# The nonexistence of an additive quaternary [15, 5, 9]-code 

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#### Abstract

We show that no additive $[15,5,9]_{4}$-code exists. As a consequence the largest dimension $k$ such that an additive quaternary $[15, k, 9]_{4^{-}}$ code exists is $k=4.5$.


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## 1 Introduction

Additive codes generalize the notion of linear codes, see also Chapter 17 of [1]. Here we concentrate on the quaternary case.

Definition 1.1. Let $k$ be such that $2 k$ is a positive integer. An additive quaternary $[n, k]_{4}$-code $\mathcal{C}$ (length $n$, dimension $k$ ) is a $2 k$-dimensional subspace of $\mathbb{F}_{2}^{2 n}$, where the coordinates come in pairs of two. We view the codewords as $n$-tuples $x=\left(x_{1}, \ldots, x_{n}\right)$ where $x_{i} \in \mathbb{F}_{2}^{2}$ and use the Hamming metric: if $y=\left(y_{1}, \ldots, y_{n}\right)$, then $d(x, y)$ is the number of indices $i$ such that $x_{i} \neq y_{i}$. A generator matrix $G$ of $\mathcal{C}$ is a binary $(2 k, 2 n)$-matrix whose rows form a basis of the binary vector space $\mathcal{C}$.

Our main result is the following.
Theorem 1.2. There is no additive $[15,5,9]_{4}$-code.
The theory of additive codes is a natural and far-reaching generalization of the classical theory of linear codes. The classical theory of cyclic and constacyclic linear codes has been generalized to additive codes in [2]. In the sequel we restrict to the case of quaternary additive codes and write $[n, k, d]$ for $[n, k, d]_{4}$. The quaternary case is of special interest, among others because of a close link to the theory of quantum stabilizer codes and their geometric representations, see $[7,9]$. The determination of the optimal parameters of additive quaternary codes of short length was initiated by Blokhuis and Brouwer [5]. In [4] we determine the optimal parameters for all lengths $n \leq 13$ except in one case. The last gap was closed by Danielsen-Parker [8] who constructed two cyclic [13, 6.5, 6]-codes. Let us concentrate on lengths $n=14$ and $n=15$ now. A cyclic [15, 4.5, 9]-code was constructed in [3], see also [2]. Together with Theorem 1.2 some more optimal parameters are determined. Table 1 collects this information.

Table 1: Maximum distance $d$ of additive $[n, k, d]_{4}$-codes of length $n \leq 15$.

| $k \backslash n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | \| 9 | 10 | 11 | 12 | \| 13 | 14 | 15 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 1.5 |  | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 12 |
| 2 |  | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 | 8 | 9 | 10 | 11 | 12 |
| 2.5 |  |  | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 | 8 | 9 | 10 | 11 |
| 3 |  |  | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 | 7 | 8 | 9 | 10 | 11 |
| 3.5 |  |  |  | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 | 8 | 9 | 10 |
| 4 |  |  |  | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 | 9 | 10 |
| 4.5 |  |  |  |  | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 | 7 | 8 | 9 |
| 5 |  |  |  |  | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 | 8 |
| 5.5 |  |  |  |  |  | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| 6 |  |  |  |  |  | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| 6.5 |  |  |  |  |  |  | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 | 7 |
| 7 |  |  |  |  |  |  | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 7.5 |  |  |  |  |  |  |  | 1 | 2 | 2 | 3 | 4 | 5 | 5-6 | 6 |
| 8 |  |  |  |  |  |  |  | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 8.5 |  |  |  |  |  |  |  |  | 1 | 2 | 2 | 3 | 4 | 4-5 | $5-6$ |
| 9 |  |  |  |  |  |  |  |  | 1 | 2 | 2 | 3 | 4 | 4 | 5 |
| 9.5 |  |  |  |  |  |  |  |  |  | 1 | 2 | 2 | 3 | 4 | 4-5 |
| 10 |  |  |  |  |  |  |  |  |  | 1 | 2 | 2 | 3 | 4 | 4 |
| 10.5, 11 |  |  |  |  |  |  |  |  |  |  | 1 | 2 | 2 | 3 | 4 |
| 11.5, 12 |  |  |  |  |  |  |  |  |  |  |  | 1 | 2 | 2 | 3 |

