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Research Article

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Coercive elliptic systems with gradient terms

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Abstract: In this paper we give a classification of positive radial solutions of the following system:

$$\Delta u = v^m, \quad \Delta v = h(|x|)g(u)f(|\nabla u|),$$

in the open ball B_R, with $m > 0$, and f, g, h nonnegative nondecreasing continuous functions. In particular, we deal with both explosive and bounded solutions. Our results involve, as in [27], a generalization of the well-known Keller–Osserman condition, namely, $\int_1^\infty (\int_0^s F(t)\,dt)^{-m/(2m+1)}\,ds < \infty$, where $F(t) = \int_0^t f(s)\,ds$. Moreover, in the second part of the paper, the p-Laplacian version, given by $\Delta_p u = v^m$, $\Delta_p v = f(|\nabla u|)$, is treated. When $p \geq 2$, we prove a necessary condition for the existence of a solution with at least a blow up component at the boundary, precisely $\int_1^\infty (\int_0^s F(t)\,dt)^{-m/(mp+p-1)} s^{(p-2)(p-1)/(mp+p-1)}\,ds < \infty$.

Keywords: Elliptic systems, explosive solutions, coercive systems

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1 Introduction

In [27], Singh investigates a semilinear elliptic system involving a mixture of power type nonlinearities and nonlinear gradient terms, given by

$$\begin{cases} \Delta u = v^m & \text{in } \Omega, \\ \Delta v = f(|\nabla u|) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω might be either a ball B_R centered at the origin and with radius R or the whole space $\mathbb{R}^N$, f is a function of class C^1 in $[0, \infty)$, nondecreasing and positive for all $t > 0$, and m is a positive real number. For $\Omega = B_R$, he considers one of the following boundary conditions:
(C1) either u and v are bounded in B_R,
(C2) u is bounded in B_R and $\lim_{|x|\to R^-} v(x) = \infty$,
(C3) $\lim_{|x|\to R^-} u(x) = \lim_{|x|\to R^-} v(x) = \infty$.
A condition where v is bounded in B_R and $\lim_{|x|\to R^-} u(x) = \infty$ cannot hold. As a matter of fact the boundness of v implies the boundness of u, thanks to the first equation in (1.1).

In particular, when $m = 1$, system (1.1) reduces to the biharmonic equation

$$\Delta^2 u = f(|\nabla u|) \quad \text{in } \Omega.$$

We mention [16] for a complete description of general problems involving the polyharmonic operator.

A subcase of system (1.1), when Ω is a ball, was analyzed in 2005 by Diaz, Lazzo and Schmidt in [10], where they considered the case with $m = 1$ and $f(t) = t^2$, related to the study of the dynamics of a viscous, heat-conducting fluid. Moreover, their study was extended to time dependent systems in [11, 12]. Other systems with nonlinearities not depending on the gradient are treated in [3, 7, 17, 21–24]. Semilinear elliptic

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problems involving gradient terms, however, have been only recently investigated. In the case of semilinear elliptic equations, we refer, for instance, to [1, 6, 13, 15, 18], whereas for systems we mention [8, 9, 14, 29]. For a complete description of singular elliptic equations we refer to [19].

Motivated by the paper of Singh, we extend some results of [27] in two directions, that is, by considering:
(a) a more general nonlinearity in (1.1),
(b) the quasilinear version of (1.1) involving the p-Laplacian operator.
Precisely, first we study the system

$$\begin{cases} \Delta u = v^m & \text{in } B_R, \\ \Delta v = h(|x|)g(u)f(|\nabla u|) & \text{in } B_R, \end{cases} \tag{1.2}$$

where throughout the paper we assume

$$h, g, f \in C(\mathbb{R}_0^+), \qquad h, g, f > 0 \quad \text{in } \mathbb{R}^+, \qquad h, g, f \quad \text{nondecreasing in } \mathbb{R}_0^+. \tag{$\mathscr{H}$}$$

As in [27], we deal with positive radial solutions of (1.2), where with positive radial solutions of (1.2) we mean couples (u, v) such that both components u and v are positive. Moreover, we give a complete classification of radial solutions of (1.2) by analyzing the problem associated to the boundary condition (C1) or to the boundary blow up conditions (C2) or (C3).

In the second part of the paper, we investigate the quasilinear version of system (1.1), that is,

$$\begin{cases} \Delta_p u = v^m & \text{in } B_R \\ \Delta_p v = f(|\nabla u|) & \text{in } B_R, \end{cases} \tag{1.3}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, is the well-known p-Laplacian operator, $m > 0$ and f is a nonnegative, nondecreasing function of class C^1 in $[0, \infty)$. As a matter of fact, due to the double nature of the p-Laplacian, singular if $1 < p < 2$ and degenerate when $p > 2$, the study of solutions of (1.3) is extremely delicate. Indeed, we are able to prove a necessary condition for the boundary blow up problem associated to system (1.3) only when $p > 2$.

One of the first work about the boundary blow up solutions is due to Bieberbach [4], who studied such solutions for the equation $\Delta u = e^u$ in a planar domain. The first general boundary blow up problem was successfully studied by Keller in [20] and Osserman in [26], who, independently, obtained an optimal condition for the system

$$\begin{cases} \Delta u = g(u) & \text{in } \Omega, \\ u(x) \to \infty & \text{as } x \to \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and g is a nonnegative function of class C^1 in $[0, \infty)$. Keller and Osserman proved that (1.4) has solutions of class $C^2(\Omega)$ if and only if

$$\int_1^\infty \frac{ds}{\sqrt{G(t)}} < \infty, \qquad G(s) = \int_0^s g(t)\,dt. \tag{1.5}$$

Condition (1.5) is the well-known Keller–Osserman condition associated to the existence of explosive or large solutions, whereas the failure of (1.5) is related to the existence of entire solutions in whole $\mathbb{R}^N$. Singh, in [27], obtained that if Ω is a ball, that is, $\Omega = B_R$, system (1.1) has positive solutions (u, v) such that v or both u and v blow up on ∂B_R if and only if

$$\int_1^\infty \frac{ds}{\left(\int_0^s F(t)\,dt\right)^{m/(2m+1)}} < \infty, \qquad F(t) = \int_0^s f(t)\,dt. \tag{1.6}$$

Condition (1.6) can be seen roughly as the analogous of (1.5), but it involves the gradient term. Several generalizations of the Keller–Osserman condition have been developed, cf. [15] and the references therein, and also [2, 8, 19, 28, 29].

Concerning system (1.2), that is in case (a), our main result, under the hypothesis mentioned about the functions f, g and h, is the following theorem.

Theorem 1.1. *Assume $h(0) > 0$ and that g is bounded in $\mathbb{R}_0^+$. Then all positive, radial solutions (u, v) of (1.2):*
(i) *satisfy* (C1) *if and only if*

$$\int_1^\infty \frac{ds}{\left(\int_0^s F(t)\,dt\right)^{m/(2m+1)}} = \infty, \tag{1.7}$$

(ii) *satisfy* (C2) *if and only if*

$$\int_1^\infty \frac{s\,ds}{\left(\int_0^s F(t)\,dt\right)^{m/(2m+1)}} < \infty,$$

(iii) *satisfy* (C3) *if and only if* (1.6) *holds and*

$$\int_1^\infty \frac{s\,ds}{\left(\int_0^s F(t)\,dt\right)^{m/(2m+1)}} = \infty. \tag{1.8}$$

Theorem 1.1 extends [27, Theorem 2.1]. Indeed, when $h \equiv 1$ and $g \equiv 1$, Theorem 1.1 reduces exactly to [27, Theorem 2.1].

