}$ if and only if $\mathbf{x} \geq \mathbf{0}$, $\sum_{j=1}^{h} x_{A_j} = 1$ and

$$\mathbf{d} = \sum_{j=1}^{h} x_{A_j} \mathbf{q}_{A_j},$$

that is $\mathbf{d}$ is the convex combination of vectors $\mathbf{q}_{A_j}$'s with weights given by $\mathbf{x}$. In other terms, condition *(ii)* is equivalent to the fact that the vector $\mathbf{d}$ belongs to the convex hull of vectors $\mathbf{q}_{A_j}$'s, that is $\mathbf{d} \in \mathbf{conv}(\{\mathbf{q}_{A_1}, \ldots, \mathbf{q}_{A_h}\})$.

(v) $\Longrightarrow$ *(iv)*. We show that condition *(v)* does not hold if condition *(iv)* does not hold. Suppose there exist $n \in \mathbb{N}$ and $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$ such that *(iv)* does not hold, that is $\mathbf{d} \notin \mathbf{conv}(\{\mathbf{q}_B : B \in \mathcal{U}_0\})$. As follows from results in [33], the squared Euclidean distance on the unit n-cube coincides with the Bregman divergence determined by the Brier quadratic scoring rule, which is a bounded (strictly) proper scoring rule (see [50]). Hence, by Proposition 3 in [50], there exists a unique element of $\mathbf{conv}(\{\mathbf{q}_B : B \in \mathcal{U}_0\})$ minimizing the squared Euclidean distance with respect to $\mathbf{d}$, said projection of $\mathbf{d}$ onto $\mathbf{conv}(\{\mathbf{q}_B : B \in \mathcal{U}_0\})$. Let $\mathbf{d}^* = (Pl^*(E_1|H_1), \ldots, Pl^*(E_n|H_n))^T$ be the projection of $\mathbf{d}$ onto $\mathbf{conv}(\{\mathbf{q}_B : B \in \mathcal{U}_0\})$, that is

$$\mathbf{d}^* = \underset{\mathbf{u} \in \mathbf{conv}(\{\mathbf{q}_B \,:\, B \in \mathcal{U}_0\})}{\arg\min} d_E(\mathbf{u}, \mathbf{d})^2.$$

By Proposition 3 in [50] it holds that, for all $B \in \mathcal{U}_0$, we have

$$d_E(\mathbf{q}_B, \mathbf{d}^*)^2 + d_E(\mathbf{d}^*, \mathbf{d})^2 \leq d_E(\mathbf{q}_B, \mathbf{d})^2,$$

moreover, since $d_E(\mathbf{d}^*, \mathbf{d})^2 > 0$, we have

$$d_E(\mathbf{q}_B, \mathbf{d}^*)^2 < d_E(\mathbf{q}_B, \mathbf{d})^2.$$

For every $B \in \mathcal{U}_0$, let $I_B = \{i \in \{1, \dots, n\} : \mathbf{1}^{\mathbf{U}}_{H_i}(B) \neq 0\}$. It holds that

$$\begin{aligned}
L_{\mathcal{F}}(B) &= \sum_{i=1}^{n} (\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) - Pl(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(B))^2 \\
&= \sum_{i=1}^{n} (Q_i(B) - Pl(E_i|H_i))^2 \\
&= \sum_{i \in I_B} (Q_i(B) - Pl(E_i|H_i))^2 = d_E(\mathbf{q}_B, \mathbf{d})^2, \\
L^*_{\mathcal{F}}(B) &= \sum_{i=1}^{n} (\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) - Pl^*(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(B))^2 \\
&= \sum_{i=1}^{n} (Q^*_i(B) - Pl^*(E_i|H_i))^2 \\
&= \sum_{i \in I_B} (Q^*_i(B) - Pl(E_i|H_i))^2 = d_E(\mathbf{q}^*_B, \mathbf{d})^2,
\end{aligned}$$

where Q^*_i and $\mathbf{q}^*_B$ are defined as in (13) and (15), using Pl^* in place of Pl.

For every $B \in \mathcal{U}_0$, since for all $i \in I_B$ it holds that $Q_i(B) = Q^*_i(B)$, we have that

$$\begin{aligned}
d_E(\mathbf{q}_B, \mathbf{d}^*)^2 &= \sum_{i=1}^{n} (Q_i(B) - Pl^*(E_i|H_i))^2 \\
&= \sum_{i \in I_B} (Q^*_i(B) - Pl^*(E_i|H_i))^2 + \sum_{i \notin I_B} (Pl(E_i|H_i) - Pl^*(E_i|H_i))^2 \\
&= d_E(\mathbf{q}^*_B, \mathbf{d}^*)^2 + d_E(\mathbf{q}_B, \mathbf{q}^*_B)^2 \geq d_E(\mathbf{q}^*_B, \mathbf{d}^*)^2.
\end{aligned}$$

From this we get that, for every $B \in \mathcal{U}_0$,

$$L^*_{\mathcal{F}}(B) = d_E(\mathbf{q}^*_B, \mathbf{d}^*)^2 \leq d_E(\mathbf{q}_B, \mathbf{d}^*)^2 < d_E(\mathbf{q}_B, \mathbf{d})^2 = L_{\mathcal{F}}(B),$$

and this implies that with such assessment $Pl^* : \mathcal{F} \to [0,1]$ we have $L^*_{\mathcal{F}}(B) < L_{\mathcal{F}}(B)$, for every $B \in \mathcal{U}_0$, and so condition *(v)* does not hold.

(iv) $\Longrightarrow$ *(v)*. Suppose that for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\} \subseteq \mathcal{G}$, setting $\mathbf{d} = (Pl(E_1|H_1), \dots, Pl(E_n|H_n))^T$, it holds that $\mathbf{d} \in \mathbf{conv}(\{\mathbf{q}_B : B \in \mathcal{U}_0\})$.

Let $Pl^* : \mathcal{F} \to [0,1]$ be a distinct assessment and denote $p_i = Pl(E_i|H_i)$ and $p^*_i = Pl(E_i|H_i)$, for $i = 1, \dots, n$. We distinguish two cases:

(a) $p^*_i \neq p_i$, for all $i = 1, \dots, n$,

(b) $p_i^* = p_i$, for some indices i.