In order to understand the entries in the two last columns corresponding to $n=14$ and $n=15$, observe that an $[n, k, d]$-code implies an $[n-1, k, d-1]$ code by puncturing and an $[n-1, k-1, d]$-code by shortening. Together with the knowledge of the optimal parameters of linear quaternary codes of short length (see for example [10]), this determines most of the entries. As an example consider the case of dimension $k=4$. It is known that $d=8$ is the optimal distance in case $n=13, k=4$. This shows that $\left[14,4, d_{1}\right]$ and $\left[15,4, d_{2}\right.$ ] have $d_{1} \leq 9$ and $d_{2} \leq 10$. As a linear quaternary [ $15,4,10$ ]code exists (it is derivable from the [17, 4,12$]$-code determined by the elliptic quadric) it follows that $d_{1}=9, d_{2}=10$ are the optimal minimum distances in those cases.

Corollary 1.3. The maximum dimension of an additive quaternary $[15, k, 9]$ code is $k=4.5$.

This follows from Theorem 1.2 and the existence of a [15, 4.5, 9]-code. This cyclic code is a direct sum $C=C_{0} \oplus C_{1} \oplus C_{2}$ of codes of binary dimensions $1,4,4$. Here, $C_{0}$ is generated by $(11)^{15}$. Let $\mathbb{F}_{16}=\mathbb{F}_{2}(\epsilon)$ where $\epsilon^{4}=\epsilon+1$, let $T: \mathbb{F}_{16} \longrightarrow \mathbb{F}_{2}$ be the trace and index the coordinates of $C$ by $\epsilon^{i}, i=0, \ldots, 14$. The 16 codewords of $C_{1}$ are indexed by $u \in \mathbb{F}_{16}$, with entry $\left(T\left(u \epsilon^{i+1}\right), T\left(u \epsilon^{i}\right)\right)$ in coordinate $\epsilon^{i}$. Likewise, the codewords of $C_{2}$ are indexed by $u \in \mathbb{F}_{16}$, with entry $\left(T\left(u \epsilon^{3 i}\right), T\left(u \epsilon^{3 i+2}\right)\right)$ in coordinate $\epsilon^{i}$. The automorphism group of this code is the cyclic group of order 15 .

In the proof of Theorem 1.2 we are going to use the information contained in the table with the exception of the non-existence of $[15,5,9]$ of course. Let a $[15,5,9]$ be described by a generator matrix $G$. The geometric description of the code is based on the multiset of 15 lines in $P G(9,2)$ which are defined by the pairs of columns of $G$ corresponding to the quaternary coordinates. In geometric language Theorem 1.2 is equivalent to the following:

Theorem 1.4. There is no multiset of 15 lines in $\mathrm{PG}(9,2)$ such that no more than 6 of those lines are in a hyperplane.

We are going to prove Theorem 1.4 in the following sections, using geometric arguments and an exhaustive computer search. Most of the geometric work is done in $\operatorname{PG}(9,2)$, a suitably chosen subspace $\operatorname{PG}(5,2)$ and its factor space $\mathrm{PG}(3,2)$. Total computing time was 250 days using a 3.2 Ghz Intel Exacore.

## 2 Some basic facts

Let $\mathcal{M}$ be a multiset of 15 lines in $\operatorname{PG}(9,2)$ with the property that no hyperplane of $\operatorname{PG}(9,2)$ contains more than 6 of those lines, in the multiset sense.

Definition 2.1. A set of $t$ lines is in general position if the space they generate has projective dimension $2 t-1$. A set of lines has strength $t$ if $t$ is maximal such that any $t$ of the lines are in general position.

The coding-theoretic meaning of the strength is the following.
Proposition 2.2. If $C$ is an $[n, k]_{4}$ additive quaternary code geometrically described by a set $L$ of $n$ lines in $\mathrm{PG}(2 k-1,2)$, then the minimum distance of the dual code $C^{\perp}$ equals $t+1$, where $t$ is the strength of $L$.

Lemma 2.3. The multiset $\mathcal{M}$ is a set and has strength 3 .
Proof. Suppose $\mathcal{M}$ is not a set. This means that some two of its members are identical; say $L_{1}=L_{2}$. Consider the subcode consisting of the codewords with entries 0 in the positions of $L_{1}$ and $L_{2}$. This yields a $[13,4,9]_{4}$-code which is known not to exist. In fact, concatenation of a $[13,4,9]_{4}$-code with the $[3,2,2]_{2}$-code would produce a binary linear $[39,8,18]_{2}$-code, which contradicts the Griesmer bound.