Furthermore, the required boundness of g can be removed if we consider problem (1.2) associated to boundary conditions (C1) and (C2), for details see Remark 3.2 and Corollary 3.3.

About the quasilinear extension, mentioned in (b), our main result for system (1.3) is a necessary condition given as follows.

Theorem 1.2. *Let $p \geq 2$. If system* (1.3) *admits a positive, radial solution (u, v) such that $\lim_{r\to R^-} v(r) = \infty$, then*

$$\int_1^\infty \frac{s^{(p-2)(p-1)/(pm+p-1)}\,ds}{\left(\int_0^s \sqrt[p]{f(t)}\,dt\right)^{mp/(mp+p-1)}} < \infty. \tag{1.9}$$

Theorem 1.2 is the extension to the p-Laplacian operator of the necessary part of [27, Theorem 2.1]. As a matter of fact, when $p = 2$, (1.9) becomes exactly

$$\int_1^\infty \frac{ds}{\left(\int_0^s \sqrt{f(t)}\,dt\right)^{2m/(2m+1)}} < \infty. \tag{1.10}$$

Actually, the necessary condition proved by Singh in [27, Theorem 2.1] is (1.6), which is essentially equivalent to (1.10), thanks to [27, Lemma 4.1]. In particular, [27, Lemma 4.1] is the key lemma, used by Singh in the case $p = 2$, to prove the main necessary and sufficient condition, that is, [27, Theorem 2.1].

For the quasilinear case, the extension of [27, Lemma 4.1] seems difficult to be obtained. Hence, the converse of Theorem 1.2 is still an open problem.

The paper is organized as follows. In Section 2 we give some qualitative properties of solutions of system (1.2) proved in Lemma 2.1. Then, in Theorems 2.3 and 2.4, we prove a classification of solutions of (1.2), under the blow up boundary conditions (C2) and (C3). In Section 3 we present the proof of Theorem 1.1, while in Section 4 we deal with the quasilinear case and we give the proof of Theorem 1.2. Finally, in Appendix A, we conclude with a local existence result for the solutions of system (1.2), that is, Proposition A.1.

2 Preliminary results for case (a)

In this section we first prove some useful inequalities for positive radial solutions of system (1.2). Since we deal with radial solutions of (1.2), that is, $(u(r), v(r))$ with $r = |x|$, we consider its radial version, namely,

$$\begin{cases} w'(r) + \dfrac{N-1}{r} w(r) = v^m(r) & \text{in } 0 < r < R,\\ v''(r) + \dfrac{N-1}{r} v'(r) = h(r) g(u(r)) f(|w(r)|) & \text{in } 0 < r < R,\\ w(0) = v'(0) = 0, \quad v(0) > 0, \end{cases} \tag{2.1}$$

where $w = u'$. For the condition $u'(0) = v'(0) = 0$, we refer to the pioneering paper by Ni and Serrin [25].

First, we will prove the following lemma.

Lemma 2.1. *Assume that* $(\mathscr{H})$ *holds. If* (u, v) *is a nonnegative radial solution of* (1.2), *then*

$$u'(r)(= w(r)) > 0 \quad \text{and} \quad v'(r) > 0 \quad \text{for all } r \in (0, R), \tag{2.2}$$

and the following inequalities are valid for all $r \in (0, R)$*:*

$$\frac{v^m(r)}{N} \le w'(r) \le v^m(r), \tag{2.3}$$

$$\frac{h(r)f(w(r))g(u(r))}{N} \le v''(r) \le h(r)f(w(r))g(u(r)). \tag{2.4}$$

Proof. The first equation of (2.1) can be written as

$$(r^{N-1}w(r))' = r^{N-1}v^m(r). \tag{2.5}$$

An integration of equation (2.5) in $[0, r]$ yields

$$w(r) = r^{1-N}\int_0^r t^{N-1}v^m(t)\,dt, \quad 0 < r < R. \tag{2.6}$$

From (2.6), we can deduce that $w > 0$ in $(0, R)$, that is, $u'(r) > 0$ in $(0, R)$. Moreover, the second equation of (2.1) is equivalent to

$$(r^{N-1}v'(r))' = r^{N-1}h(r)g(u(r))f(w(r)). \tag{2.7}$$

Integrating (2.7) in $[0, r]$, we get

$$v'(r) = r^{1-N}\int_0^r t^{N-1}h(t)g(u(t))f(w(t))\,dt, \quad 0 < r < R, \tag{2.8}$$

which gives $v' > 0$ in $(0, R)$, thanks to $(\mathscr{H})$, and thus (2.2) is proved, since u and v are strictly increasing in $(0, R)$.

Next, from (2.6), we can deduce

$$w(r) \le \frac{r}{N}v^m(r), \tag{2.9}$$

and by putting (2.9) in the first equation of (2.1), we get

$$\frac{v^m(r)}{N} \le w'(r). \tag{2.10}$$

In addition, using that $w > 0$ in the first equation of (2.1), we have

$$w'(r) \le v^m(r). \tag{2.11}$$

Inequalities (2.10) and (2.11) give (2.3), as required.

Now thanks to (2.3) we obtain that $w' > 0$, namely, that w is increasing. This fact, used in (2.8), together with the monotonicity of h, g, f, yields

$$v'(r) \le \frac{r}{N}h(r)g(u(r))f(w(r)), \quad 0 < r < R. \tag{2.12}$$

Putting (2.12) in the second equation of (2.1), we obtain

$$\frac{h(r)f(w(r))g(u(r))}{N} \le v''(r). \tag{2.13}$$

Furthermore, since $v' > 0$, from the second equation in (2.1), we have

$$v''(r) \le h(r)f(w(r))g(u(r)), \tag{2.14}$$

and combining (2.13) and (2.14) we get (2.4), which completes the proof. □

Now, we remind the crucial lemma, [27, Lemma 4.1], which is related to the analogous of the Keller–Osserman condition but for system (1.1). Actually we state it in a slightly different form, which will be useful later.

Lemma 2.2. *Let f be a continuous positive nondecreasing function in $\mathbb{R}^+$ and let ν be a positive number. Then we have*

$$\Big(\int_0^{2s} F(t)\,dt\Big)^{\nu} \ge \Big(\int_0^{s} \sqrt{f(t)}\,dt\Big)^{2\nu} \tag{2.15}$$

and

$$\Big(\int_0^{s} \sqrt{f(t)}\,dt\Big)^{2\nu} \ge \Big(2\int_0^{s} F(t)\,dt\Big)^{\nu},$$

where $F(t) = \int_0^t f(s)\,ds$. Consequently,

$$\int_1^{\infty} \frac{ds}{(\int_0^s F(t)\,dt)^{\nu}} < \infty \iff \int_1^{\infty} \frac{ds}{(\int_0^s \sqrt{f(t)}\,dt)^{2\nu}} < \infty.$$

In particular, we will use this lemma by considering $\nu = \frac{m}{2m+1}$.

Now we prove a classification result for positive radial solutions (u, v) of problem (1.2), having at least one explosive component. Hence, we can fall within the boundary conditions (C2) or (C3).