Case (a). For every $B \in \mathcal{U}_0$, we have that

$$L_{\mathcal{F}}(B) - L^*_{\mathcal{F}}(B) = 2\sum_{i=1}^{n} \mathbf{1}^{\mathbf{U}}_{H_i}(B)(p_i^* - p_i)[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) - p_i] - \delta_B,$$

where $\delta_B = \sum_{i=1}^{n}(p_i^* - p_i)^2 \mathbf{1}^{\mathbf{U}}_{H_i}(B) \geq 0$. The hypothesis $p_i^* \neq p_i$, for all $i = 1, \ldots, n$, implies that $\delta_B > 0$, for every $B \in \mathcal{U}_0$. Moreover, since $\mathbf{d} = \sum_{B \in \mathcal{U}_0} x_B \mathbf{q}_B$, we have that, for all $i = 1, \ldots, n$,

$$p_i = Pl(E_i|H_i) = \sum_{B \in \mathcal{U}_0} x_B Q_i(B).$$

Hence, it follows that

$$\begin{aligned} \sum_{B \in \mathcal{U}_0} x_B[L_{\mathcal{F}}(B) - L^*_{\mathcal{F}}(B)] &= 2\sum_{i=1}^{n}(p_i^* - p_i) \sum_{B \in \mathcal{U}_0} x_B \mathbf{1}^{\mathbf{U}}_{H_i}(B)[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) - p_i] - \sum_{B \in \mathcal{U}_0} x_B \delta_B \\ &= -\sum_{B \in \mathcal{U}_0} x_B \delta_B, \end{aligned}$$

where the last equality is due to

$$\begin{aligned} \sum_{B \in \mathcal{U}_0} x_B \mathbf{1}^{\mathbf{U}}_{H_i}(B)[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) - p_i] &= \sum_{B \in \mathcal{U}_0} x_B[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(B) - p_i \mathbf{1}^{\mathbf{U}}_{H_i}(B)] \\ &= \sum_{B \in \mathcal{U}_0} x_B[Q_i(B) - p_i] \\ &= \sum_{B \in \mathcal{U}_0} x_B Q_i(B) - \sum_{B \in \mathcal{U}_0} x_B p_i = p_i - p_i = 0. \end{aligned}$$

Thus, since $\delta_B > 0$, for all $B \in \mathcal{U}_0$, we get

$$\sum_{B \in \mathcal{U}_0} x_B[L_{\mathcal{F}}(B) - L^*_{\mathcal{F}}(B)] = -\sum_{B \in \mathcal{U}_0} x_B \delta_B < 0,$$

so there exists at least a $B \in \mathcal{U}_0$ such that $L_{\mathcal{F}}(B) < L^*_{\mathcal{F}}(B)$, that is it cannot be $L^*_{\mathcal{F}}(B) < L_{\mathcal{F}}(B)$, for every $B \in \mathcal{U}_0$.

Case (b). Let $J = \{i \in \{1, \ldots, n\} : Pl^*(E_i|H_i) \neq Pl(E_i|H_i)\}$. The restriction of Pl to $\mathcal{F}' = \{E_j|H_j\}_{j \in J} \subseteq \mathcal{F}$ satisfies condition *(iv)*. Hence, denoting $H_0' = \bigcup_{j \in J} H_j$ and $\mathcal{U}_0' = \{B \in \mathcal{U}_{\mathcal{F}'} : B \subseteq H_0'\}$, there exists

$\mathbf{x}' \in \mathbb{R}^{\mathcal{U}_0'}$ with $\sum_{B \in \mathcal{U}_0'} x'_B = 1$ and $x'_B \geq 0$ for all $B \in \mathcal{U}_0'$ such that, setting $\mathbf{d}' = (Pl(E_j|H_j))^T_{j \in J}$, we have that

$$\mathbf{d}' = \sum_{B \in \mathcal{U}_0'} x'_B \mathbf{q}'_B$$

where, for every $B \in \mathcal{U}_0'$, $\mathbf{q}'_B = (Q_j(B))^T_{j \in J}$. Following the same steps of case *(a)* we have that there exists at least a $B \in \mathcal{U}_0'$ such that $L_{\mathcal{F}'}(B) < L^*_{\mathcal{F}'}(B)$.

Finally, since $\mathcal{F}' \subseteq \mathcal{F}$ implies $\mathcal{A}_{\mathcal{F}'} \subseteq \mathcal{A}_{\mathcal{F}}$ which, in turn, implies $\mathcal{U}_0' \subseteq \mathcal{U}_0$, we have that for all $B \in \mathcal{U}_0' \subseteq \mathcal{U}_0$

$$\begin{aligned} L_{\mathcal{F}}(B) &= \sum_{j \notin J} (\mathbf{1}^{\mathbf{U}}_{E_j \cap H_j}(B) - p_j \mathbf{1}^{\mathbf{U}}_{H_j}(B))^2 + \sum_{j \in J} (\mathbf{1}^{\mathbf{U}}_{E_j \cap H_j}(B) - p_j \mathbf{1}^{\mathbf{U}}_{H_j}(B))^2, \\ L^*_{\mathcal{F}}(B) &= \sum_{j \notin J} (\mathbf{1}^{\mathbf{U}}_{E_j \cap H_j}(B) - p_j \mathbf{1}^{\mathbf{U}}_{H_j}(B))^2 + \sum_{j \in J} (\mathbf{1}^{\mathbf{U}}_{E_j \cap H_j}(B) - p^*_j \mathbf{1}^{\mathbf{U}}_{H_j}(B))^2. \end{aligned}$$

This implies that

$$\begin{aligned} L_{\mathcal{F}}(B) - L^*_{\mathcal{F}}(B) &= \sum_{j \in J} (\mathbf{1}^{\mathbf{U}}_{E_j \cap H_j}(B) - p_j \mathbf{1}^{\mathbf{U}}_{H_j}(B))^2 - \sum_{j \in J} (\mathbf{1}^{\mathbf{U}}_{E_j \cap H_j}(B) - p^*_j \mathbf{1}^{\mathbf{U}}_{H_j}(B))^2 \\ &= L_{\mathcal{F}'}(B) - L^*_{\mathcal{F}'}(B), \end{aligned}$$

so there exists a $B \in \mathcal{U}_0' \subseteq \mathcal{U}_0$ such that $L_{\mathcal{F}}(B) < L^*_{\mathcal{F}}(B)$, that is it cannot be $L^*_{\mathcal{F}}(B) < L_{\mathcal{F}}(B)$, for every $B \in \mathcal{U}_0$. □

By virtue of Theorem 3, we say that a conditional plausibility assessment $Pl : \mathcal{G} \to [0, 1]$ is *Pl-coherent* if one (and hence all) of the previous notions of coherence holds, otherwise it is said *Pl-incoherent*.

We want to stress that, for a finite $\mathcal{F} \subseteq \mathcal{G}$, the projection $\mathbf{d}^*$ of a Pl-incoherent $\mathbf{d}$ onto $\mathbf{conv}(\{\mathbf{q}_B \ B \in \mathcal{U}_0\})$ does not generally give rise to a Pl-coherent assessment on $\mathcal{F}$. On the other hand, analogously to what happens for unconditional belief assessments [47], if all the events in $\mathcal{F}$ are conditional on the same conditioning event $H \neq \emptyset$, then $\mathbf{d}^*$ turns out to be Pl-coherent. This is due to the fact that $H_0 = H$, vectors $\mathbf{q}_B$'s take values in $\{0, 1\}$, and $\mathbf{d}^*$ is actually the restriction to $\mathcal{F}$ of a conditional plausibility $Pl'(\cdot|H)$ defined on $\mathcal{A}_{\mathcal{F}}$.