Now, suppose that some three lines $L_{1}, L_{2}, L_{3}$ of $\mathcal{M}$ are not in general position. The subcode with entries 0 in those three coordinate positions has length 12 , dimension $\geq 5-2.5=2.5$ and minimum distance $\geq 9$. Such a code does not exist. In fact, concatenation yields a binary linear $[36,5,18]_{2}$-code, which again contradicts the Griesmer bound. It follows that the strength is $\geq 3$. Assume it is $\geq 4$. Then the dual of the code $C$ described by $\mathcal{M}$ is a $[15,10,5]_{4}$-code by Proposition 2.2. Such a code does not exist. In fact, shortening results in a $[12,7,5]_{4}$-code whose nonexistence has been shown in [4]. It follows that the strength of $\mathcal{M}$ is precisely 3 .

Definition 2.4. (1) The lines of $\mathcal{M}$ are called codelines.
(2) A point in $\operatorname{PG}(9,2)$ is a codepoint if it is on some codeline.
(3) If $M$ is the set of 45 codepoints and $U$ is a subspace of $\operatorname{PG}(9,2)$, then the weight of $U$ is $w(U)=|U \cap M|$.
(4) We will write $V_{i}$ for the subspaces $\operatorname{PG}(i-1,2)$ in our $\operatorname{PG}(9,2)$.
(5) A subspace $V_{i}$ of weight $m$ is also called an $m-V_{i}$.

Lemma 2.5. (1) There are at most 27 codepoints on a hyperplane, 18 points on a $V_{8}, 13$ points on a $V_{7}, 10$ points on a $V_{6}$ and 8 points on a $V_{5}$.
(2) All these upper bounds can be attained.
(3) Each $18-V_{8}$ is contained in three 27-hyperplanes.
(4) There is a $10-V_{6}$ containing three codelines and an isolated point.

Proof. Let $H$ be a hyperplane. Each codeline meets $H$ either in 3 points or in one point. Because of the defining condition, $w(H) \leq 6 \times 3+9 \times 1=27$. This bound is reached with equality, as otherwise $\mathcal{M}$ would describe a $[15,5,10]_{4^{-}}$ code, which does not exist.

Let $S$ be a $V_{8}$ and $w(S)=i$. The distribution of the codepoints on the hyperplanes containing $S$ shows that $i+3(27-i) \geq 45$, which implies $i \leq 18$. The argument also implies that in case $i=18$ all hyperplanes containing $S$ have weight 27.

The remaining bounds follow from similar elementary counting arguments. The $V_{7}$-bound follows from $14+7 \times 4=42<45$, the $V_{6}$-bound from $11+15 \times 2<45$ and the $V_{5}$-bound from $7+31<45$.

Let us show that all those bounds are reached. As the strength is 3 we find some four lines in a $V_{7}$. We know that they are not contained in a $V_{6}$; so they generate the $V_{7}$. Without loss of generality, these lines can be chosen as

$$
\left\langle v_{1}, v_{2}\right\rangle,\left\langle v_{3}, v_{4}\right\rangle,\left\langle v_{5}, v_{6}\right\rangle,\left\langle v_{7}, v_{1}+v_{3}+v_{5}\right\rangle .
$$

This shows that $\left\langle v_{1}, \ldots, v_{5}\right\rangle$ is an $8-V_{5}$ and $\left\langle v_{1}, \ldots, v_{6}\right\rangle$ is a $10-V_{6}$ containing three codelines and an isolated point. This in turn is contained in a $13-V_{7}$ and so on.

A more accurate count can be made using geometric arguments in factor spaces.

Definition 2.6. Let $U$ be a subspace and $\Pi_{U}$ the factor space. If $U$ is a $V_{i}$, then $\Pi_{U}$ is a $P G(9-i, 2)$. Let $P \in \Pi_{U}$ be a point and $\langle U, P\rangle$ the $V_{i+1}$ whose factor space $\bmod U$ is the point $P$. Define $w(P)=w(\langle U, P\rangle)-w(U)$. For each subspace $X \subseteq \Pi_{U}$ define $w(X)=\sum_{P \in X} w(P)$.

Observe that the weight $w(P)$ equals the number of codepoints which are in the preimage of $P \bmod U$ but not in $U$. In particular,

$$
\sum_{P \in \Pi_{U}} w(P)=45-w(U) .
$$

Lemma 2.7. Let $U_{7}$ be a $13-V_{7}$ and $\Pi_{7}$ the factor space (a Fano plane). Then $\Pi_{7}$ has three collinear points of weight 4, the remaining four points having weight 5 .