Theorem 2.3. *Let h, g, f be functions satisfying $(\mathscr{H})$ and such that*

$$\lim_{t\to\infty} g(t) = l \in \mathbb{R}^+. \tag{2.16}$$

If system (1.2) *admits a positive radial solutions* (u, v) *such that*

$$\lim_{r\to R^-} v(r) = \infty, \tag{2.17}$$

then

$$\int_1^{\infty} \frac{ds}{(\int_0^s \sqrt{f(t)}\,dt)^{2m/(2m+1)}} < \infty. \tag{2.18}$$

Proof. Let (u, v) be a positive radial solution of (1.2) satisfying (2.17). By Lemma 2.1, $v' > 0$, so multiplying inequality (2.4) by $v'(r)$, and then integrating in $[0, r]$, we obtain, thanks to the monotonicity of h, g, f,

$$\frac{(v'(r))^2}{2} \le h(r)g(u(r))f(w(r)) \int_0^r v'(s)\,ds = h(r)g(u(r))f(w(r))v(r),$$

which yields

$$v'(r)(v(r))^{-1/2} \le C\sqrt{h(r)f(w(r))g(u(r))}, \quad 0 < r < R, \tag{2.19}$$

where C is a positive constant. Multiplying inequality (2.19) by $w'(r)$, which is positive by (2.10), we have

$$w'(r)v'(r)(v(r))^{-1/2} \le Cw'(r)\sqrt{h(r)f(w(r))g(u(r))}, \quad 0 < r < R,$$

and, by using (2.3), we get

$$\frac{v'(r)v^{m-1/2}(r)}{N} \le Cw'(r)\sqrt{h(r)f(w(r))g(u(r))}, \quad 0 < r < R,$$

which can be written as

$$\Big(\frac{v^{m+1/2}(r)}{m+\frac{1}{2}}\Big)' \le Cw'(r)\sqrt{h(r)f(w(r))g(u(r))}.$$

An integration in $[0, r]$ gives

$$v^{m+1/2}(r) - v^{m+1/2}(0) \le C \int_0^r w'(s)\sqrt{h(s)g(u(s))f(w(s))}\,ds. \tag{2.20}$$

Inequality (2.20) becomes

$$v^{m+1/2}(r) - v^{m+1/2}(0) \le C\sqrt{h(r)}\sqrt{g(u(r))} \int_{0=w(0)}^{w(r)} \sqrt{f(t)}\,dt,$$

and since $\lim_{r\to R^-} v(r) = \infty$, there exists $\rho \in (0, R)$ such that we can, roughly, give up the term $v^{m+1/2}(0)$, obtaining

$$(v^m(r))^{(2m+1)/2m} \le C\sqrt{h(r)}\sqrt{g(u(r))} \int_0^{w(r)} \sqrt{f(t)}\,dt, \quad \rho < r < R. \tag{2.21}$$

Using again (2.3), inequality (2.21) yields

$$(w'(r))^{(2m+1)/2m} \le C\sqrt{h(r)}\sqrt{g(u(r))} \int_0^{w(r)} \sqrt{f(t)}\,dt, \quad \rho < r < R,$$

which implies

$$\frac{w'(r)}{(\sqrt{h(r)}\sqrt{g(u(r))})^{2m/(2m+1)}(\int_0^{w(r)} \sqrt{f(t)}\,dt)^{2m/(2m+1)}} \le C, \quad \rho < r < R.$$

Integrating in $[\rho, r]$, we obtain

$$\int_\rho^r \frac{w'(t)\,dt}{(\sqrt{h(t)}\sqrt{g(u(t))})^{2m/(2m+1)}(\int_0^{w(r)} \sqrt{f(t)}\,dt)^{2m/(2m+1)}} \le C(r-\rho) \le Cr,$$

which implies, thanks to the monotonicity of h and g,

$$\frac{1}{(h(r)g(u(r)))^{m/(2m+1)}} \int_{w(\rho)}^{w(r)} \frac{ds}{(\int_0^s \sqrt{f(t)}\,dt)^{2m/(2m+1)}} \le Cr. \tag{2.22}$$

Now, when u is bounded as $r \to R^-$ or unbounded, thanks to (2.16), we deduce

$$\lim_{r\to R^-} \frac{1}{g(u(r))^{m/(2m+1)}} = L \in \mathbb{R}^+.$$

Consequently, letting $r \to R^-$ in (2.22), since $w(r) \to \infty$ as $r \to R^-$ by (2.6) and (2.17), we have

$$\int_{w(\rho)}^{\infty} \frac{ds}{(\int_0^s \sqrt{f(t)}\,dt)^{2m/(2m+1)}} \le CR < \infty, \tag{2.23}$$

where we have also used that h is positive in $\mathbb{R}^+$. Inequality (2.23) gives (2.18). □

Theorem 2.4. *Let h, g, f be functions satisfying all the hypothesis of Theorem 2.3 and let $h(0) > 0$. Then system* (1.2) *admits a positive radial solutions* (u, v) *satisfying* (2.17) *if and only if* (1.6) *holds.*

Proof. In order to prove the necessary part, we have to verify (1.6). This follows directly from condition (2.18), by using Lemma 2.2.

Now we prove the sufficient part. We assume that condition (1.6) holds, and we prove that system (1.2) has a positive radial solution satisfying (2.17).

We look for radial solution of system (2.1), which is equivalent to

$$\begin{cases} w(r) = r^{1-N} \int_0^r t^{N-1} v^m(t)\,dt, \\ v(r) = v(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} h(s) g(u(s)) f(w(s))\,ds\,dt, \\ w(0) = v'(0) = 0, \\ u(0) > 0, \quad v(0) > 0, \end{cases} \tag{2.24}$$

Since $(\mathscr{H})$ holds, an application of a point fix theorem gives us the existence of a solution (u, v) of (2.24) defined in a maximal interval $[0, R_{\max})$. (See Proposition A.1 in Appendix A, in the case $p = 2$, for more details). We claim that $R_{\max} < \infty$. From inequalities (2.3) and (2.4), we have

$$w'(r) \le v^m(r) \tag{2.25}$$

and

$$h(r) f(w(r)) g(u(r)) \le N v''(r). \tag{2.26}$$

Multiplying inequalities (2.25) and (2.26) and then integrating in $[0, r]$, we arrive at

$$\int_0^r w'(t) h(t) f(w(t)) g(u(t))\,dt \le N \int_0^r v^m(t) v''(t)\,dt, \quad 0 < r < R_{\max},$$

and this yields

$$h(0) g(u(0)) F(w(r)) \le N v^m(r) v'(r), \quad 0 < r < R_{\max}.$$

Multiplying this inequality by $w'(r)$ and using (2.3), we get

$$h(0) g(u(0)) F(w(r)) w'(r) \le N v^{2m}(r) v'(r), \quad 0 < r < R_{\max}. \tag{2.27}$$

Now we fix a $\rho \in (0, R_{\max})$ and let G be the function defined by

$$G(r) := \int_\rho^r F(t)\,dt, \quad \rho \le r < R_{\max}.$$

Integrating (2.27) over $[\rho, r]$ yields

$$h(0) g(u(0)) \int_\rho^r F(w(s)) w'(s) ds \le N \int_\rho^r v^{2m}(s) v'(s)\,ds,$$

which can be written as

$$h(0) g(u(0)) \int_{w(\rho)}^{w(r)} F(t)\,dt \le N \int_\rho^r v^{2m}(t) v'(t)\,dt. \tag{2.28}$$

Now, using (2.3) in (2.28) and the fact that v is increasing, we obtain

$$h(0) g(u(0)) \int_{w(\rho)}^{w(r)} F(t)\,dt \le C (v^m(r))^{(2m+1)/m} \le C (w'(r))^{(2m+1)/m}.$$