Example 1. *Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider the Pl-incoherent assessment $Pl(A|\Omega) = Pl(B|\Omega) = \frac{1}{4}$ and $Pl(A \cup B|\Omega) = \frac{3}{4}$, where $A = \{\omega_1\}$ and $B = \{\omega_3\}$.*

Take $\mathcal{F} = \{A|\Omega, B|\Omega, A \cup B|\Omega\}$ with $\mathcal{A}_{\mathcal{F}} = \mathcal{P}(\Omega)$, $H_0 = \Omega$ and $\mathcal{U}_0 = \mathcal{A}_{\mathcal{F}} \setminus \{\emptyset\}$. Denote $A_i = \{\omega_i\}$, $A_{ij} = \{\omega_i, \omega_j\}$.

	$\mathbf{d}$	$\mathbf{q}_{A_1}$	$\mathbf{q}_{A_2}$	$\mathbf{q}_{A_3}$	$\mathbf{q}_{A_{12}}$	$\mathbf{q}_{A_{13}}$	$\mathbf{q}_{A_{23}}$	$\mathbf{q}_{\Omega}$
$A\|\Omega$	$\frac{1}{4}$	1	0	0	1	1	0	1
$B\|\Omega$	$\frac{1}{4}$	0	0	1	0	1	1	1
$A \cup B\|\Omega$	$\frac{3}{4}$	1	0	1	1	1	1	1

Since $\mathbf{q}_{A_1} = \mathbf{q}_{A_{12}}$, $\mathbf{q}_{A_3} = \mathbf{q}_{A_{23}}$ *and* $\mathbf{q}_{A_{13}} = \mathbf{q}_{A_\Omega}$, *denoting by* $\mathbf{a}^1 = (1,0,1)^T$, $\mathbf{a}^2 = (0,0,0)^T$, $\mathbf{a}^3 = (0,1,1)^T$ *and* $\mathbf{a}^4 = (1,1,1)^T$, *it holds that* $\mathbf{conv}(\{\mathbf{q}_B : B \in \mathcal{U}_0\}) = \mathbf{conv}(\{\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3,\mathbf{a}^4\})$. *A simple verification (see Figure 1) shows that* $\mathbf{d} \notin \mathbf{conv}(\{\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3,\mathbf{a}^4\})$.

We consider the projection

$$\mathbf{d}^* = \arg\min_{\mathbf{u}\in\mathbf{conv}(\{\mathbf{q}_B\,:\,B\in\mathcal{U}_0\})} d_E(\mathbf{u},\mathbf{d})^2,$$

that turns out to be

$$\begin{aligned}\mathbf{d}^* &= \tfrac{1}{3}\cdot\mathbf{a}^1 + \tfrac{1}{3}\cdot\mathbf{a}^2 + \tfrac{1}{3}\cdot\mathbf{a}^3 + 0\cdot\mathbf{a}^4\\ &= \alpha\cdot\mathbf{q}_{A_1} + \tfrac{1}{3}\cdot\mathbf{q}_{A_2} + \beta\cdot\mathbf{q}_{A_3} + \left(\tfrac{1}{3}-\alpha\right)\cdot\mathbf{q}_{A_{12}} + 0\cdot\mathbf{q}_{A_{13}} + \left(\tfrac{1}{3}-\beta\right)\cdot\mathbf{q}_{A_{23}} + 0\cdot\mathbf{q}_{A_\Omega}\\ &= \left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{2}{3}\right)^T,\end{aligned}$$

where $\alpha,\beta \in \left[0,\frac{1}{3}\right]$. *Figure 1 shows the points* $\mathbf{d}$ *and* $\mathbf{d}^*$ *together with the convex polytope corresponding to* $\mathbf{conv}(\{\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3,\mathbf{a}^4\})$.

We have an entire class of plausibility functions $Pl'(\cdot|\Omega)$ *on* $\mathcal{A}_\mathcal{F}$ *correcting the assessment* Pl, *with* $\alpha,\beta \in \left[0,\frac{1}{3}\right]$:

$\mathcal{A}_\mathcal{F}$	$\emptyset$	A_1	A_2	A_3	A_{12}	A_{13}	A_{23}	Ω
$Pl'(\cdot\|\Omega)$	0	$\frac{1}{3}$	$1-\alpha-\beta$	$\frac{1}{3}$	$1-\beta$	$\frac{2}{3}$	$1-\alpha$	1

Independently of $\alpha,\beta \in \left[0,\frac{1}{3}\right]$, *we have* $d_E(\mathbf{d}^*,\mathbf{d})^2 = \frac{1}{48}$ *and*

$$Pl'(A|\Omega) = \frac{1}{3},\ Pl'(B|\Omega) = \frac{1}{3},\ Pl'(A\cup B|\Omega) = \frac{2}{3}.$$

In particular, we notice that for $\alpha = \beta = \frac{1}{3}$, *then* $Pl'(\cdot|\Omega)$ *reduces to a probability measure, while for no choice of* $\alpha,\beta \in \left[0,\frac{1}{3}\right]$ *it reduces to a possibility measure.*

The next result shows that Pl-coherence is a necessary and sufficient condition for the extendibility of an assessment Pl on $\mathcal{G}$ to a full conditional plausibility Pl on the entire $\mathcal{P}(\Omega)\times\mathcal{P}(\Omega)^0$. As a by-product, such result assures that every Pl-coherent unconditional plausibility assessment can be extended to a full conditional plausibility on the entire $\mathcal{P}(\Omega)\times\mathcal{P}(\Omega)^0$.

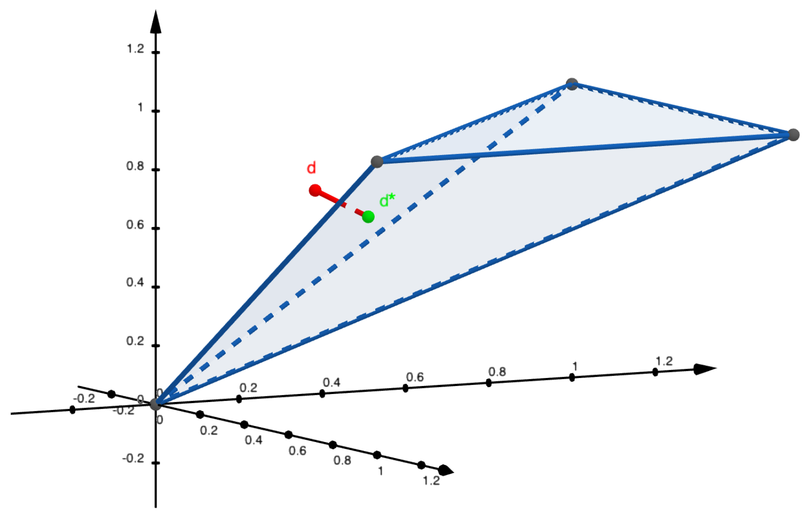

Figure 1: Projection $\mathbf{d}^*$ (green point) of $\mathbf{d}$ (red point) onto $\mathbf{conv}(\{\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3, \mathbf{a}^4\})$ (blue polytope).