Proof. By Lemma 2.5, each point of $\Pi_{7}$ has weight $\leq 5$, each line has weight $\leq 14$ and the sum of all weights is $45-13=32$. Let $B$ be the set of points of weight $<5$ in $\Pi_{7}$. Then $B$ is a blocking set for the lines and $|B| \leq 3$. It follows that $|B|=3$ and $B$ is a line. This gives the result.

## 3 The geometric setting

In Lemma 2.5 we saw that there is a $10-V_{6}$ containing 3 codelines and an isolated point. We concentrate on such a subspace and its factor space $P G(3,2)$.

Definition 3.1. Let $U$ be a $10-V_{6}$ containing three codelines $L_{1}, L_{2}, L_{3}$ and an isolated codepoint. Let $P_{0} \in \Pi_{U}$ be the unique point whose preimage contains another codeline $L_{4}$.

For each line $l$ of $\Pi_{U}$ define the $h$-weight $h(l)$ as the number of codelines different from $L_{1}, \ldots, L_{4}$ which are in the preimage of $l \bmod U$. For each plane $E$, let $h(E)$ be the number of codelines different from $L_{1}, \ldots, L_{4}$ which are contained in the preimage of $E$.

Lemma 3.2. (1) The points of $\Pi_{U}$ have weights $\leq 3$, lines have weights $\leq 8$, planes have odd weights $\leq 17$.
(2) Each 8-line of $\Pi_{U}$ is contained in three 17-planes.
(3) Each 17-plane of $\Pi_{U}$ contains either precisely 3 or precisely 4 points of weight 3 . There is no point of weight 0 on a 17-plane.

Proof. (1) This follows from Lemma 2.5.
(2) Let $E$ be a plane of $\Pi_{U}$ and $H$ the preimage of $E$. Then $H$ is a hyperplane of $P G(9,2)$. If $H$ contains $i \leq 6$ codelines, then $w(H)=15+2 i$
and $w(E)=w(H)-10=5+2 i$, an odd number $\leq 17$. Lemma 2.5 shows that each 8-line of $\Pi_{U}$ is contained in three 17-planes.
(3) Let $E$ be a 17 -plane. As $7 \times 2=14$, the plane $E$ contains at least three points of weight 3 . Assume it has five such points. Then there is a line $g$ all of whose points have weight 3 . This is a line of weight 9 , contradiction.

Lemma 3.3. All points of $\Pi_{U}$ have weights 1,2 or 3 .
Proof. Assume that there is a point $P$ of weight 0 . Let $P \in l$. As every plane containing $l$ has weight at most 15 we must have $w(l) \leq 5$. The pencil of lines through $P$ shows that all lines $l$ through $P$ must have weight 5 . It follows that there are 7 points of weight 3 and 7 of weight 2 . Consider the line $h$ through two of the points of weight 3 . The third point on that line must have weight 2 or 3 . The latter is excluded as $w(h) \leq 8$. It follows that $w(h)=8$. The plane generated by $P$ and the line $h$ of weight 8 has weight 15 , which contradicts a statement from Lemma 2.5.

Definition 3.4. Let $m_{i}$ be the number of points of $\Pi_{U}$ of weight $i$.
In particular, $m_{1}+m_{2}+m_{3}=15$.
Lemma 3.5. A point $P$ of weight 3 is contained in one plane of weight 15 and in six planes of weight 17. It is contained in three lines of weight 7 and in four lines of weight 8 . The plane of weight 15 is the union of $P$ and the lines of weight 7 through $P$.

Proof. Consider the pencil of 7 lines through $P$. If $P$ is on a plane of weight 13 , then the sum of the weights of the points on some three of the lines is 13. As the total weight is 35 , one of the remaining four lines of the pencil must have weight $>8$, contradiction. It follows that the planes containing $P$ have weights 15 or 17 . Assume they all have weight 17 . Then all lines through $P$ must have weight 8 , which leads to the contradiction $35=3+7 \times 5$. It follows that the weights of the points on some three lines of the pencil add to 15 . The remaining four lines of the pencil have weight 8 each. It follows that all remaining planes containing $P$ have weight 17 . Also, any two lines of weight 8 through $P$ determine a line of weight 7 as the third line through $P$ on the same plane (of weight 17).