In other words,

$$\int_{w(\rho)}^{w(r)} F(t)\,dt \le C (w'(r))^{(2m+1)/m}. \tag{2.29}$$

Now, if $w(\rho) \le \rho$, then inequality (2.29) becomes

$$G(w(r)) = \int_{\rho}^{w(r)} F(t)\,dt \le \int_{w(\rho)}^{w(r)} F(t)\,dt \le C(w'(r))^{(2m+1)/m},$$

hence

$$C \le \frac{(w'(r))^{(2m+1)/m}}{G(w(r))}, \quad \rho \le r < R_{\max}. \tag{2.30}$$

Otherwise, if $w(\rho) > \rho$, then by adding $\int_{\rho}^{w(\rho)} F(t)\,dt$ in equation (2.29), we get

$$G(w(r)) = \int_{\rho}^{w(\rho)} F(t)\,dt + \int_{w(\rho)}^{w(r)} F(t)\,dt \le \int_{\rho}^{w(\rho)} F(t)\,dt + C(w'(r))^{(2m+1)/m},$$

which implies

$$1 \le \frac{G(w(\rho))}{G(w(r))} + C\frac{(w'(r))^{(2m+1)/m}}{G(w(r))}, \quad \rho \le r < R_{\max}.$$

Since $w(0) = 0$, we have $\lim_{\rho\to 0^+} G(w(\rho)) = 0$, hence, there exists a positive constant C such that (2.30) holds. Thus, in both cases we obtain

$$C \le \frac{w'(r)}{(G(w(r)))^{m/(2m+1)}}, \quad \rho \le r < R_{\max}. \tag{2.31}$$

Integrating equation (2.31) in $[\rho, r]$, we arrive at

$$\begin{aligned} C(r-\rho) &\le \int_{w(\rho)}^{w(r)} \frac{ds}{(G(s))^{m/(2m+1)}} \\ &\le \int_{w(\rho)}^{\infty} \frac{ds}{\big(\int_{\rho}^{s} F(t)\,dt\big)^{m/(2m+1)}} \\ &= \int_{w(\rho)}^{\infty} \frac{ds}{\big(\int_{0}^{s} F(t)\,dt - \int_{0}^{\rho} F(t)\,dt\big)^{m/(2m+1)}}. \end{aligned} \tag{2.32}$$

Since

$$\int_0^s F(t)\,dt - \int_0^{\rho} F(t)\,dt \sim \int_0^s F(t)\,dt \quad \text{as } s \to \infty,$$

inequality (2.32) implies

$$C(r-\rho) \le \int_{w(\rho)}^{\infty} \frac{ds}{\big(\int_0^s F(t)\,dt\big)^{m/(2m+1)}}. \tag{2.33}$$

Now, if $w(\rho) > 1$, then trivially

$$C(r-\rho) \le \int_{w(\rho)}^{\infty} \frac{ds}{\big(\int_0^s F(t)\,dt\big)^{m/(2m+1)}} \le \int_{1}^{\infty} \frac{ds}{\big(\int_0^s F(t)\,dt\big)^{m/(2m+1)}} < \infty.$$

Otherwise, if $w(\rho) < 1$, then from inequality (2.33) it follows that

$$C(r-\rho) \le \int_{w(\rho)}^{1} \frac{ds}{\big(\int_0^s F(t)\,dt\big)^{m/(2m+1)}} + \int_{1}^{\infty} \frac{ds}{\big(\int_0^s F(t)\,dt\big)^{m/(2m+1)}} < \infty.$$

Hence, in both cases, by letting $r \to R_{\max}$, we obtain the claim $R_{\max} < \infty$.

Thus, we have just proved the existence of a positive radial solution (u, v) defined in $[0, R_{\max})$, with $R_{\max} < \infty$. Now we need to show that the solution of (1.2) is defined in $[0, R)$, where R is the required radius. Let

$$\tilde{f} = \alpha^{2(1+1/m)} f\Big(\frac{t}{\alpha}\Big) \quad \text{for all } t \geq 0,$$

where $\alpha = \frac{R}{R_{\max}}$. The function $\tilde{f}$ satisfies (1.6) so, due to previous arguments, there exists a solution $(\tilde{u}, \tilde{v})$ of the system

$$\begin{cases} \Delta \tilde{u} = \tilde{v}^m & \text{in } B_{R_{\max}}, \\ \Delta \tilde{v} = \tilde{h}(|x|)\tilde{g}(u)\tilde{f}(|\nabla \tilde{u}|) & \text{in } B_{R_{\max}}, \end{cases}$$

where $\tilde{g}(u) = g(\tilde{u})$ and $\tilde{h}(|x|) = h(|\tilde{x}|)$. Now, by setting

$$u(r) = \tilde{u}\Big(\frac{r}{\alpha}\Big), \quad v(r) = \alpha^{-2/m}\tilde{v}\Big(\frac{r}{\alpha}\Big) \quad \text{in } B_R,$$

we have the required solution (u, v) in B_R.

It remains to prove condition (2.17). Since v is strictly increasing, the following limit exists:

$$\lim_{r \to R^-} v(r) = \sup_{[0,R)} v(r) = \ell_1.$$

Assume by contradiction that $\ell_1 \in \mathbb{R}^+$. From inequality (2.3), we have

$$w'(r) \leq v^m(r) \quad \text{for all } r \in [0, R),$$

and, since v is bounded, we can deduce that w' is bounded in $[0, R)$. Hence, we get that $w = u'$ is bounded in $[0, R)$ and, analogously, u is bounded in $[0, R)$, i.e.,

$$\lim_{r \to R^-} u(r) < \infty \quad \text{and} \quad \lim_{r \to R^-} u'(r) < \infty.$$

In addition, we have

$$(r^{N-1}v'(r))' = r^{N-1}h(r)g(u(r))f(w(r)) > 0 \quad \text{for all } r \in [0, R),$$

which implies that $r^{N-1}v'(r)$ is increasing in $[0, R)$, and so

$$\lim_{r \to R^-} r^{N-1}v'(r) = \ell_2 \in (0, \infty].$$

Since $R < \infty$,

$$\lim_{r \to R^-} v'(r) = \ell_3 \in (0, \infty].$$

We now prove that $\ell_3 < \infty$. Indeed,

$$(r^{N-1}v'(r))' = r^{N-1}h(r)g(u(r))f(w(r)),$$

and by integrating in $[0, r]$, with $r < R$, we get

$$v'(r) = \int_0^r \Big(\frac{s}{r}\Big)^{N-1} h(s)g(u(s))f(w(s))\,ds \leq \int_0^R h(s)g(u(s))f(w(s))\,ds.$$

Since w and u are bounded, by continuity, we have that $h(s)$, $g(u(s))$ and $f(w(s))$ are bounded, and since $v' > 0$, from Lemma 2.1, we obtain

$$0 \leq \lim_{r \to R^-} v'(r) \leq \int_0^R h(s)g(u(s))f(w(s))\,ds < \infty,$$

which contradicts the maximality of B_R. This completes the proof. □

3 Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1, which gives a complete classification of the solutions of system (1.2).

Proof of Theorem 1.1. The proof is divided into two steps.