Theorem 4. *Let $\mathcal{G}$ be an arbitrary set of conditional events and $Pl : \mathcal{G} \to [0, 1]$ be a conditional plausibility assessment. Then Pl can be extended to a full conditional plausibility function Pl' on $\mathcal{P}(\Omega)$ if and only if Pl is Pl-coherent.*

Proof. The only if part is trivial since if Pl can be extended to a full conditional plausibility function on $\mathcal{P}(\Omega)$, then it is Pl-coherent. Thus, we only prove the if part.

For every finite sub-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, let $\mathbf{Q}_{\mathcal{A}}$ be the set of mappings from $\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0$ to $[0, 1]$ whose restriction to $\mathcal{A} \times \mathcal{A}^0$ is a full conditional plausibility function extending $Pl_{|(\mathcal{A} \times \mathcal{A}^0) \cap \mathcal{G}}$. If $(\mathcal{A} \times \mathcal{A}^0) \cap \mathcal{G} = \emptyset$, then $\mathbf{Q}_{\mathcal{A}}$ is trivially non-empty since we do not have any constraint for the full conditional plausibility functions on $\mathcal{A} \times \mathcal{A}^0$. If $(\mathcal{A} \times \mathcal{A}^0) \cap \mathcal{G} \neq \emptyset$, then since $\mathcal{F} = (\mathcal{A} \times \mathcal{A}^0) \cap \mathcal{G}$ is finite, Theorems 3 and 2 imply that $\mathbf{Q}_{\mathcal{A}}$ is not empty.

Hence, for every finite sub-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, the set $\mathbf{Q}_{\mathcal{A}}$ is a non-empty closed subset of the compact and Hausdorff space $[0, 1]^{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0}$ endowed with the product topology. Indeed, for every net $\{\nu_\lambda\}_{\lambda \in \Lambda}$ in $\mathbf{Q}_{\mathcal{A}}$ converging pointwise to ν, a simple application of properties of limits of real nets shows that ν is a $[0,1]$-valued function whose restriction to $\mathcal{A}\times\mathcal{A}^0$ is a conditional plausibility and $\nu(E|H) = Pl(E|H)$ for every $E|H \in (\mathcal{A}\times\mathcal{A}^0)\cap\mathcal{G}$. Thus, $\nu \in \mathbf{Q}_\mathcal{A}$. It is easily seen that the family

$$\{\mathbf{Q}_\mathcal{A} \,:\, \mathcal{A} \subseteq \mathcal{P}(\Omega), \text{finite sub-algebra}\},$$

possesses the finite intersection property, thus it holds that

$$\bigcap\{\mathbf{Q}_\mathcal{A} \,:\, \mathcal{A} \subseteq \mathcal{P}(\Omega), \text{finite sub-algebra}\} \neq \emptyset$$

and so there exists $Pl' \in \bigcap\{\mathbf{Q}_\mathcal{A} \,:\, \mathcal{A} \subseteq \mathcal{P}(\Omega), \text{finite sub-algebra}\}$ which is a full conditional plausibility function on $\mathcal{P}(\Omega)$ extending Pl. □

Another consequence of Theorem 4 is that a Pl-coherent conditional plausibility assessment Pl on an arbitrary $\mathcal{G}$ can be extended, generally not in a unique way, to a *conditional completely alternating Choquet expectation* $\mathbb{C}_{Pl'} : \mathcal{L}(\Omega)\times\mathcal{P}(\Omega)^0 \to \mathbb{R}$, where $\mathcal{L}(\Omega)$ is the set of bounded real-valued functions on Ω, and Pl' is a full conditional plausibility function on $\mathcal{P}(\Omega)$ extending Pl. Also in this context, in agreement with Remark 1 (see also [60]), a *conditional gamble* is simply intended to be a pair (X, H), where $H \neq \emptyset$, that we denote as $X|H$. The conditional functional $\mathbb{C}_{Pl'}$, for which a betting scheme notion of coherence has been given in [13], is defined through the Choquet integral [5, 35], for every $X|H \in \mathcal{L}(\Omega)\times\mathcal{P}(\Omega)^0$, as

$$\mathbb{C}_{Pl'}(X|H) = \oint X(\omega)\,\mathrm{d}Pl'(\omega|H). \tag{20}$$

We stress that, denoting by $\mathbf{P}_H$ the class of finitely additive probability measures on $\mathcal{P}(\Omega)$ pointwise dominated by $Pl'(\cdot|H)$, the properties of the Choquet integral (see, e.g., [35, 60, 55]) imply that

$$\mathbb{C}_{Pl'}(X|H) = \max_{P\in\mathbf{P}_H} \int X\,\mathrm{d}P, \tag{21}$$

where the integrals appearing in the maximum are of Stieltjes type [1]. Equation (21) shows that the conditional functional $\mathbb{C}_{Pl'}$ has an upper expectation interpretation, locally on every conditioning event $H \in \mathcal{P}(\Omega)^0$. Besides satisfying all the well-known properties of the Choquet integral with respect to a normalized completely alternating capacity $Pl'(\cdot|H)$ (see, e.g., [22, 35, 60]), the conditional functional $\mathbb{C}_{Pl'}(\cdot|H)$ satisfies also the two following properties that are implied by the adopted notion of conditioning.

Proposition 1. *For every $H \in \mathcal{P}(\Omega)^0$, the conditional functional $\mathbb{C}_{Pl'}$ satisfies the properties:*

(i) $\mathbb{C}_{Pl'}(X|H) = \mathbb{C}_{Pl'}(X\mathbf{1}_H|H)$*, for every* $X \in \mathcal{L}(\Omega)$*;*

(ii) $\mathbb{C}_{Pl'}(X\mathbf{1}_E|H) = \mathbb{C}_{Pl'}(\mathbf{1}_E|H) \cdot \mathbb{C}_{Pl'}(X|E \cap H)$*, for every non-negative* $X \in \mathcal{L}(\Omega)$ *and* $E, E \cap H \in \mathcal{P}(\Omega)^0$*.*

Proof. Property *(i)* is due to the fact that, for all $t \in \mathbb{R}$,

$$\begin{aligned} Pl'(X \geq t|H) &= Pl'(\{X \geq t\} \cap H|H) \\ &= Pl'(\{X\mathbf{1}_H \geq t\} \cap H|H) = Pl'(X\mathbf{1}_H \geq t|H). \end{aligned}$$

Finally, property *(ii)* holds since, being $X \geq 0$, for all $t > 0$, we have

$$Pl'(X\mathbf{1}_E \geq t|H) = Pl'(\{X \geq t\} \cap E|H) = Pl'(E|H) \cdot Pl'(X \geq t|E \cap H),$$

and this implies

$$\begin{aligned} \mathbb{C}_{Pl'}(X\mathbf{1}_E|H) &= \oint X(\omega)\mathbf{1}_E(\omega)\,\mathrm{d}Pl'(\omega|H) = \int_0^{+\infty} Pl'(X\mathbf{1}_E \geq t|H)\,\mathrm{d}t \\ &= Pl'(E|H) \cdot \int_0^{+\infty} Pl'(X \geq t|E \cap H)\,\mathrm{d}t \\ &= Pl'(E|H) \cdot \oint X(\omega)\,\mathrm{d}Pl'(\omega|E \cap H) \\ &= \mathbb{C}_{Pl'}(\mathbf{1}_E|H) \cdot \mathbb{C}_{Pl'}(X|E \cap H). \end{aligned}$$

□

We point out that property *(ii)* in Proposition 1 holds for a non-negative gamble X and may fail if X takes negative values, as shown in Example 6 in [48], where the conditional submodular capacity is actually a conditional plausibility in the sense of Definition 1.