Lemma 3.6. (1) The sum of all $h$-weights of lines is 11.
(2) The sum of all $h$-weights of lines through $P_{0}$ is $w\left(P_{0}\right)-2$.
(3) For $P \neq P_{0}$ the sum of the $h$-weights of lines through $P$ is $w(P)$.
(4) Let $E$ be a plane not containing $P_{0}$. Then the sum of the $h$-weights of lines in $E$ is $(w(E)-11) / 2$.
(5) Let $E$ be a plane containing $P_{0}$. Then the sum of the $h$-weights of lines in $E$ is $(w(E)-13) / 2$.

Proof. (1) This is immediate.
(3) This follows from the fact that each of the remaining 11 lines different from $L_{1}, \ldots, L_{4}$ whose image mod $U$ passes through $P \neq P_{0}$ contributes exactly one codepoint to the weight of $P$. When $P=P_{0}$ two of the codepoints contributing to $w\left(P_{0}\right)$ come from line $L_{4}$. This shows (2).
(4) The number of codelines contained in the preimage $H$ of $E$ is then $3+h(E)$. It follows

$$
|H \cap M|=3(3+h(E))+(15-3-h(E))=2 h(E)+21=10+w(E),
$$

which implies $h(E)=(w(E)-11) / 2$, as claimed.
(5) In the case when $P_{0} \in E$ the argument is analogous.

In particular each plane has odd weight between 11 and 17. In what follows, three cases are distinguished.

Case 1: There is a plane of weight 11.
Case 2: There is a plane of weight 13.
Case 3: All planes have weight 15 or 17.
In each case we attempt to construct a generator matrix $G$ of the code $C$. Here $G$ has 10 rows and 15 pairs of columns. The column pairs correspond to codelines, where each codeline is represented by two of its three points. The top 6 rows correspond to the parameters $x_{1}, \ldots, x_{6}$ of $U$, the remaining four rows to the parameters $y_{1}, y_{2}, y_{3}, y_{4}$ of $\Pi_{U}$, where all those parameters are read from top to bottom. In each case we start from our lines $L_{1}, L_{2}, L_{3}, L_{4}$ and a system of 11 lines in $\Pi_{U}=P G(3,2)$ which determine the last 4 rows of $G$. A computer program then decides that this cannot be completed to a generator matrix of a code with minimum distance 9 .

### 3.1 Case 1

Assume there is a plane $E_{0}$ of weight 11 . Then all points in $\Pi_{U} \backslash E_{0}$ have weight 3 . All points of $E_{0}$ have weight 1 or 2 as otherwise a line of weight 9 would exist. It follows that $E_{0}$ has 3 points of weight 1 and 4 points of weight 2 , consequently $m_{1}=3, m_{2}=4, m_{3}=8$. No three points of weight 2 can be collinear as otherwise some plane would have even weight. It follows that the four points of weight 2 form a quadrangle in $E_{0}$, and those of weight 1 are collinear on a line $l_{0}$. The planes $E \neq E_{0}$ intersecting $E_{0}$ in $l_{0}$ have weight 15 , the others have weight 17 . Write $E_{0}$ as $y_{1}=0$. The points of weight 2 are $0100,0010,0001,0111$, the remaining points of $E_{0}$ having weight 1 form the line $l_{0}=\{0110,0101,0011\}$ and all affine points have weight 3 . As $w\left(E_{0}\right)=11$, the preimage of $E_{0}$ contains 3 codelines. It follows that all lines of $E_{0}$ have $h$-weight 0 and the point $P_{0}$ which defines a $V_{7}$ with four lines is not on $E_{0}$. We choose $P_{0}=1000$ as the corresponding point, and $L_{4}=\left\langle v_{1}+v_{3}+v_{5}, 0^{6} 1000\right\rangle$. As $w\left(P_{0}\right)=3$ there is a unique line $g_{0}$ of $h$-weight 1 through $P_{0}$. The line $g_{0}$ through $P_{0}$ of $h$-weight 1 can be chosen as either $P_{0} \cdot 0100$ or $P_{0} \cdot 0110$.

A computer calculation shows that there are 12 solutions for the resulting system of 11 lines in $\Pi_{U}$. In all cases $g_{0}=P_{0} \cdot 0110$. In Figures 1,2 we list the lines $L_{5}, L_{6}, R_{1}, \ldots, R_{9}$ for those 12 solutions.

