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SMART-or and *SMART-and* Fuzzy Average Operators: a generalized proposal

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Abstract

In this paper we introduce new average operators for merging any number of fuzzy numbers, without any exogenous components. The proposed n-ary operators are based on a specific adaptation of Marzullo's algorithm, and depart from the usual fuzzy arithmetic mean according to the degree of agreement or disagreement among the memberships of input fuzzy numbers. Such merging operators are suitable to be applied in any model where the same quantity (usually a parameter) can be measured (estimated) through different fuzzy memberships stemming by different sources of information. The special case of two fuzzy memberships was the focus of our previous contributions that were elicited in order to estimate the fuzzy volatility parameter in an hybrid fuzzy-stochastic model for option pricing. In this paper we generalize the setting to the case of n fuzzy inputs to be merged and also remove exogenous factors from the definition of the operators. In order to have an application at hand we consider the same example treated in the quoted paper and we compare the outcomes obtained via the new operators, named $SMART$, with the fuzzy arithmetic mean as a canonical benchmark.

Keywords Smart Average Operators, Fuzzy Mean, Merging, Fuzzy Option Pricing

1 Introduction and motivation

The problem of aggregating evidences from several sources of information is of concern in many fields, from engineering to decision theory. As pointed out in [19] some examples are

- to aggregate pieces of information coming from different sensors i.e. in engineering,

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- to aggregate multiple source interrogation systems where several databases can provide precise, imprecise or uncertain information about values of interest,
- to take into account expert opinions, when different individual statements have to be aggregated to a synthetic single value.

In the last decade, due to the increase of availability and variety of data, the need for merging information becomes very strong. The choice of a fusion operator is not unique and, above all, is heavily context-dependent. Authors in [33] affirm that there are more than 90 different fuzzy operators proposed in the literature for fuzzy set operations and there is a wide family of aggregation functions with predictable and tailored properties (e.g. those analyzed in [27] [28]) related to different areas and disciplines and different basic properties they could satisfy (see e.g. [16]).

Classes of aggregation functions include triangular norms and conorms, copulas, means and averages, and those based on nonadditive integrals [26]. To distinguish properly among all the suggestions in the literature, it is worth to remark that the problem of merging information can be classified in two broad classes: either as a way for extending filtering and estimation techniques to fuzzy data or as a tool to extract the most reliable information out of imprecise data, [19].

In the former case, the extension principle is usually applied to classical estimators as well as methods of fuzzy arithmetic (see [15]); in the latter, the purpose of aggregation is pursued by applying fuzzy set-theoretic operations and the choice of conjunctive versus disjunctive connectives depends upon assumptions on whether all sources are reliable or not (see also [23] for a survey). An interesting interpretation of the difference among the two broad classes of aggregation is given in [19] " *(Estimation) differs from fusion in the sense that the aggregation is performed on the measurement scale (horizontal view) while in the fusion process the aggregation is performed on the scale of membership values, in practice degrees of plausibility (vertical view). A bridge between these two views of merging information is ensured using the concept of constrained fusion.*" A more detailed illustration of these two different merging approaches is reported in the Appendix of this contribution.

In this paper we focus on the estimation problem, which is of substantial interest in many domains. Indeed, in our contribution [5] we introduced a methodology for the elicitation of the fuzzy membership of the volatility parameter of a risky asset; by applying the method on either the historical volatility estimator $\widehat{\sigma}$ or to the estimator $\nu = \mathrm{VIX}/100$, based on the volatility index VIX of the $S\&P500$ market index, see [40], we obtained two different memberships for the fuzzy volatility. The peculiarity of our procedure was to deal with alternative sources of information and raised the need to define suitable *operators* to merge the memberships stemmed by the different sources, which was left as an open problem. Indeed, the membership elicitation is an extension to fuzzy numbers of the estimation of the crisp volatility parameter, hence we look for proper aggregations within the class of *averaging operators*. In this case, the problem is to combine outcomes coming from different sources of information to obtain one representative value which is consistent with the observations. Many aggregation operations are introduced within Fuzzy set theory to perform estimation, especially within the class of generalized means, which can be directly applied to fuzzy numbers.

As claimed in [19] a bridge between the "estimation" and the "fusion" is represented by the fuzzy arithmetic mean (named simply "fuzzy mean" in the sequel) which is, indeed, a basic operation for estimation and also a fuzzy set-theoretic connective. However, it is well known that the fuzzy mean, even more than the crisp one, is *insensitive with respect to the agreement or disagreement among original values* (we can have the same mean between two very close values ad well as between two very distant).