4. Special cases

As pointed out in Section 2, two distinguished subclasses of conditional plausibility functions are given by conditional probabilities and conditional possibilities. The notions of Pl-coherence introduced in Section 3 can be adapted to work inside these two frameworks. As it will be shown below, working with conditional probabilities requires to assume *completely resolving uncertainty* while working with conditional possibilities amounts to assume partially resolving uncertainty and a systematically optimistic behavior, together with *consonance*.

4.1. Conditional probabilities

Coherent conditional probability theory rests upon the principle of *completely resolving uncertainty*. In turn, this consists in assuming that the basic piece of information we may acquire when uncertainty is resolved is the truth of one of the elements of the finest partition of Ω determined by the events at hand. For this, in what follows, if $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ is a finite subset of $\mathcal{G}$, then $\mathcal{C}_\mathcal{F}$ denotes the set of *atoms* generated by them, i.e., $\mathcal{C}_\mathcal{F} = \textbf{atoms}(\textbf{algebra}(\mathcal{O}(\mathcal{F})))$. We further denote by $\mathcal{C}_0$ the set of atoms in $\mathcal{C}_\mathcal{F}$ contained in $H_0 = \bigcup_{i=1}^n H_i$, that is

$$\mathcal{C}_0 = \{C \in \mathcal{C}_\mathcal{F} \ : \ C \subseteq H_0\}. \tag{22}$$

In practice, conditions Pl-coherence-2, Pl-coherence-3, Pl-coherence-4 and Pl-coherence-5 are specialized substituting $\mathcal{U}_0$ with $\mathcal{C}_0$. From a semantic point of view, this has a particular impact on Pl-coherence-3 and Pl-coherence-5 in which all the speculations on the betting scheme and the penalization are carried out focusing on $\mathcal{C}_0$.

A conditional probability assessment $P : \mathcal{G} \to [0, 1]$ is said to be:

P-coherent-1: if there exists a conditional probability $P' : \mathcal{A}_\mathcal{G} \times \mathcal{H}_\mathcal{G} \to [0, 1]$ such that $P'_{|\mathcal{G}} = P$.

P-coherent-2: if for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, the following system with unknowns x_C for all $C \in \mathcal{C}_0$

$$\mathcal{S}_\mathcal{F} : \begin{cases} \displaystyle\sum_{\substack{C \subseteq E_i \cap H_i \\ C \in \mathcal{C}_0}} x_C = P(E_i|H_i) \cdot \sum_{\substack{C \subseteq H_i \\ C \in \mathcal{C}_0}} x_C, & \text{for } i = 1, \ldots, n, \\ \displaystyle\sum_{C \in \mathcal{C}_0} x_C = 1, & \\ x_C \geq 0, & \text{for all } C \in \mathcal{C}_0, \end{cases} \tag{23}$$

is compatible.

P-coherent-3: if for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, for every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, the function $G_\mathcal{F} : \mathcal{C}_0 \to \mathbb{R}$ defined, for every $C \in \mathcal{C}_0$, as

$$G_\mathcal{F}(C) = \sum_{i=1}^n \lambda_i \cdot \left[\mathbf{1}^\mathbf{U}_{E_i \cap H_i}(C) - P(E_i|H_i) \cdot \mathbf{1}^\mathbf{U}_{H_i}(C)\right] \tag{24}$$

satisfies $\max_{C \in \mathcal{C}_0} G_\mathcal{F}(C) \geq 0$.

P-coherent-4: if for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, the vectors in $\mathbb{R}^n$

$$\begin{aligned} \mathbf{d} &= (P(E_1|H_1), \ldots, P(E_n|H_n))^T, & (25)\\ \mathbf{q}_C &= (Q_1(C), \ldots, Q_n(C))^T, \quad \text{for all } C \in \mathcal{C}_0, & (26) \end{aligned}$$

are such that $\mathbf{d} \in \mathbf{conv}(\{\mathbf{q}_C \,:\, C \in \mathcal{C}_0\})$.

P-coherent-5: if for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, there is no distinct assessment $P^* : \mathcal{F} \to [0,1]$ such that the functions $L_{\mathcal{F}}, L^*_{\mathcal{F}} : \mathcal{C}_0 \to \mathbb{R}$ defined, for every $C \in \mathcal{C}_0$, as

$$\begin{aligned} L_{\mathcal{F}}(C) &= \sum_{i=1}^{n} \left[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(C) - P(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(C)\right]^2, & (27)\\ L^*_{\mathcal{F}}(C) &= \sum_{i=1}^{n} \left[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(C) - P^*(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(C)\right]^2, & (28) \end{aligned}$$

are such that $L^*_{\mathcal{F}}(C) < L_{\mathcal{F}}(C)$ for every $C \in \mathcal{C}_0$.

Since the indicators of events in $\mathcal{O}(\mathcal{F})$ are constant on the elements of $\mathcal{C}_{\mathcal{F}}$, the gain function $G_{\mathcal{F}}$ in condition P-coherence-3 can be replaced by the function $\tilde{G}_{\mathcal{F}} : \Omega \to \mathbb{R}$ defined, for every $\omega \in \Omega$, as

$$\tilde{G}_{\mathcal{F}}(\omega) = \sum_{i=1}^{n} \lambda_i \cdot \mathbf{1}_{H_i}(\omega) \cdot \left[\mathbf{1}_{E_i}(\omega) - P(E_i|H_i)\right]. \quad (29)$$

The function $\tilde{G}_{\mathcal{F}}$ is constant on the elements of $\mathcal{C}_{\mathcal{F}}$ and takes the same values of $G_{\mathcal{F}}$ on the elements of $\mathcal{C}_0$, thus $\max\limits_{\omega \in H_0} \tilde{G}_{\mathcal{F}}(\omega) = \max\limits_{C \in \mathcal{C}_0} G_{\mathcal{F}}(C)$. The notion of coherence for a conditional probability assessment relying on (29) has been considered in [37, 38, 52, 65] and coincides, therefore, with our condition P-coherence-3.

In the case of conditional probability, an analog of Lemma 1 can be proved as well as an analog of Remark 4 can be introduced (see, e.g., [15, 6]). Based on previous considerations, the proof of equivalence among the conditions P-coherence-1, P-coherence-2 and P-coherence-3 goes essentially along to the proofs of analogous results in [15, 6].