Here is what we know about the generator matrix $G$ in the first of these 12 cases:
$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}L_{1} & L_{2} & L_{3} & L_{4} & L_{5} & L_{6} & R_{1} & R_{2} & R_{3} & R_{4} & R_{5} & R_{6} & R_{7} & R_{8} & R_{9} \\ 10 & 00 & 00 & 10 & & & & & & & & & & & \\ 01 & 00 & 00 & 00 & & & & & & & & & & & \\ 00 & 10 & 00 & 10 & & & & & & & & & & & \\ 00 & 01 & 00 & 00 & & & & & & & & & & & \\ 00 & 00 & 10 & 10 & & & & & & & & & & & \\ 00 & 00 & 01 & 00 & & & & & & & & & & & \\ \hline 00 & 00 & 00 & 01 & 10 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 \\ 00 & 00 & 00 & 00 & 01 & 01 & 01 & 11 & 01 & 00 & 11 & 11 & 11 & 11 & 00 \\ 00 & 00 & 00 & 00 & 01 & 01 & 10 & 10 & 10 & 01 & 10 & 00 & 01 & 00 & 11 \\ 00 & 00 & 00 & 00 & 00 & 10 & 01 & 10 & 00 & 10 & 11 & 10 & 01 & 01 & 10\end{array}\right]$

Use Gaussian elimination: as the bottom part of lines $L_{5}, R_{1}$ has full rank 4 we can manage to have the upper part of those two lines identically zero.

Figure 1: Case 1, the first 6 solutions

| $L_{5}$ | $L_{6}$ | $R_{1}$ | $R_{2}$ | $R_{3}$ | $R_{4}$ | $R_{5}$ | $R_{6}$ | $R_{7}$ | $R_{8}$ | $R_{9}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 | 01 | 01 | 01 | 01 | 01 | 01 | 01 | 01 | 01 | 01 |
| 01 | 01 | 01 | 11 | 01 | 00 | 11 | 11 | 11 | 11 | 00 |
| 01 | 01 | 10 | 10 | 10 | 01 | 10 | 00 | 01 | 00 | 11 |
| 00 | 10 | 01 | 10 | 00 | 10 | 11 | 10 | 01 | 01 | 10 |
| 10 | 01 | 01 | 01 | 01 | 10 | 01 | 11 | 11 | 01 | 01 |
| 01 | 01 | 01 | 01 | 00 | 11 | 00 | 10 | 10 | 11 | 11 |
| 01 | 00 | 01 | 10 | 11 | 00 | 11 | 01 | 01 | 01 | 01 |
| 00 | 10 | 10 | 00 | 01 | 01 | 10 | 10 | 10 | 01 | 01 |
| 10 | 01 | 01 | 01 | 01 | 11 | 10 | 01 | 10 | 01 | 01 |
| 01 | 01 | 01 | 01 | 00 | 10 | 11 | 00 | 11 | 11 | 11 |
| 01 | 00 | 10 | 10 | 01 | 01 | 00 | 11 | 11 | 01 | 01 |
| 00 | 10 | 01 | 00 | 10 | 10 | 01 | 10 | 01 | 01 | 01 |
| 10 | 01 | 01 | 01 | 10 | 01 | 01 | 01 | 01 | 10 | 10 |
| 01 | 01 | 10 | 00 | 11 | 11 | 00 | 01 | 01 | 11 | 11 |
| 01 | 01 | 01 | 01 | 00 | 00 | 11 | 10 | 10 | 01 | 01 |
| 00 | 10 | 00 | 10 | 01 | 01 | 10 | 01 | 01 | 01 | 01 |
| 10 | 01 | 01 | 01 | 01 | 10 | 01 | 11 | 10 | 01 | 01 |
| 01 | 01 | 01 | 01 | 10 | 11 | 00 | 10 | 11 | 11 | 00 |
| 01 | 00 | 01 | 10 | 01 | 01 | 11 | 01 | 00 | 01 | 11 |
| 00 | 10 | 10 | 01 | 00 | 01 | 01 | 10 | 01 | 01 | 10 |
| 10 | 01 | 01 | 10 | 01 | 10 | 01 | 01 | 10 | 01 | 01 |
| 01 | 01 | 10 | 11 | 00 | 11 | 11 | 00 | 11 | 01 | 01 |
| 01 | 00 | 01 | 01 | 01 | 00 | 01 | 11 | 11 | 10 | 10 |
| 00 | 10 | 00 | 01 | 10 | 01 | 01 | 10 | 01 | 01 | 01 |