Step 1: We start by proving case (iii). Note that the boundness of g implies the validity of (2.16). Thus, let (u,v) be a positive radial solution of system (1.2) satisfying (C3). Since $\lim_{r\to R^-} v(r)=\infty$, condition (1.6) holds thanks to Theorem 2.4. By letting $r\to R^-$ in (2.33), we get

$$C_2(R-\rho)\le\int_{w(\rho)}^{\infty}\frac{ds}{\big(\int_0^s F(t)\,dt\big)^{m/(2m+1)}},\qquad \rho\in(0,R). \tag{3.1}$$

In addition, from (2.15) we have

$$\bigg(\int_0^s\sqrt{f(t)}\,dt\bigg)^{2m/(2m+1)}\le\bigg(\int_0^{2s}F(t)\,dt\bigg)^{m/(2m+1)}\qquad\text{for all } s\ge 0,$$

which by integration gives

$$\int_{w(r)}^{\infty}\frac{ds}{\big(\int_0^{2s}F(t)\,dt\big)^{m/(2m+1)}}\le\int_{w(r)}^{\infty}\frac{ds}{\big(\int_0^s\sqrt{f(t)}\,dt\big)^{2m/(2m+1)}}.$$

Consequently, by (2.23), we obtain

$$\int_{w(r)}^{\infty}\frac{ds}{\big(\int_0^{2s}F(t)\,dt\big)^{m/(2m+1)}}\le C_1(R-r). \tag{3.2}$$

Let $\Gamma\colon(0,\infty)\to(0,\infty)$ be a function such that

$$\Gamma(t)=\int_t^{\infty}\frac{ds}{\big(\int_0^s F(\tau)\,d\tau\big)^{m/(2m+1)}}. \tag{3.3}$$

Clearly, Γ is strictly decreasing and $\lim_{t\to\infty}\Gamma(t)=0$, by (1.6). Furthermore, we can deduce, from (3.1) with $\rho=r$, and (3.2) that

$$\Gamma(2w(r))\le C_1(R-r)\quad\text{and}\quad C_2(R-r)\le\Gamma(w(r)).$$

Since Γ is strictly decreasing, the last two inequalities yield

$$\Gamma^{-1}(C_1(R-r))\le 2w(r),\qquad w(r)\le\Gamma^{-1}(C_2(R-r))\quad\text{for all }\rho\le r<R. \tag{3.4}$$

Now, since $w=u'$, from

$$u(r)=u(\rho)+\int_\rho^r w(t)\,dt\quad\text{for all }\rho\le r<R,$$

we deduce that $\lim_{r\to R^-}u(r)=\infty$ if and only if $\int_\rho^R w(t)\,dt=\infty$, namely, thanks to (3.4), if and only if

$$\int_\rho^R\Gamma^{-1}(C(R-t))\,dt=\infty$$

for a positive constant C. With a change of variables, we obtain $\lim_{r\to R}u(r)=\infty$ if and only if

$$\int_0^{C(R-\rho)}\Gamma^{-1}(\sigma)\,d\sigma=\infty,$$

which, since Γ^{-1} is well defined in $(0,\infty)$, is equivalent to

$$\int_0^1\Gamma^{-1}(\sigma)\,d\sigma=\infty. \tag{3.5}$$

Now, due to (3.3), by setting $t = \Gamma^{-1}(\sigma)$, we observe that

$$\lim_{\sigma\to 0^+} \Gamma^{-1}(\sigma) = \lim_{r\to R^-} \Gamma^{-1}(C(R-r)) \geq \lim_{r\to R^-} w(r) = \infty,$$

where we have used (3.4) and (2.6) together with $\lim_{r\to R^-} v(r)$. Hence, (3.5) becomes

$$\int_1^\infty \frac{s\,ds}{\big(\int_0^s F(t)\,dt\big)^{m/(2m+1)}} = \infty,$$

which is exactly condition (1.8). This concludes the proof of the necessary part.

To sum up, we have just obtained that if $v \to \infty$ as $r \to R^-$, then

$$\lim_{r\to R^-} u(r) = \infty \iff \int_1^\infty \frac{s\,ds}{\big(\int_0^s F(t)\,dt\big)^{m/(2m+1)}} = \infty. \tag{3.6}$$

To prove the sufficient part under conditions (1.6) and (1.8), it is enough to observe that (1.6) implies the existence of a solution of problem (1.2) satisfying (2.17), thanks to Theorem 2.4. Furthermore, the other requirement, that is, $\lim_{r\to R^-} u(r) = \infty$, follows from (1.8) using the same arguments as in the last lines of the necessary part.

Step 2: We now prove cases (i) and (ii).

If condition (C3) does not hold, then there are two possible cases:

(i) condition (C2) holds,

(ii) both u and v are bounded, that is, condition (C1).

The first case leads to (1.6), using Theorem 2.4. Moreover, since u is bounded in this case, using (3.6), we obtain

$$\int_1^\infty \frac{s\,ds}{\big(\int_0^s F(t)\,dt\big)^{m/(2m+1)}} < \infty.$$

Thus, case (ii) is proved.

Finally, if both u and v are bounded, by Theorem 2.4, condition (1.6) cannot hold, hence (1.7) is verified. This completes the proof of the theorem. □

Example 3.1. Theorem 1.1 can be applied, for instance, to the problem

$$\begin{cases} \Delta u = v^m & \text{in } B_R, \\ \Delta v = e^{|x|} \arctan u\, f(|\nabla u|) & \text{in } B_R, \end{cases}$$

or to the system

$$\begin{cases} \Delta u = v^m & \text{in } B_R, \\ \Delta v = (|x|+1)^\alpha (1 - e^{-u}) f(|\nabla u|) & \text{in } B_R, \end{cases}$$

where $\alpha > 0$.

Remark 3.2. Assumption (2.16), that is, the boundness of g in $\mathbb{R}^+$, can be removed if we consider problem (1.2) under the boundary conditions (C1) and (C2). Indeed, (2.16) is used in the proof of Theorem 2.4, and consequently in the proof of Theorem 1.1, in order to manage the case when $\lim_{r\to R^-} u(r) = \infty$.

Precisely, the following corollary holds.

Corollary 3.3. *Let h, g, f be functions satisfying ($\mathscr{H}$) and such that $h > 0$ in $[0, \infty)$ and* (2.16) *holds. Then system* (1.2) *has:*

(i) *positive, radial and bounded solutions (u, v) if and only if*

$$\int_1^\infty \frac{ds}{\big(\int_0^s F(t)\,dt\big)^{m/(2m+1)}} = \infty,$$

(ii) *positive radial solutions* (u, v) *such that* $\lim_{r\to R^-} v(r) = \infty$ *if and only if*

$$\int_1^\infty \frac{ds}{\left(\int_0^s F(t)\,dt\right)^{m/2m+1}} < \infty.$$

4 The quasilinear case

In this section we analyze the quasilinear version of system (1.1). Precisely, we deal with system (1.3), which is a more general problem involving the p-Laplacian operator. Again, we are interested in finding optimal conditions in order to classify its positive radial solutions (u, v).

We start by giving the extension of Lemma 2.1 relative to solutions of system (1.3).

