Worthiness-based Scale Quantifying
Re-interpretare scale ordinali equi-distanziate

Giulio D'Epifanio[1]

Abstract The construction of an ordinal scale Y, to be associated to a performance index in evaluating social-agents, is outlined which is quantified, adopting an "intrinsic worthiness" criterion, standardized on a chosen reference-agent, eg that which is intended as representing a (actual or perhaps hypothetical) "best practice". The usual practice of using the equi-spaced scale is re-interpreted. The ordinal levels of Y are identified by design through scheduling a hierarchical sequence of increasingly stringent "goals to be achieved". The index, to be associated to Y, borrows the structure of the Yaari-Quiggin functional, from the RDEU theory. But, the concept of "value increases", in advancing on the scheduled goals, is meaningfully re-interpreted herein, besides utility-based meaning, as "social worthiness". These "value increases" may be extracted, fully normalized on the data-behavior of the chosen "reference social-agent", upon a probabilistic formal setting using data-analysis (perhaps pseudo-Bayesian) tools. Thus, the ordinal levels of Y remain quantified, alternatively to various other approaches.
Abstract Si delinea la costruzione di una scala ordinale (da associare ad un indice di prestazione), quantificata adottando un principio di "merito intrinseco", normalizzata sulla scelta di un agente di riferimento, ad es. quello il cui comportamento è da intendersi come una "buona pratica", reale o ipotetica che sia. I livelli ordinali of Y sono identificati, per disegno, attraverso la pianificazione di una seguenza di obiettivi da raggiungere, progressivamente più severi. L'indice, associato alla quantificazione di Y, è formalmente strutturato sul funzionale di Yaari-Quiggin, ripreso dalla teoria RDEU. Ma il concetto di "incremento di valore", nelle transizioni sulla catena di obiettivi, è qui re-interpretato come "merito sociale", piuttosto che utilità. Questi incrementi potranno essere operativamente estratti previa formalizzazione in un quadro probabilistico, attraverso strumenti di data-analysis.

Key words: indexing, social agent evaluation, scale quantifying, worthiness

University of the Study of Perugia, Department of Political Science, ggiuliodd@gmail.com

1 Introduction and methodological outline

[The higher level question] Some agent-schools $\{A_1, A_2, \ldots, A_q\}$ have to be benchmarked, from the view of the National Instruction Authority (the policy maker), with respect to *"the ability to address students*[1] *in achieving outcomes in learning"* on a certain outcome scale Y which classifies student-performance (eg by using a certain *learning test*) on the outcome-levels labeled as: "*very bad*" , "*bad*", "*almost enough*", "*sufficient*, "*more than sufficient*", "*good*", "*excellent*"). The policy-maker (PM) demands the agent-assessments be standardized on the behavior of a certain school A^* (the reference agent), perhaps chosen by experts to represent an actual instance of "*what should be intended as a, reasonable and desirable, best practice*". Performance-data are reported in table (1). More complex ex. are in [3].

level of performance Y:	I	II	III	V	V	VI	VII
actual agents:							
agent A1	0	16	24	31	12	0	4
agent A2	4	28	65	107	26	1	3
agent A3	4	42	71	102	33	0	2
agent A4	4	71	112	194	53	0	4

(a) Data by the agents to be benchmarked

Y:	I	II	III	IV	V	VI	VII
reference agent A^*	12	157	272	434	124	1	13

(b) Data by reference stand. agent A^*

Table 1: Example data

The PM[2] would need now an evaluation-machinery (the index) which, whenever applied to any agent A, it takes into input the performance-data of A to provide a certain performance-value, on a properly quantified ordinal scale Y, so that such a value is meaningful upon the conceptual framework which the PM has adopted, conditional on a set of design specifications. These specifications also including the choice of reference-agent A^* on the which the PM would normalize its evaluation-machinery.

[Ordinal scaling] In order to design an ordinal scale Y, on the which the evaluation-machine has to be constructed, we refer to the following scheme. Suppose that the PM, in pursuing its purposes, has been able to schedule a sequential hierarchy of, increasingly stringent, (Guttman-like ordered) goals[3]

[1] in a specified social domain $\mathcal{D}$, e.g. "18 year old female with a certain social background"

[2] To him, in setting value-levels, it seems be excessively "naive" considering equi-distanced scale; but, not clearly structured levels-score choices seem difficult to be justified in institutional benchmarking. On the other hand, economic utility-based interpretations seem lacking of meaning in social assessments (for an operative-research-based approach, see [2]). Worse still, merely data-analysis-criteria based methods (eg see [4]), "*di per se*" seem of little relevance at the PM's higher level question (eg see also [6]).