Figure 2: Case 1, the remaining 6 solutions

| $L_{5}$ | $L_{6}$ | $R_{1}$ | $R_{2}$ | $R_{3}$ | $R_{4}$ | $R_{5}$ | $R_{6}$ | $R_{7}$ | $R_{8}$ | $R_{9}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 10 | 01 | 01 | 01 | 01 | 10 | 01 | 10 | 01 | 11 | 11 |
| 01 | 01 | 01 | 00 | 00 | 11 | 11 | 10 | 11 | 10 | 10 |
| 01 | 01 | 10 | 01 | 11 | 00 | 01 | 01 | 00 | 01 | 01 |
| 00 | 10 | 00 | 10 | 01 | 01 | 01 | 01 | 01 | 10 | 10 |
| 10 | 01 | 01 | 11 | 10 | 01 | 10 | 01 | 10 | 01 | 01 |
| 01 | 01 | 01 | 10 | 11 | 11 | 10 | 11 | 11 | 00 | 00 |
| 01 | 10 | 10 | 01 | 00 | 01 | 01 | 00 | 11 | 01 | 01 |
| 00 | 01 | 00 | 10 | 01 | 01 | 01 | 01 | 01 | 10 | 10 |
| 10 | 01 | 01 | 01 | 10 | 01 | 01 | 11 | 10 | 10 | 01 |
| 01 | 01 | 01 | 10 | 11 | 00 | 00 | 10 | 11 | 10 | 11 |
| 01 | 01 | 10 | 01 | 01 | 01 | 11 | 01 | 00 | 01 | 00 |
| 00 | 10 | 01 | 00 | 01 | 10 | 01 | 10 | 01 | 01 | 01 |
| 10 | 01 | 10 | 10 | 10 | 01 | 10 | 01 | 01 | 01 | 01 |
| 01 | 10 | 11 | 11 | 10 | 11 | 11 | 01 | 01 | 00 | 00 |
| 01 | 01 | 01 | 00 | 01 | 00 | 11 | 10 | 10 | 01 | 01 |
| 00 | 00 | 01 | 01 | 01 | 01 | 01 | 01 | 01 | 10 | 10 |
| 10 | 01 | 01 | 01 | 10 | 01 | 10 | 01 | 01 | 11 | 11 |
| 01 | 01 | 01 | 10 | 11 | 11 | 10 | 00 | 00 | 10 | 10 |
| 01 | 00 | 01 | 01 | 00 | 01 | 01 | 11 | 11 | 01 | 01 |
| 00 | 10 | 10 | 00 | 01 | 01 | 01 | 01 | 01 | 10 | 10 |
| 10 | 01 | 01 | 01 | 01 | 01 | 11 | 10 | 01 | 10 | 10 |
| 01 | 01 | 01 | 10 | 00 | 00 | 10 | 11 | 11 | 10 | 11 |
| 01 | 00 | 10 | 01 | 01 | 11 | 01 | 00 | 01 | 01 | 11 |
| 00 | 10 | 01 | 00 | 10 | 01 | 10 | 01 | 01 | 01 | 01 |

This will not change the left column of $L_{4}$, but the right column is destroyed. The following form is obtained. Here we can choose the top entry of the right column of $L_{4}$ to be zero as we could replace it by the sum of the columns.
$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}L_{1} & L_{2} & L_{3} & L_{4} & L_{5} & L_{6} & R_{1} & R_{2} & R_{3} & R_{4} & R_{5} & R_{6} & R_{7} & R_{8} & R_{9} \\ 10 & 00 & 00 & 10 & 00 & & 00 & & & & & & & & \\ 01 & 00 & 00 & 0 & 00 & & 00 & & & & & & & & \\ 00 & 10 & 00 & 1 & 00 & & 00 & & & & & & & & \\ 00 & 01 & 00 & 0 & 00 & & 00 & & & & & & & & \\ 00 & 00 & 10 & 1 & 00 & & 00 & & & & & & & & \\ 00 & 00 & 01 & 0 & 00 & & 00 & & & & & & & & \\ \hline 00 & 00 & 00 & 01 & 10 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 \\ 00 & 00 & 00 & 00 & 01 & 01 & 01 & 11 & 01 & 00 & 11 & 11 & 11 & 11 & 00 \\ 00 & 00 & 00 & 00 & 01 & 01 & 10 & 10 & 10 & 01 & 10 & 00 & 01 & 00 & 11 \\ 00 & 00 & 00 & 00 & 00 & 10 & 01 & 10 & 00 & 10 & 11 & 10 & 01 & 01 & 10\end{array}\right]$

A computer search showed that this cannot be completed to a generator matrix of the putative [15, 5, 9]-code. The same procedure excludes the remaining 11 solutions.