Lemma 4.1. *If system* (1.3) *has a positive radial solution* (u, v)*, then*

$$u' = w > 0, \quad v' > 0 \quad \text{in } (0, R), \tag{4.1}$$

and the following inequalities hold for all $r \in (0, R)$*:*

$$\frac{v^m(r)}{N} \le (w^{p-1}(r))' \le v^m(r), \qquad \frac{f(w(r))}{N} \le ((v')^{p-1}(r))' \le f(w(r)).$$

Proof. First of all, letting $w = u'$, we can rewrite system (1.3) as

$$\begin{cases} (|w(r)|^{p-2}w(r))' + \dfrac{N-1}{r}|w(r)|^{p-2}w(r) = v^m(r), & r \in (0, R), \\ (|v'(r)|^{p-2}v'(r))' + \dfrac{N-1}{r}|v'(r)|^{p-2}v'(r) = f(|w(r)|), & r \in (0, R). \end{cases} \tag{4.2}$$

This reduces to

$$\begin{cases} (r^{N-1}|w(r)|^{p-2}w(r))' = r^{N-1}v^m(r), & r \in (0, R), \\ (r^{N-1}|v'(r)|^{p-2}v'(r))' = r^{N-1}f(|w(r)|), & r \in (0, R). \end{cases} \tag{4.3}$$

Integrating these two equations over $[0, r]$ with $r > 0$, and using the fact that $w(0) = v'(0) = 0$, from (4.3), we obtain

$$\begin{cases} |w(r)|^{p-2}w(r) = r^{1-N}\displaystyle\int_0^r t^{N-1}v^m(t)\,dt, & r \in (0, R), \\ |v'(r)|^{p-2}v'(r) = r^{1-N}\displaystyle\int_0^r t^{N-1}f(|w(t)|)\,dt, & r \in (0, R). \end{cases} \tag{4.4}$$

Since $v > 0$ and $f > 0$, thanks to the two equalities in (4.4), we have $w > 0$ and $v' > 0$, so both u and v are increasing in $(0, R)$. In addition, system (4.3) becomes

$$\begin{cases} (r^{N-1}w^{p-1}(r))' = r^{N-1}v^m(r), & r \in (0, R), \\ (r^{N-1}(v')^{p-1}(r))' = r^{N-1}f(w(r)), & r \in (0, R), \end{cases}$$

while system (4.4) becomes

$$\begin{cases} w^{p-1}(r) = r^{1-N}\displaystyle\int_0^r t^{N-1}v^m(t)\,dt, & r \in (0, R), \\ (v')^{p-1}(r) = r^{1-N}\displaystyle\int_0^r t^{N-1}f(w(t))\,dt, & r \in (0, R). \end{cases} \tag{4.5}$$

Since v is increasing, from the first equation in (4.5), we obtain

$$w^{p-1}(r) \leq \frac{r}{N} v^m(r),$$

and inserting this inequality in the first equation of (4.2), we get

$$v^m(r) = (w^{p-1}(r))' + \frac{N-1}{r} w^{p-1}(r) \leq (w^{p-1}(r))' + \frac{N-1}{r}\frac{r}{N} v^m(r).$$

This implies

$$\frac{v^m(r)}{N} \leq (w^{p-1}(r))'. \tag{4.6}$$

Now, again from the first equation in (4.2), we are able to deduce

$$(w^{p-1}(r))' \leq v^m(r). \tag{4.7}$$

Combining (4.6) and (4.7), we obtain

$$\frac{v^m(r)}{N} \leq (w^{p-1}(r))' \leq v^m(r). \tag{4.8}$$

In turn

$$\frac{v^m(r)}{N} \leq (p-1) w^{p-2}(r) w'(r),$$

thus, since $w > 0$ and $p > 1$, we can deduce that $w' > 0$. Hence, w is increasing in $(0, R)$, and since f is non-decreasing over $(0, R)$, the second equation in (4.5) yields

$$(v')^{p-1} \leq \frac{r}{N} f(w(r)). \tag{4.9}$$

Using (4.9) in the second equation of (4.2), we have

$$f(w(r)) = ((v')^{p-1}(r))' + \frac{N-1}{r}((v')^{p-1}(r)) \leq ((v')^{p-1}(r))' + \frac{N-1}{r}\frac{r}{N} f(w(r)).$$

This implies

$$\frac{f(w(r))}{N} \leq ((v')^{p-1}(r))'. \tag{4.10}$$

Furthermore, from the second equation in (4.2), we can deduce

$$((v')^{p-1}(r))' \leq f(w(r)), \tag{4.11}$$

and by combining (4.10) with (4.11), we obtain

$$\frac{f(w(r))}{N} \leq ((v')^{p-1}(r))' \leq f(w(r)).$$ □

Remark 4.2. Note that if $p = 2$, Lemma 4.1 gives the same inequalities of Lemma 2.1 when $g \equiv 1$ and $h \equiv 1$.

Now we can prove Theorem 1.2, which gives a necessary condition in order to have a solution (u, v) of system (1.3) that verifies (2.17), extending Theorem 2.3 to the quasilinear case.

Proof of Theorem 1.2. We start by assuming the existence of a positive radial solution (u, v) such that $\lim_{r \to R^-} v(r) = \infty$.

Multiplying inequality (4.11) by $v'(r)$ and then integrating over $[0, r]$, with $r < R$, yields

$$\int_0^r ((v')^{p-1}(t))' v'(t)\, dt \leq \int_0^r f(w(t)) v'(t)\, dt.$$

The fact that f is non-decreasing while w is increasing over $(0, r)$, leads to

$$(p-1) \int_0^r (v'(t))^{p-1} v''(t)\, dt \leq \int_0^r f(w(t)) v'(t)\, dt \leq v(r) f(w(r)).$$

This inequality becomes

$$\frac{p-1}{p}(v'(r))^p \le v(r)f(w(r)),$$

which yields

$$v'(r)(v(r))^{-1/p} \le C\sqrt[p]{f(w(r))}, \quad 0 < r < R, \tag{4.12}$$

where C is a positive constant changing from line to line. Multiplying inequality (4.12) by $(w^{p-1}(r))'$, which is positive, we obtain

$$v'(r)(v(r))^{-1/p}(w^{p-1}(r))' \le C(w^{p-1}(r))'\sqrt[p]{f(w(r))}, \quad 0 < r < R,$$

and using (4.8) we get

$$\frac{v'(r)v^{m-1/p}(r)}{N} \le C(w^{p-1}(r))'\sqrt[p]{f(w(r))},$$

which can be written as

$$\left(\frac{v^{m+(p-1)/p}(r)}{m+(p-1)/p}\right)' \le C(w^{p-1}(r))'\sqrt[p]{f(w(r))}.$$

With an integration over $[0, r]$, we obtain

$$v^{m+(p-1)/p}(r) - v^{m+(p-1)/p}(0) \le C\int_0^r (w^{p-1}(s))'\sqrt[p]{f(w(s))}\,ds,$$

that is,

$$v^{m+(p-1)/p}(r) - v^{m+(p-1)/p}(0) \le C(p-1)\int_0^r w^{p-2}(s)w'(s)\sqrt[p]{f(w(s))}\,ds.$$

Since w is increasing over $(0, r)$ and $p > 2$, by changing the variable, we get

$$v^{m+(p-1)/p}(r) - v^{m+(p-1)/p}(0) \le Cw^{p-2}(r)\int_{0=w(0)}^{w(r)} \sqrt[p]{f(t)}\,dt.$$

Now, since $\lim_{r\to R^-} v(r) = \infty$ and $m + \frac{p-1}{p} > 0$, there exists $\rho \in (0, R)$ such that

$$(v^m(r))^{[m+(p-1)/p]/m} = v^{m+(p-1)/p}(r) \le Cw^{p-2}(r)\int_0^{w(r)} \sqrt[p]{f(t)}\,dt, \quad \rho < r < R.$$

Using (4.8), this yields

$$\frac{(w^{p-1}(r))'}{(w^{p-2}(r))^{m/[m+(p-1)/p]}\left(\int_0^{w(r)}\sqrt[p]{f(t)}\,dt\right)^{m/[m+(p-1)/p]}} \le C,$$

in other words

$$\frac{w'(r)}{(w^{p-2}(r))^{m/[m+(p-1)/p]-1}\left(\int_0^{w(r)}\sqrt[p]{f(t)}\,dt\right)^{m/[m+(p-1)/p]}} \le C.$$