[3] Guttman order: $O_l \preceq O_{l+1} \Leftrightarrow$ "whenever goal O_{l+1} is achieved, also O_l has been achieved"

$$O_0 \preceq O_1 \preceq O_2 \preceq ... \preceq O_l \preceq ... \preceq \ldots \preceq O_{L-1} \preceq O_L := O_{Full}, \tag{1}$$

Assume that a verbal ordinal scale for the outcome exists so that (unless of recoding it as a sequence of integers $0, 1, \ldots, l, \ldots, L$) the intrinsic meaning is established through the following identification[4]: $O_l \leftrightarrow (Y \geq l)$, $l := 0, \ldots, L$, where $O_0 := (Y \geq 0)$ represents the "tautological-goal" (ie the dummy goal always achieved by anyone). Therefore, an ordinal scale $Y \in \{0, 1, ..., l, ..., L\}$ remains identified with goals sequence (1) from the which it will inherit semantics, and vice versa.

[The formal evaluation machine] Suppose that the PM is able to assign, for any transitions in advancing sequence (1), the value-increase[5] $\omega_l := \Delta_{l-1}Val := Val(O_l) - Val(O_{l-1}) \geq 0$, $l := 1, \ldots, L$. Here, $Val(.)$ denotes a (non-negative, not decreasing in value by crossing goals-sequence (1)) value-function, which is initialized at $\omega_0 := Val(O_0) = 0$. The utility-value theory would provide the formal platform (e.g. see [1], pp. 559) from the which it could be inherited (as an instance of a more general class of Yaari-Quiggin-like value-functionals, eg see [1]) the following index-structure:

$$A \in \{A_1, .., A_q\} :\mapsto W[A] := \sum_{l:=1}^{L} \omega_l \cdot (1 - F_Y[p[A]](l)) = \sum_{l:=1}^{L} s(l; \omega) \cdot p_l[A] \tag{2}$$

unless of parameters list $\omega = (\omega_1, .., \omega_L)$ of value-increases on the transitions between adjacent levels of Y (ie in advancing performance on the scheduled ordered goals-sequence (1)). Here, $p[A] := (p_0, p_1, \ldots, p_L)[A]$ denotes the relative distribution which describes the behavior of agent A; $F_Y[p]$ the cumulative distribution such that $F_Y[p(A)](l) = p_0(A) + p_1(A) + \cdots + p_{l-1}(A)$; $s(l;\, \omega) := \omega_1 + \omega_2 + \cdots + \omega_l$ the Y-levels-quantifier.[6]

[The core methodological question] At the higher-level of the PM's decisions, the practical question now arises, which has an intrinsic methodological interest with regard to the operative way for specifying parameters ω_l, $l := 1, \ldots, L$, which will enter index structure (2) above, in a way that they will be actually meaningful and useful to the PM in evaluating social performance, besides formal utility-based settings.

[What it is proposed herein] Value-increases parameters ω_l on chain (1), which will enter (2), are re-interpreted with the meaning of "social worthiness-

[4] it is set here the logical identification of proposition *"goal O_l is achieved"* with that of "*the outcome-level of Y is at least l*"

[5] Here, $\omega_l := \Delta_{l-1}Val$ may be interpreted as the reference-value which would be gained by any social agent (due to its political activity in addressing the governed individuals) which is able to improve the condition-level of a certain "*standard individual*", from the current $(l-1)$th level to the next lth one

[6] Formally, $s(.;\, \omega)$ can be viewed as the quantification-function of the ordinal-levels $0, 1, \ldots, L$ of Y, which is obtained by (Choquet-)integrating value-increases ω_l, so that:$s(0;\, \omega) := Val(O_0) = 0 \leq\ s(1;\, \omega) := Val(O_1) = \omega_1 \leq\ s(2;\, \omega) := Val(O_2) = \omega_1 + \omega_2 \leq\ \ldots \leq\ s(L;\, \omega) := Val(O_L) = \omega_1 + \omega_2 + \cdots + \omega_L$

increases", by recalling a "principle of intrinsic worthiness". These increases are then specified upon a probabilistic formal setting, which may be also advanced enough to include complex contexts (eg see [3]). Hence, parameters ω_l could be numerically elicited from the reference-agent A^* data, conditionally on the PM's design-specifications, perhaps using pseudo-Bayesian data-analysis tools (eg as in [3]). The methodological point of arrival (unless of more complex extensions and advancing) is that an ordinal quantified scale Y (by Choquet-integrating increases ω_l) will remain operatively quantified, fully normalized on that specific "meaning of worthiness" which is intrinsic into the PM's choice of assuming, as reference-behavior among various alternatives, the behavior of agent A^* (eg a presumed "best-practice") .