### 3.2 Case 2

Assume there is a plane $E_{0}$ of weight 13 . By Lemma 3.5, $E_{0}$ does not contain points of weight 3 . It follows that $E_{0}$ contains a unique point of weight 1 , all remaining points having weight 2 . Let $l \subset E$ be a line of type $(2,2,2)$. Then $l$ is contained in two 17-planes, each containing one further point of weight 2 . It follows $m_{1}=1, m_{2}=8, m_{3}=6$. Let $l^{\prime} \subset E$ be a line of type $(1,2,2)$. Then $l^{\prime}$ is on exactly one 15 -plane, and this plane contains the two affine points (off $E$ ) of weight 2 . They are collinear with the weight 1 point. This determines the distribution of weights uniquely. There are three planes of weight 15 and eleven of weight 17 . The latter come in three containing a weight 1 point and eight of the other type ( 3 points of weight 3 and four of weight 2). Here is a concrete description: the weight 1 point is 0100 , the remaining points on the weight 13 plane $E_{0}=\left(y_{1}=0\right)$ have weight 2 . The two remaining points of weight 2 are 1000,1100 and the remaining six points are of weight 3 . The point $P_{0}$ which defines a $V_{7}$ with four lines must have weight $\geq 2$. It follows that we have without loss of generality three subcases:

Subcase (2,1): $P_{0}=1000$ (of weight $2, \notin E_{0}$ )
Subcase (2,2): $P_{0}=0010$ (of weight $2, \in E_{0}$ )
Subcase (2,3): $P_{0}=1010$ (of weight 3)
Subcase $(2,1)$ has 12 solutions, subcase $(2,2)$ has 40 solutions and subcase $(2,3)$ has 101 solutions. In each of those 153 cases a computer program checked that there is no completion to a generator matrix of a $[15,5,9]_{4^{-}}$ code.

### 3.3 Case 3

We have $m_{1}+m_{2}+m_{3}=15, m_{1}+2 m_{2}+3 m_{3}=35$ which implies $m_{3}=5+m_{1}$. As lines have weight $<9$, the points of weight 3 form a cap in $\operatorname{PG}(3,2)$. Observe also that each cap of size 6 or more is contained in an affine space (avoids some plane). This shows that in case $m_{1}>0$ there is a subplane without points of weight 3 . Such a plane has weight $\leq 13$, contradiction. We have $m_{1}=0$, hence $m_{2}=10, m_{3}=5$. The only 5 -cap in $\operatorname{PG}(3,2)$, which is not affine, consists of 5 points any four of which are in general position, a coordinate frame which we choose as $1000,0100,0010,0001,1111$. Those are the points of weight 3 , all others have weight 2 . There are 10 planes of weight 17 (containing three points of the frame) and 5 planes of weight 15 (containing one point of the frame). The point $P_{0}$ which describes a $V_{7}$ with four lines may have weight 2 or 3 . Observe that the stabilizer of the frame in $G L(4,2)$ is the full $S_{5}$. This leads to two subcases as follows:

Subcase (3,1): $P_{0}=1100$ (of weight 2)
Subcase (3,2): $P_{0}=1000$ (of weight 3)
There are 43 solutions in Subcase $(3,1)$ and 70 solutions in Subcase $(3,2)$. In each of those cases a computer program checked that there is no completion to a generator matrix of a $[15,5,9]_{4}$-code.

### 3.4 The search algorithm

To construct a generator matrix $G$ of the code $C$, we started from partial matrices obtained as described in the previous Subsections. Then we tried
to complete the matrices in all possible ways. We proceeded adding one row at time using a backtracking algorithm. At each step we tested if the obtained submatrix generates a subcode with minimum distance not less than 9 , otherwise the last added row is discarded.

We observe that in all three cases, consideration of the tenth line of the generator matrix has never been necessary. In other words, it never happened that the nine rows that have been chosen formed the generator matrix of a [15, 4.5, 9]-code.

This means that the geometrical approach we adopted gives constraints about the intersections with subspaces of the putative [ $15,5,9]$-code that allowed us to restrict the computer search. In this way the algorithm is more effective with respect to classifying all [15, 4.5, 9]-codes and extending them trying to obtain a $[15,5,9]$-code.

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