Analogously, functions $L_{\mathcal{F}}, L^*_{\mathcal{F}}$ in condition P-coherence-5 can be replaced by the two functions $\tilde{L}_{\mathcal{F}}, \tilde{L}^*_{\mathcal{F}} : \Omega \to \mathbb{R}$ defined, for every $\omega \in \Omega$, as

$$\tilde{L}_{\mathcal{F}}(\omega) = \sum_{i=1}^{n} \mathbf{1}_{H_i}(\omega) \cdot [\mathbf{1}_{E_i}(\omega) - P(E_i|H_i)]^2 , \quad (30)$$

$$\tilde{L}^*_{\mathcal{F}}(\omega) = \sum_{i=1}^{n} \mathbf{1}_{H_i}(\omega) \cdot [\mathbf{1}_{E_i}(\omega) - P^*(E_i|H_i)]^2 . \quad (31)$$

Also in this case, the functions $\tilde{L}_{\mathcal{F}}, \tilde{L}^*_{\mathcal{F}}$ are constant on the elements of $\mathcal{C}_{\mathcal{F}}$ and take the same values of $L_{\mathcal{F}}, L^*_{\mathcal{F}}$, respectively, on the elements of $\mathcal{C}_0$. The functions $\tilde{L}_{\mathcal{F}}, \tilde{L}^*_{\mathcal{F}}$ are used in [32] to define a notion of coherence that requires the non-existence of a distinct assessment $P^* : \mathcal{F} \to [0,1]$, such that $\tilde{L}^*_{\mathcal{F}} \leq \tilde{L}_{\mathcal{F}}$ and $\tilde{L}^*_{\mathcal{F}} \neq \tilde{L}_{\mathcal{F}}$, where comparisons are pointwise. We point out that the condition given in [32] does not restrict the functions $\tilde{L}_{\mathcal{F}}, \tilde{L}^*_{\mathcal{F}}$ to H_0, nevertheless, as follows by our Theorem 5 below and Theorem 5.1 in [32] this condition is equivalent to our condition P-coherence-5.

Theorem 5. *For a conditional probability assessment $P : \mathcal{G} \to [0,1]$, the following statements are equivalent:*

(i) P is P-coherent-1;

(ii) P is P-coherent-2;

(iii) P is P-coherent-3;

(iv) P is P-coherent-4;

(v) P is P-coherent-5.

Proof. The proof of *(i)* $\Longleftrightarrow$ *(ii)* is analogous to the proof of the corresponding equivalence in Theorem 3. The proof of *(i)* $\Longleftrightarrow$ *(iii)* is obtained from Theorem 5.7 in [38] by considering (29). Alternatively, the proof of *(ii)* $\Longleftrightarrow$ *(iii)* is analogous to the proof of the corresponding equivalence in Theorem 3, working with $\mathcal{C}_0$ in place of $\mathcal{U}_0$. Finally, the proofs of *(ii)* $\Longleftrightarrow$ *(iv)* and *(iv)* $\Longleftrightarrow$ *(v)* are analogous to the proofs of the corresponding equivalences in Theorem 3, working with $\mathcal{C}_0$ in place of $\mathcal{U}_0$. Indeed, assume that system $\mathcal{S}_{\mathcal{F}}$ is compatible. Proceeding as in the proof of Theorem 3, $\mathcal{S}_{\mathcal{F}}$ is compatible if and only if $\max\limits_{C \in \mathcal{C}_0} G_{\mathcal{F}}(C) \geq 0$. Further, $\mathcal{S}_{\mathcal{F}}$ is compatible if and only if $\mathbf{d} \in \mathbf{conv}(\{\mathbf{q}_C : C \in \mathcal{C}_0\})$ which, in turn, is equivalent to the non-existence of a distinct assessment $P^* : \mathcal{F} \to [0,1]$ such that $L^*_{\mathcal{F}}(C) < L_{\mathcal{F}}(C)$ for every $C \in \mathcal{C}_0$. □

By virtue of Theorem 5, we say that a conditional probability assessment $P : \mathcal{G} \to [0,1]$ is *P-coherent* if one (and hence all) of the previous notions of coherence holds, otherwise it is said *P-incoherent*.

Also in this case, the next result shows that P-coherence is a necessary and sufficient condition for the extendibility of an assessment P on $\mathcal{G}$ to a full conditional probability P' on the entire $\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0$. As a by-product, such result assures that every P-coherent unconditional probability assessment can be extended to a full conditional probability on the entire $\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0$.

Theorem 6 (Theorem 5.7 in [38])**.** *Let $\mathcal{G}$ be an arbitrary set of conditional events and $P : \mathcal{G} \to [0,1]$ be a conditional probability assessment. Then P can be extended to a full conditional probability P' on $\mathcal{P}(\Omega)$ if and only if P is P-coherent.*

4.2. Conditional possibilities

In the original work by Shafer [56], limiting to a finite Ω and $\mathcal{A} = \mathcal{P}(\Omega)$, necessity/possibility measures are introduced as particular belief/plausibility functions satisfying the further property of *consonance*. In particular, in such a finite setting, this translates into a Möbius inverse with nested focal elements (see, e.g., [26, 35, 56]).

As highlighted in Section 2, conditional possibility measures can be considered as particular conditional plausibility functions, where the conditional rule is specified by condition *(iii)* of Definition 1. In particular, conditions *(i)* and *(ii**)* of Definition 1 assure that for any conditioning event $H \in \mathcal{H}$ the measure $\Pi(\cdot|H)$ is finitely maxitive and normalized, as it holds that $\Pi(E|H) = 1$, for every $E \in \mathcal{A}$ with $H \subseteq E$, and $\Pi(E|H) = 0$, for every $E \in \mathcal{A}$ with $H \cap E = \emptyset$. In practice, this has a direct impact on conditions Pl-coherence-2, Pl-coherence-3, Pl-coherence-4 and Pl-coherence-5 where $\mathcal{U}_0$, that collects all the possible partial information we may acquire on the occurrence of an event generated from $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ under the hypothesis $H_0 = \bigcup_{i=1}^n H_i$ that a least one of the H_i's comes true, is replaced by a family $\mathcal{D}_0 \in \mathbf{chains}(\mathcal{U}_\mathcal{F}, H_0)$. In other terms, here we assume partially resolving uncertainty and a systematically optimistic behavior but we further require consonance, i.e., we focus on nested sets of events generated from the events in $\mathcal{F}$ and having H_0 as top element. From a semantic point of view, this has a particular impact on conditions Pl-coherence-3 and Pl-coherence-5 in which all the speculations on the betting scheme and the penalization are carried out focusing on a family in $\mathbf{chains}(\mathcal{U}_\mathcal{F}, H_0)$.

A conditional possibility assessment $\Pi : \mathcal{G} \to [0,1]$ is said to be:

Π-coherent-1: if there exists a conditional possibility $\Pi' : \mathcal{A}_\mathcal{G} \times \mathcal{H}_\mathcal{G} \to [0,1]$ such that $\Pi'_{|\mathcal{G}} = \Pi$.