2 Worthiness based indexing

Recalling (eg see [3]) the criterion of intrinsic worthiness[7], the "increases of worthiness"$\omega_l := \Delta_{l-1}Val(.)$ may be interpreted as follows. Let $\mathcal{P}^*$denote the population of the (real or perhaps virtual) individuals which are governed by the "reference agent" A^*.

> For any actual individual i (eg a student of a school in tab.1a), which has achieved goal O_{l-1} moving up goals-chain (1), *the higher* the $\mathcal{P}^*-$standardized risk of failing the next goal O_l, *the greater* the $\mathcal{P}^*-$standardized "increase of worthiness" $\Delta_{l-1}Val(.)$ which such an individual i gains whenever it also achieves goal O_l.

By adopting now a probabilistic interpretative setting, such a $\mathcal{P}^*-$standardized risk could be related to non-transition probability[8] $Pr\{Y = l-1|Y \geq l-1; \mathcal{P}^*\}$, up to some monotone transformation $\varphi_l(.)$ (eg see [3]). Thus, by setting value-increases as:

$$\Delta_{l-1}Val := \omega_l^* := \varphi_l(Pr\{Y = l-1|Y \geq l-1; \mathcal{P}^*\}),\ l := 1, \ldots L \tag{3}$$

[7] Consider hierarchical chain of goals (1). Given that a certain goal O_{l-1} has been achieved, the greater the resistance, with reference to the evaluation framework, to also achieve the next pursued goal O_l, by continuing to improve, the greater the increment of value $\omega_l := \Delta_{l-1}Val(.)$ due to the "intrinsic worthiness" of who, effectively, is able to achieve it.

[8] of course, $Pr\{Y{=}l{-}1|Y{\geq}l{-}1;\mathcal{P}^*\}{=}\frac{Pr\{Y=l-1;\mathcal{P}^*\}}{Pr\{Y\geq l-1;\mathcal{P}^*\}} = \frac{p^*_{l-1}}{p^*_{l-1}+p^*_l+\cdots+p^*_L}$.

Example, by data-table (1b): $\omega_0{=}Pr\{O_0\ fails\}{=}0$, $\omega_1{=}Pr\{O_1\ fails\}{\simeq}\frac{12}{12+157+272+434+124+1+13}$ $\omega_2{=}Pr\{O_2\ fails|O_1\ achieved\}{\simeq}\frac{157}{157+272+434+124+1+13}$, $\omega_3{=}Pr\{O_3\ fails|O_2\ achieved\}{\simeq}\frac{272}{272+434+124+1+13}$, $\omega_4{\simeq}Pr\{O_4\ fails|O_3\ achieved\}{=}\frac{434}{434+124+1+13}$, $\omega_5{=}Pr\{O_5\ fails|O_4\ achieved\}{\simeq}\frac{124}{124+1+13}$, $\omega_6{\simeq}Pr\{O_6\ fails|O_5\ achieved\}{=}\frac{1}{1+13}$. Then (let here $\varphi_l(.)$ be the identity), by accumulating value-increases yields:

level-score of Y:	0	1	2	3	4	5	6
	0	0.011846	0.168689	0.490964	1.249705	2.148256	2.219685

it yields index structure (2) to be instanced as[9]

$$A :\longmapsto W[A;\, \omega^*] := \sum_{l:=1}^{L} \varphi_l\left(\frac{Pr\{Y = l-1; \mathcal{P}^*\}}{Pr\{Y \geq l-1; \mathcal{P}^*\}}\right) \cdot (1 - F_Y[p[A]](l)) \qquad (4)$$

From the which, the re-ranged on interval $[0, 100\%]$, normalized version[10]

$$A \in \{A_1, .., A_p\} :\longmapsto W^*[A;\, \omega^*] := \frac{W[A;\, \omega^*] - W[A_{worst};\, \omega^*]}{W[A_{best};\, \omega^*] - W[A_{worst};\, \omega^*]} \qquad (5)$$

3 Advancing and notes on equidistant scaling

[***Multi-domain indexing***] The PM might want consider worthiness-based performance difference, among the social agents, conditional on status $x := X \in \{x_1, \ldots, x_r, \ldots, x_R\}$ of the governed individuals, normalized on standard agent A^*. Here, $\{x_1, \ldots, x_r, \ldots, x_R\}$ represents a set of *"reference domains"*. A global worthiness-index, integrating on domains X from (4), could have the following structure:

$$A \longmapsto \sum_{r:=1}^{R} q_r \cdot W_{x_r}[p_r([A]; \omega_r(\mathcal{P}^*)] = \sum_{r:=1}^{R} q_r \cdot \{\sum_{l:=1}^{L} \varphi_l\left(\frac{\exp(\widehat{a}_l + \hat{b}_l x_r)}{1 + \exp(\hat{a}_l + \hat{b}_l x_r)}\right) \cdot (1 - F_{Y|x_r}[p[A]](l))\}$$

where parameters $\hat{a}$ and $\widehat{b}$ were determined, using a sequence of logistic models to model value-increases (3) conditional on status x, given reference-population $\mathcal{P}^*$associated to A^*. Here, $q_r \geq 0$ weights[11] ($\sum_{i:=1}^{R} q_r = 1$) the reference domains. A more complex example is presented in [3].