Integrating this last inequality over $[\rho, r]$, we arrive at

$$\int_\rho^r \frac{w'(t)\,dt}{(w^{p-2}(t))^{m/[m+(p-1)/p]-1}\left(\int_0^{w(r)}\sqrt[p]{f(t)}\,dt\right)^{m/[m+(p-1)/p]}} \le C(r-\rho) \le Cr.$$

Now, by changing the variable and letting $r \to R$, we obtain

$$\int_{w(\rho)}^{\infty} \frac{ds}{(s^{p-2})^{m/[m+(p-1)/p]-1}\left(\int_0^{s}\sqrt[p]{f(t)}\,dt\right)^{m/[m+(p-1)/p]}} \le C(R-\rho) < \infty.$$

This inequality gives claim (1.9). □

Remark 4.3. If $p = 2$, then Theorem 1.2 reduces to Theorem 2.3.

Remark 4.4. The exponent $\alpha_p = \frac{(p-1)(p-2)}{pm+p-1}$ is positive, since $p > 2$. In particular, if $2 < p \le 3$, we have $\alpha_p < 1$, whereas if $p > 3$, we have two possible cases:
(i) $\alpha_p < 1$ if $m > \frac{(p-3)(p-1)}{p}$,
(ii) $\alpha_p > 1$ if $m < \frac{(p-3)(p-1)}{p}$.

The sufficient condition for the existence of a large solution of (1.3) is still an open problem. First of all, it is necessary to prove an extension of Lemma 2.2, whose proof (see [27]) is strongly based on the fact that the operator is linear, namely, that $p = 2$. In addition, even if one succeeds in proving it, the main target is still rather distant. Indeed, proceeding as in Theorem 2.4, a great difficulty, in this case, relies on the definition of the auxiliary function G. As a matter of fact, by defining G as in Theorem 2.4, we cannot manage to reach the claim due to the presence of terms which appear to be difficult to estimate.

A Local existence

In this section, for completeness, we give a proof of local existence for radial solutions of (1.2). We follow the proof of [5, Proposition 9] in the case $\varphi(t) = t^{p-1}$ but in our setting there is no need to assume $f(0) > 0$ as in [5, Proposition 9], since (4.1) holds.

Proposition A.1. *Let $p > 1$. Then the problem*

$$\begin{cases} [r^{N-1}|u'|^{p-2}u']' = r^{N-1}v^m & \text{in } \mathbb{R}^+, \\ [r^{N-1}|v'|^{p-2}v']' = r^{N-1}h(r)g(u)f(|u'|) & \text{in } \mathbb{R}^+, \\ u(0) = u_0 > 0, \quad v(0) = v_0 > 0, \\ u'(0) = v'(0) = 0 \end{cases} \tag{A.1}$$

has a solution on some interval $[0, \tau]$, $\tau > 0$.

Proof. Any local solution (u, v) of (A.1) is strictly positive with $u' > 0$ and $v' > 0$ for $r > 0$, thanks to Lemma 4.1, and for small $r > 0$, it must be a fixed point of the operator

$$\mathcal{T}[u, v](r) = \begin{pmatrix} \mathcal{T}_1[u, v](r) \\ \mathcal{T}_2[u, v](r) \end{pmatrix}, \tag{A.2}$$

where

$$\mathcal{T}_1[u, v](r) = u_0 + \int_0^r \left(s^{1-N} \int_0^s \tau^{N-1} v^m(\tau)\, d\tau \right)^{1/(p-1)} ds,$$

$$\mathcal{T}_2[u, v](r) = v_0 + \int_0^r \left(s^{1-N} \int_0^s \tau^{N-1} h(\tau) g(u(\tau)) f(u'(\tau))\, d\tau \right)^{1/(p-1)} ds.$$

Fix now $\varepsilon > 0$ so small that $[u_0 - \varepsilon, u_0 + \varepsilon]$, $[v_0 - \varepsilon, v_0 + \varepsilon] \subset \mathbb{R}^+$, so that by $(\mathscr{H})$,

$$0 \le j = \min_{[0,\varepsilon]} h(t) \le \max_{[0,\varepsilon]} h(t) = H < \infty,$$

$$0 < i = \min_{[u_0-\varepsilon, u_0+\varepsilon]} g(u) \le \max_{[u_0-\varepsilon, u_0+\varepsilon]} g(u) = M < \infty,$$

$$0 \le l = \min_{[0,\varepsilon]} f(t) \le \max_{[0,\varepsilon]} f(t) = L < \infty.$$

Let $C^1[0, r_0]$, $r_0 > 0$, be the usual Banach space of real functions of class C^1 in $[0, r_0]$, endowed with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$. Put $u_0(r) \equiv u_0 \in C^1[0, r_0]$ and $v_0(r) \equiv v_0 \in C^1[0, r_0]$ and let

$$C = \{(u, v) \in C^1[0, r_0] \times C^1[0, r_0] : \|(u, v) - (u_0, v_0)\| \le \varepsilon\},$$

with $\|(u, v)\| = \|u\| + \|v\|$. Then $(u, v) \in C$ if and only if

$$\|u - u_0\|_\infty + \|v - v_0\|_\infty + \|u'\|_\infty + \|v'\|_\infty \le \varepsilon.$$

Clearly, C is the closed ball in $C^1[0,r_0]\times C^1[0,r_0]$ of center (u_0,v_0) and radius $\varepsilon>0$, so that C is closed, convex and bounded in $C^1[0,r_0]\times C^1[0,r_0]$.

If $(u,v)\in C$, then

$$\begin{aligned}&u([0,r_0])\subset[u_0-\varepsilon,u_0+\varepsilon],\quad v([0,r_0])\subset[v_0-\varepsilon,v_0+\varepsilon],\\&u'([0,r_0])\subset[-\varepsilon,\varepsilon],\quad v'([0,r_0])\subset[-\varepsilon,\varepsilon],\\&0\le h(r)\le H,\quad 0<g(u(r))\le M,\quad 0\le f(u'(r))\le L\end{aligned}$$

for all $r\in[0,r_0]$. Furthermore, it is easy to prove, by considering $(\mathscr{H})$, that the operator $\mathcal{T}$ in (A.2) is well defined.

Now we show that $\mathcal{T}\colon C\to C$, provided that r_0 is sufficiently small. Indeed, for $(u,v)\in C$, we have

$$\begin{aligned}\|\mathcal{T}[u,v]-(u_0,v_0)\|_\infty&=\int_0^{r_0}\bigg(\int_0^s\Big(\frac{\tau}{s}\Big)^{N-1}v^m(\tau)\,d\tau\bigg)^{1/(p-1)}ds+\int_0^{r_0}\bigg(\int_0^s\Big(\frac{\tau}{s}\Big)^{N-1}h(\tau)g(u(\tau))f(u'(\tau))\,d\tau\bigg)^{1/(p-1)}ds\\&\le r_0^{p/(p-1)}(v^{m/(p-1)}(r_0)+(\mathrm{HLM})^{1/(p-1)}),\\\|\mathcal{T}[u,v]'\|_\infty&\le\bigg(\int_0^{r_0}v^m(\tau)\,d\tau\bigg)^{1/(p-1)}+\bigg(\int_0^{r_0}h(\tau)g(u(\tau))f(u'(\tau))\,d\tau\bigg)^{1/(p-1)}\\&\le r_0^{1/(p-1)}(v^{m/(p-1)}(r_0)+(\mathrm{HLM})^{1/(p-1)}).\end{aligned}$$

Thus, there exists $r_0=r_0(\varepsilon)>0$ so small that

$$(r_0^{p/(p-1)}+r_0^{1/(p-1)})(v^{m/(p-1)}(r_0)+(\mathrm{HLM})^{1/(p-1)})\le\varepsilon.$$

This in turn implies $\mathcal{T}[u,v]\in C$, and so $\mathcal{T}(C)\subset C$.