Π-coherent-2: if for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, there exists $\mathcal{D}_0 \in \mathbf{chains}(\mathcal{U}_\mathcal{F}, H_0)$, such that the following system with unknowns x_D for all $D \in \mathcal{D}_0$

$$\mathcal{S}_\mathcal{F} : \begin{cases} \sum\limits_{\substack{D \cap E_i \cap H_i \neq \emptyset \\ D \in \mathcal{D}_0}} x_D = \Pi(E_i|H_i) \cdot \sum\limits_{\substack{D \cap H_i \neq \emptyset \\ D \in \mathcal{D}_0}} x_D, & \text{for } i = 1, \ldots, n, \\ \sum\limits_{D \in \mathcal{D}_0} x_D = 1, & \\ x_D \geq 0, & \text{for all } D \in \mathcal{D}_0, \end{cases} \tag{32}$$

is compatible.

Π-coherent-3: if for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, there exists $\mathcal{D}_0 \in \mathbf{chains}(\mathcal{U}_\mathcal{F}, H_0)$, such that for every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, the function $G_\mathcal{F} : \mathcal{D}_0 \to \mathbb{R}$ defined, for every $D \in \mathcal{D}_0$, as

$$G_\mathcal{F}(D) = \sum_{i=1}^{n} \lambda_i \cdot \left[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(D) - \Pi(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(D)\right] \tag{33}$$

satisfies $\max\limits_{D \in \mathcal{D}_0} G_\mathcal{F}(D) \geq 0$.

Π-coherent-4: if for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, there exists $\mathcal{D}_0 \in \mathbf{chains}(\mathcal{U}_\mathcal{F}, H_0)$, such that the vectors in $\mathbb{R}^n$

$$\mathbf{d} = (\Pi(E_1|H_1), \ldots, \Pi(E_n|H_n))^T, \tag{34}$$

$$\mathbf{q}_D = (Q_1(D), \ldots, Q_n(D))^T, \quad \text{for all } D \in \mathcal{D}_0, \tag{35}$$

are such that $\mathbf{d} \in \mathbf{conv}(\{\mathbf{q}_D \,:\, D \in \mathcal{D}_0\})$.

Π-coherent-5: if for every $n \in \mathbb{N}$ and every $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, there exists $\mathcal{D}_0 \in \mathbf{chains}(\mathcal{U}_\mathcal{F}, H_0)$, such that there is no distinct assessment $\Pi^* : \mathcal{F} \to [0,1]$ such that the functions $L_\mathcal{F}, L^*_\mathcal{F} : \mathcal{D}_0 \to \mathbb{R}$ defined, for every $D \in \mathcal{D}_0$, as

$$L_\mathcal{F}(D) = \sum_{i=1}^{n} \left[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(D) - \Pi(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(D)\right]^2, \tag{36}$$

$$L^*_\mathcal{F}(D) = \sum_{i=1}^{n} \left[\mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(D) - \Pi^*(E_i|H_i) \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(D)\right]^2, \tag{37}$$

are such that $L^*_\mathcal{F}(D) < L_\mathcal{F}(D)$ for every $D \in \mathcal{D}_0$.

Limiting to a finite Ω, condition Π-coherence-3 has been introduced in [49] where also an analogous notion of coherence is given for a comparative conditional possibility assessment.

Lemma 2. *Let $\mathcal{G}$ be finite and $\Pi : \mathcal{G} \to [0,1]$. Then the following statements are equivalent:*

(i) Π *is* Π*-coherent-1;*

(ii) Π *is* Π*-coherent-2.*

Proof. (i) $\Longrightarrow$ (ii). If the assessment Π is Π-coherent-1, there exists a conditional possibility $\Pi' : \mathcal{A}_\mathcal{G} \times \mathcal{H}_\mathcal{G} \to [0,1]$, extending Π. For every finite subset $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{G}$, denote by Π'' the restriction to $\mathcal{A}_\mathcal{F}$ of the possibility measure $\Pi'(\cdot|H_0)$. Such possibility measure is such that, for $i = 1, \ldots, n$,

$$\Pi''(E_i \cap H_i) = \Pi(E_i|H_i) \cdot \Pi''(H_i),$$

and $\Pi''(H_0) = 1$ and $\Pi''(H_0^c) = 0$. Let $m'' : \mathcal{A}_\mathcal{F} \to [0,1]$ be the Möbius inverse of the dual necessity measure of Π''. We have that (see [35]) there exists $\mathcal{D}_0 \in \mathbf{chains}(\mathcal{U}_\mathcal{F}, H_0)$ such that m'' has focal elements contained in $\mathcal{D}_0$, thus setting $x_D = m''(D)$, for every $D \in \mathcal{D}_0$, we get a solution of system $\mathcal{S}_\mathcal{F}$. This implies that Π is Π-coherent-2.

(ii) $\Longrightarrow$ (i). Suppose Π is Π-coherent-2. Let $\mathcal{F}_0 = \mathcal{G} = \{E_1|H_1, \ldots, E_n|H_n\}$ and denote $H_0^0 = \bigcup_{i=1}^n H_i$. Since Π is Π-coherent-2, then there exists $\mathcal{D}_0^0 \in \mathbf{chains}(\mathcal{U}_{\mathcal{F}_0}, H_0^0)$ such that the corresponding system $\mathcal{S}_{\mathcal{F}_0}$ has a solution $\mathbf{x}^0$ with components x_D^0, for all $D \in \mathcal{D}_0^0$. Define $m_0 : \mathcal{A}_\mathcal{G} \to [0,1]$ setting $m_0(D) = x_D^0$, for all $D \in \mathcal{D}_0^0$, and 0 otherwise. The function m_0 is the Möbius inverse of a necessity measure whose dual possibility measure Π_0 on $\mathcal{A}_\mathcal{G}$ is such that, for $i = 1, \ldots, n$,

$$\Pi_0(E_i \cap H_i) = \Pi(E_i|H_i) \cdot \Pi_0(H_i),$$

and $\Pi_0(H_0^0) = 1$ and $\Pi_0((H_0^0)^c) = 0$.

For $\alpha > 0$, let $I_\alpha = \{i \in \{1, \ldots, n\} : \Pi_\beta(H_i) = 0, \beta = 0, \ldots, \alpha - 1\}$. If $I_\alpha = \emptyset$ the construction stops, otherwise denote $\mathcal{F}_\alpha = \{E_i|H_i\}_{i \in I_\alpha}$, $H_0^\alpha = \bigcup_{i \in I_\alpha} H_i$. Since Π is Π-coherent-2, then there exists $\mathcal{D}_0^\alpha \in \mathbf{chains}(\mathcal{U}_{\mathcal{F}_\alpha}, H_0^\alpha)$ such that the corresponding system $\mathcal{S}_{\mathcal{F}_\alpha}$ has a solution $\mathbf{x}^\alpha$ with components x_D^α, for all $D \in \mathcal{D}_0^\alpha$. Define $m_\alpha : \mathcal{A}_\mathcal{G} \to [0,1]$ setting $m_\alpha(D) = x_D^\alpha$, for all $D \in \mathcal{D}_0^\alpha$, and 0 otherwise. The function m_α is the Möbius inverse of a necessity measure whose dual plausibility function Π_α on $\mathcal{A}_\mathcal{G}$ is such that, for $i \in I_\alpha$,

$$\Pi_\alpha(E_i \cap H_i) = \Pi(E_i|H_i) \cdot \Pi_\alpha(H_i),$$

and $\Pi_\alpha(H_0^\alpha) = 1$ and $\Pi_\alpha((H_0^\alpha)^c) = 0$.