Agreement among the sources of information has been considered in fuzzy aggregation literature, like e.g. in [39] where averaging is based on a so called *coordination index*; this proposal operates, as a *fusion*, among membership values.

Attempts to address such issue for fuzzy *estimation* are the *relative agreement degree* proposed in [21] and the *constrained merging* developed in [19, §6.6.2] which allow to overcome the above insensitivity of the mean, while maintaining the aforementioned double role of estimator and set-theoretic connective.

Since we are interested in estimation and filtering, we look at proper averaging operators in order to merge fuzzy memberships which are derived according to different sources of information (i.e. historical volatility an VIX in the numerical example). In order o take into account the level of agreement or disagreement among the n fuzzy inputs to be merged, we adopt a similar view in this research and build on generalizations of the fuzzy mean such as the Fuzzy Operator Weighted Average (FOWA, see e.g. [35]) or the various combination techniques described in [1].

We define merging operators differently in case the estimators give alternative ("disjunctive/disagreeing") or concomitant ("conjunctive/agreeing") information, in accordance with one of the basic principle of information merging reported in [16]. In this view, though out operators are fuzzy weighted averages and operate "horizontally" on the α-cuts of the inputs, we refer them as disjunctive and conjunctive operators with a little abuse of langu age. In fact, from a technical perspective, our proposals are hybrid in that they are based on suitable deformations of the fuzzy mean towards canonical conjunctive (i.e. min) and disjunctive (i.e. max) connectives, by looking at their "vertical" values.

The aforementioned *constrained merging* is based on an original Yager's "intelligent" component [43]; we borrow from the above quoted paper the motivation of including a "smart" component in the averaging process to address conflicts in the data to be fused and, similarly, we name our operators as "SMART". SMART is in fact both a synonym of "intelligent" and a, commonly used, acronym for "**S**pecific, **M**easurable, **A**chievable, **R**ealistic and **T**ime-related", most of which are also goals of our approach. The main difference with respect Yager's proposal is that we do not make use of any exogenous "combinability function" that was instead used in [43].

It is worth to stress that we do not look for t-norms and t-conorms, as e.g. done in [10], or to any of their generalizations, like e.g. overlap and grouping functions (see e.g. [2]), which belong to the class of fusion connectives, since they are not suitable to merge same claim given by different subjects over an unknown parameter, which is the aim of our paper, but rather they are tailored to merge different claims given by the same subject.

As already pointed out, our first proposal of merging operators based on the level of agreement/disagreement has been given in [6]; however, it suffered from being restricted to merging only two fuzzy numbers, and ad-hoc binary deformations were adopted to define the weights. Here, we consistently overcome both these drawbacks: the new suggested operators, namely *SMART-or* and *SMART-and*, can fuse any number n of fuzzy numbers and we avoid any external component in the definition of these new merging operators, which would be arbitrary by construction.

For the sake of clarity, we set up our framework by giving in Subsection 2.1 the preliminary concepts underlying our proposal; then, in Subsections 2.2-2.4 we define the *SMART-or* and *SMART-and* n-ary operators, respectively. Section 3 is devoted to a numerical application gathering a detailed description on the entailment, on the pricing for options, of using the new operators to merge the elicited fuzzy volatility of the $S\&P500$ Market Index according to several sources of information. Finally, in Section 4 some concluding remarks are collected.

2 The general *SMART-or* and *SMART-and* averages

2.1 Preliminaries

We recall that membership functions $\mu : \mathbb{R} \to [0, 1]$ of the fuzzy set of possible values of a random variable X are usually viewed as imprecise values.

From a practical point of view, we profit from membership characterization through α-cuts, that for a generic j-th membership result as

$$\mu j^{\alpha} = \{x \in \mathbb{R} : \mu(x) \geq \alpha\}, \quad \alpha \in [0, 1]. \tag{1}$$

In particular, since we deal with fuzzy numbers, i.e. memberships with nested, compact and close α-cuts and with a unique core value with maximum membership 1, (1) reduces to closed intervals characterized by a left and a right extreme:

$$\mu j^{\alpha} = [\mu j_l^{\alpha}, \mu j_r^{\alpha}]. \tag{2}$$

The agreement/disagreement among α-cuts, for the same level α, for n different fuzzy numbers can be expressed by exploiting the q-relaxed intersection computation applied in [30] and based on the Marzullo's algorithm [34] which efficiently computes the shortest interval shared by the maximum number of intervals. This choice avoids any external, hence arbitrary, imposition.

The average operators we introduce in this paper are based on the above assessment of agreement/disagreement which is measured α-cut by

α-cut, hence"horizontally" with respect to the membership function definition, in line with [29].