Now we prove that the operator $\mathcal{T}$ is compact. Let $(u_k,v_k)_k$ be a sequence in C, and let r,t be two points in $[0,r_0]$. Obviously, $(u_k,v_k)_k\in C$ is bounded because it belongs to C. Since $\mathcal{T}[u,v](r)\colon[0,r_0]\to\mathbb{R}^2$, we denote by $\|\cdot\|_2$ the usual euclidean norm in $\mathbb{R}^2$. We have

$$\begin{aligned}\|\mathcal{T}[u_k,v_k](r)-\mathcal{T}[u_k,v_k](t)\|_2&\le\bigg|\int_t^r\bigg(\int_0^s v_k^m(\tau)\,d\tau\bigg)^{1/(p-1)}ds\bigg|+\bigg|\int_t^r\bigg(\int_0^s h(\tau)g(u_k(\tau))f(u_k'(\tau))\,d\tau\bigg)^{1/(p-1)}ds\bigg|\\&\le r_0^{1/(p-1)}(v_k^{m/(p-1)}(r_0)+(\mathrm{HLM})^{1/(p-1)})|r-t|.\end{aligned}$$

Furthermore, by setting

$$\mathcal{J}_k(r)=\int_0^r\tau^{N-1}v_k^m(\tau)\,d\tau\quad\text{and}\quad\mathcal{I}_k(r)=\int_0^r\tau^{N-1}h(\tau)g(u_k(\tau))f(u_k'(\tau))\,d\tau,$$

we get

$$\big\|\mathcal{T}[u_k,v_k]'(r)-\mathcal{T}[u_k,v_k]'(t)\big\|_2=\bigg|\Big(\frac{\mathcal{J}_k(r)}{r^{N-1}}\Big)^{1/(p-1)}-\Big(\frac{\mathcal{J}_k(t)}{t^{N-1}}\Big)^{1/(p-1)}\bigg|+\bigg|\Big(\frac{\mathcal{I}_k(r)}{r^{N-1}}\Big)^{1/(p-1)}-\Big(\frac{\mathcal{I}_k(t)}{t^{N-1}}\Big)^{1/(p-1)}\bigg|.$$

Now, if $p\ge2$, then, using [5, Lemma 6] (case (i) with $\varphi(t)=t^{p-1}$), we obtain

$$\begin{aligned}\bigg|\Big(\frac{\mathcal{J}_k(r)}{r^{N-1}}\Big)^{1/(p-1)}-\Big(\frac{\mathcal{J}_k(t)}{t^{N-1}}\Big)^{1/(p-1)}\bigg|^{p-1}&\le\bigg|\frac{\mathcal{J}_k(r)}{r^{N-1}}-\frac{\mathcal{J}_k(t)}{t^{N-1}}\bigg|\le Nv_k^m(r_0)|r-t|,\\\bigg|\Big(\frac{\mathcal{I}_k(r)}{r^{N-1}}\Big)^{1/(p-1)}-\Big(\frac{\mathcal{I}_k(t)}{t^{N-1}}\Big)^{1/(p-1)}\bigg|^{p-1}&\le\bigg|\frac{\mathcal{I}_k(r)}{r^{N-1}}-\frac{\mathcal{I}_k(t)}{t^{N-1}}\bigg|\le\mathrm{NHLM}|r-t|.\end{aligned}$$

On the other hand, if $1<p<2$, then, using again [5, Lemma 6] (case (ii) with $\varphi(t)=t^{p-1}$), we have

$$\begin{aligned}\bigg|\Big(\frac{\mathcal{J}_k(r)}{r^{N-1}}\Big)^{1/(p-1)}-\Big(\frac{\mathcal{J}_k(t)}{t^{N-1}}\Big)^{1/(p-1)}\bigg|&\le M_1\bigg|\frac{\mathcal{J}_k(r)}{r^{N-1}}-\frac{\mathcal{J}_k(t)}{t^{N-1}}\bigg|\le M_1Nv_k^m(r_0)|r-t|,\\\bigg|\Big(\frac{\mathcal{I}_k(r)}{r^{N-1}}\Big)^{1/(p-1)}-\Big(\frac{\mathcal{I}_k(t)}{t^{N-1}}\Big)^{1/(p-1)}\bigg|&\le M_2\bigg|\frac{\mathcal{I}_k(r)}{r^{N-1}}-\frac{\mathcal{I}_k(t)}{t^{N-1}}\bigg|\le M_2\mathrm{NHLM}|r-t|,\end{aligned}$$

where

$$M_1 = \frac{1}{p-1}\max\Big\{\Big(\frac{\mathcal{J}_k(r)}{r^{N-1}}\Big)^{(2-p)/(p-1)}, \Big(\frac{\mathcal{J}_k(t)}{t^{N-1}}\Big)^{(2-p)/(p-1)}\Big\},$$
$$M_2 = \frac{1}{p-1}\max\Big\{\Big(\frac{\mathfrak{I}_k(r)}{r^{N-1}}\Big)^{(2-p)/(p-1)}, \Big(\frac{\mathfrak{I}_k(t)}{t^{N-1}}\Big)^{(2-p)/(p-1)}\Big\}.$$

Therefore, in both cases, by the Ascoli–Arzelà theorem, $\mathcal{T}$ maps bounded sequences into relatively compact sequences, with limit points in C, since C is closed.

Finally, $\mathcal{T}$ is continuous, because if $(u, v) \in C$ and $(u_k, v_k)_k \subset C$ are such that $\|(u_k, v_k) - (u, v)\| \to 0$ as $k \to \infty$, then by the Lebesgue dominated convergence theorem, we can pass under the sign of integrals twice in (A.2), and so $\mathcal{T}[u_k, v_k]$ tends to $\mathcal{T}[u, v]$ pointwise in $[0, r_0]$ as $k \to \infty$. By the above argument, it is obvious that $\|\mathcal{T}[u_k, v_k] - \mathcal{T}[u, v]\| \to 0$ as $k \to \infty$ as claimed.

By the Schauder fixed point theorem, $\mathcal{T}$ possesses a fixed point (u, v) in C, namely, $\mathcal{T}[u, v] = (u, v)$. Clearly, $(u, v) \in C^1[0, r_0] \times C^1[0, r_0]$ by the representation formula (A.2), that is,

$$\begin{cases} u(r) = u_0 + \displaystyle\int_0^r s^{1-N}\Big(\int_0^s t^{N-1} v^m(t)\,dt\Big)^{1/(p-1)} ds, \\ v(r) = v_0 + \displaystyle\int_0^r s^{1-N}\Big(\int_0^s t^{N-1} h(t) g(u(t)) f(|u'(t)|)\,dt\Big)^{1/(p-1)} ds, \end{cases} \tag{A.3}$$

as desired. □

Once it is known that a solution (u, v) of (A.1) exists, then (u, v) necessarily obeys (A.3).

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