[***Equi-distanced quantifying***] Process any individual, governed by standard-agent A^* in a certain condition-domain $\mathcal{D}$ (ie in reference-population $\mathcal{P}^*_{|\mathcal{D}}$), sequentially against goal-achievement detectors associated to goals-hierarchy

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[9] Here, a continuous monotone functions $\varphi_l(.)$ could be chosen for specifying some types of design-requirements (e.g. the additivity) on the worthiness-scale.

[10] Here, the performance of agent $A \in \{A_1, .., A_p\}$ is graduated on the percentage of gained worthiness, in advancing from the complete social-failure (represented by distribution $p[A_{worst}] := (1, 0, \ldots, 0)$ associated to the "worst-virtual agent" named A_{worst}) toward the full achievement of the social overall-goal (represented by distribution $p[A_{best}] := (0, 0, \ldots, 0, 1)$ associated to the best virtual agent named A_{best})

[11] the weights should represent the political relevancy of the "social reference domains" to the overall purpose of the PM

(1). Then[12], recalling worthiness-increments (3), the value-increases will be provided (for simplicity use identity $\varphi_l(t) = t$) by:

$$\omega_l^* = \frac{q_1 q_2 \cdots q_l (1-q_{l+1})}{q_1 q_2 \cdots q_l (1-q_{l+1}) + q_1 q_2 \cdots q_l \cdot q_{l+1}(1-q_{l+2}) + \ldots + q_1 q_2 \cdots q_l \cdot q_{l+1} \cdots q_L} = \frac{(1-q_{l+1})}{(1-q_{l+1}) + q_{l+1}(1-q_{l+2}) + \ldots + q_{l+1} \cdots q_L}$$

Suppose now that, in particular, A^* has been chosen (to be put into example-table 1b) such that transition probabilities $q_l := Pr\{Y > l-1 | Y \geq l-1; \mathcal{P}^*\} = q,\ l := 1, \ldots L$ ($0 < q < 1$ constant) were invariant through chain (1). Then, it will happen[13] that worthiness increases $\omega_l^* = 1-q,\ l := 1, \ldots L$ remain constant. Thus, normalized on A^*, it will be induced an equi-distanced scale, with some curious interpretation[14]with respect to the PM's choice of considering A^* as a "best practice".

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[12] Let $q_l := Pr\{Y > l-1 | Y \geq l-1; \mathcal{P}^*\}$, $l := 1, \ldots, L$, denote the "transition probability" which an individual, once arrived at level $(l-1)$th of Y, is also able to pass over beside. Of course, the expected "social behavior" of A^*, with respect to outcome-classifier $Y \in \{0, 1, \ldots, L\}$, is represented by parameters $p[A^*] := (p_0, p_1, \ldots, p_L)[A^*]$, where $p_0 := 1-q_1$, $p_1 := q_1(1-q_2)$, ..., $p_l := q_1 q_2 \cdots q_l (1-q_{l+1})$,.., $p_L := q_1 q_2 \cdots q_l \cdot q_{l+1} \cdots q_L$; here, $p_l[A^*]$ represents the probability of definitively remaining at level lth of Y.

[13] $\omega_l^* = \frac{1-q}{(1-q)+q(1-q)+q^2(1-q)+\ldots+q^{L-1}(1-q)+q^L} = \frac{1-q}{1-q+q-q^2+q^2-q^3+\ldots+q^{L-1}-q^L+q^L} = 1-q$

[14] Vice versa, the choice of using an equi-distanced ordinal scale for Y would be equivalent, in our interpretative setting in schools evaluations, to choose as reference standard-agent A^* that "virtual" school (the distribution $p[A^*] := (1-q, q(1-q), \ldots, q^{L-1}(1-q), q^L)[A^*]$ to be put into example-table 1b) whose students constitute that reference population $\mathcal{P}^*$ where everyone, subjected to a sequence of k-outcome-formatted learning-tests (intended as goal-achievement detectors on goals-hierarchy (1)) until a goal is failed, tries to guess, among the k (k constant positive integer) proposed answers, the correct one. That is to say that, in particular using a sequence of $k := 2$ outcome-formatted learning-tests, it would be "*as if*" any student in $\mathcal{P}^*$ would respond according to the random outcome from tossing a certain coin.