Let k be the first index such that $I_{k+1} = \emptyset$. Then $\{\Pi_0, \ldots, \Pi_k\}$ is by construction a $\mathcal{H}_\mathcal{G}$-minimal agreeing class of possibility measures on $\mathcal{A}_\mathcal{G}$ corresponding (see Subsection 2.1) to a conditional possibility $\Pi' : \mathcal{A}_\mathcal{G} \times \mathcal{H}_\mathcal{G} \to [0,1]$ that extends Π. This implies that Π is Π-coherent-1. $\square$

The following theorem shows that, also in the subclass of conditional possibilities, all the notions of Π-coherence introduced above are equivalent.

Theorem 7. *For a conditional possibility assessment* $\Pi : \mathcal{G} \to [0,1]$*, the following statements are equivalent:*

(i) Π *is* Π*-coherent-1;*

(ii) Π *is* Π*-coherent-2;*

(iii) Π *is* Π*-coherent-3;*

(iv) Π *is* Π*-coherent-4;*

(v) Π *is* Π*-coherent-5.*

Proof. The proof of *(i)* $\Longleftrightarrow$ *(ii)* is analogous to the proof of the corresponding equivalence in Theorem 3 and relies on Lemma 2: Π-coherence-2 can be reformulated in terms of solvability of a sequence of systems $\mathcal{S}_{\mathcal{F}_0}, \ldots, \mathcal{S}_{\mathcal{F}_k}$ with progressively less constraints and unknowns. We point out that $\mathcal{F}_0, \ldots, \mathcal{F}_k$ are decreasingly ordered by set inclusion, with $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_{k+1} = \emptyset$. This implies that, for $\alpha > 0$, the elements of $\mathbf{chains}(\mathcal{U}_{\mathcal{F}_\alpha}, H_0^\alpha)$ are sub-chains of elements of $\mathbf{chains}(\mathcal{U}_{\mathcal{F}_{\alpha-1}}, H_0^{\alpha-1})$. Further, the structure of systems depends on the sequence of solutions $\mathbf{x}^0, \ldots, \mathbf{x}^k$, that actually are restrictions of Möbius inverses defined on $\mathcal{A}_\mathcal{F}$ corresponding to a $\mathcal{H}_\mathcal{F}$-minimal agreeing class $\{\Pi_0, \ldots, \Pi_k\}$ of possibility measures on $\mathcal{A}_\mathcal{F}$.

For a finite $\mathcal{F}$, it is easy to show that in systems $\mathcal{S}_{\mathcal{F}_0}, \ldots, \mathcal{S}_{\mathcal{F}_k}$ it is equivalent to take, for $\alpha = 0, \ldots, k$, $\mathcal{D}_0^\alpha \in \mathbf{chains}(\mathcal{U}_\mathcal{F}, H_0^\alpha)$. This translates in considering more unknowns without affecting solvability and allows to obtain all $\mathcal{H}_\mathcal{F}$-minimal agreeing classes of possibility measures on $\mathcal{A}_\mathcal{F}$ compatible with the given assessment.

The proofs of *(ii)* $\Longleftrightarrow$ *(iii)*, *(ii)* $\Longleftrightarrow$ *(iv)* and *(iv)* $\Longleftrightarrow$ *(v)* are analogous to the proofs of the corresponding equivalences in Theorem 3, working with $\mathcal{D}_0 \in \mathbf{chains}(\mathcal{U}_\mathcal{F}, H_0)$ in place of $\mathcal{U}_0$. Indeed, assume that there exists $\mathcal{D}_0 \in \mathbf{chains}(\mathcal{U}_\mathcal{F}, H_0)$ such that system $\mathcal{S}_\mathcal{F}$ is compatible. Proceeding as in the proof of Theorem 3, $\mathcal{S}_\mathcal{F}$ is compatible if and only if $\max\limits_{D \in \mathcal{D}_0} G_\mathcal{F}(D) \geq 0$. Further

$\mathcal{S}_{\mathcal{F}}$ is compatible if and only if $\mathbf{d} \in \mathbf{conv}(\{\mathbf{q}_D : D \in \mathcal{D}_0\})$ which, in turn, is equivalent to the non-existence of a distinct assessment $\Pi^* : \mathcal{F} \to [0,1]$ such that $L^*_{\mathcal{F}}(D) < L_{\mathcal{F}}(D)$ for every $D \in \mathcal{D}_0$. □

By virtue of Theorem 7, we say that a conditional possibility assessment $\Pi : \mathcal{G} \to [0,1]$ is Π-*coherent* if one (and hence all) of the previous notions of coherence holds, otherwise it is said Π-*incoherent*.

In analogy to what happens for conditional plausibility functions and conditional probabilities, the next result shows that Π-coherence is a necessary and sufficient condition for the extendibility of an assessment Π on $\mathcal{G}$ to a full conditional possibility Π' on the entire $\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0$. As a by-product, such result assures that every Π-coherent unconditional possibility assessment can be extended to a full conditional possibility on the entire $\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)^0$.

Theorem 8. *Let $\mathcal{G}$ be an arbitrary set of conditional events and $\Pi : \mathcal{G} \to [0,1]$ be a conditional possibility assessment. Then Π can be extended to a full conditional possibility Π' on $\mathcal{P}(\Omega)$ if and only if Π is Π-coherent.*

Proof. The only if part is trivial since if Π can be extended to a full conditional possibility on $\mathcal{P}(\Omega)$, then it is Π-coherent. Thus, we only prove the if part.

If Π is Π-coherent, then Π can be extended to a full conditional possibility Π' on $\mathcal{P}(\Omega)$ by Theorem 12 in [7] (or the more general Theorem 2 in [10]) considering the algebraic product triangular norm. □

5. Conclusions

We develop a theory of coherent conditional plausibilities, by presenting different equivalent notions of Pl-coherence: Pl-coherence as consistency, Pl-coherence as betting scheme and Pl-coherence as penalty criterion. The resulting notions of Pl-coherence are based on the principle of partially resolving uncertainty due to Jaffray [39] and a systematically optimistic behavior of the agent. We stress that the introduced notions of Pl-coherence provide different interpretations and permit to assess "subjective" conditional plausibilities.

We also show that Pl-coherence is a necessary and sufficient condition for the extendibility of a conditional plausibility assessment to a conditional completely alternating Choquet expectation defined on the set of all bounded conditional gambles.

Finally, we specialize all the results to work inside the subclasses of conditional probabilities and conditional possibilities. In the first case, the notions of Pl-coherence are adapted requiring completely resolving uncertainty. In the second case, the notions of Pl-coherence are adapted requiring partially resolving uncertainty, a systematically optimistic behavior, and consonance.

An extension of present work could regard the wider framework of conditional 2-alternating capacities, by referring to the axiomatic definition an characterization given in [12].

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