Figure 1 shows such quantities for two memberships, together with other specific quantities that will be involved in our averaging operators as described below.

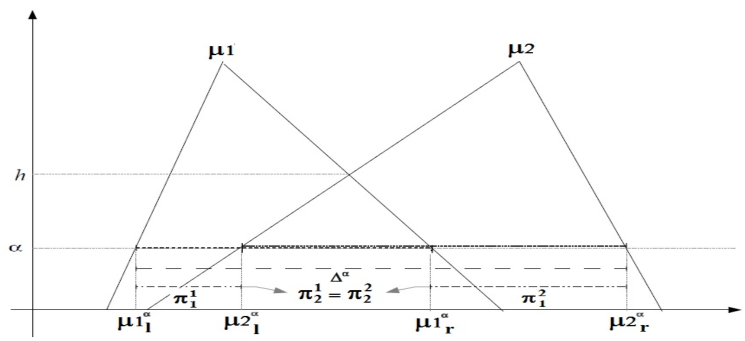

Figure 1: α-cuts of two memberships $\mu 1$ and $\mu 2$ and their characteristic values for their SMART averages.

More precisely, the original Marzullo's algorithm returns the optimal, i.e. shortest, interval which is consistent with the maximum number of inputs. For example, among $\mu 1^\alpha = [8, 9]$, $\mu 2^\alpha = [8, 12]$ and $\mu 3^\alpha = [10, 12]$ it produces $[8, 9]$ as the shortest intersection between two of the original intervals.

Actually, we want to control for intersections among all subsets of the n α cuts; to this aim we can modify the algorithm by taking trace of the different numbers of intersecting intervals, so that the results are now specific weights π_f^j representing the overlap lengths among f α-cuts, $f = 1, \ldots, n$, inside the j-th α-cut, $j = 1, \ldots, n$.

By applying the modification of the algorithm suggested below we obtain, for the the three α-cuts mentioned before:

$$\begin{array}{ccc} \pi_1^1 = 0 & \pi_1^2 = 10 - 9 = 1 & \pi_1^3 = 0 \\ \pi_2^1 = (9 - 8) = 1 & \pi_2^2 = (9 - 8) + (12 - 10) = 3 & \pi_2^3 = (12 - 10) = 2 \\ \pi_3^1 = 0 & \pi_3^2 = 0 & \pi_3^3 = 0. \end{array}$$

In few words, we can say that each π_f^j measures the part of the j-th α-cut involved into the intersection among f α-cuts. It comes straightforwardly that the sum of the various weights associated to a specific α-cut gives its length:

$$\sum_{f=1}^{n} \pi_f^j = \delta_j = \mu j_r^\alpha - \mu j_l^\alpha. \tag{3}$$

The detailed algorithm that permits to obtain weights π_f^j is reported in Tab. 1.

As already outlined, the SMART-or $\veebar$ and SMART-and $\barwedge$ operators, formally defined in what follows, are modifications of the fuzzy mean

Table 1: Pseudo R code of an adapted Marzullo's algorithm, with input ext = list of left and right extremes, i =list of memberships belongings, type = -1 if left ext; +1 otherwise.

```
relaxint = function(ext,i,type)
{
n = length(ext)/2
pi = matrix(0,n,n)
lambda = numeric(2*n)
lambda[1] = - type[1]
j=list()
j[[1]] = c(i[1])
   for (l in 1:(2*n-1))
    { for (k in j[[l]]){
    pi[lambda[l],k] = pi[lambda[l],k] + (ext[l+1] - ext[l])}
     lambda[l+1] = lambda[l] - type[l+1]
     if (type[l+1] == -1)  j[[l+1]]=c(j[[l]], i[l+1])
     else j[[l+1]] = setdiff(j[[l]], c(i[l+1]))
   }
return(pi)
}
```

based on full/partial overlap among the α-cuts $[\mu j_l^\alpha, \mu j_r^\alpha]$, $j = 1, 2, \ldots, n$, of the fuzzy memberships to be merged. The formal definition bases on the weights π_f^j obtained by the adapted Marzullo's algorithm in order to emphasize the partial agreement or disagreement among the different memberships. This is a peculiarity of our approach since usually in weighted averages, like FOWA or S-FOWA, the weights are n fixed parameters the value of which is left to the decision maker (for the sake of completeness we briefly recall definitions of FOWA and S-FOWA operators in the Appendix A); here weights are endogenously determined by the input fuzzy numbers. In [21] weights are also obtained according to the *relative agreement degree* that is an average of pairwise agreements among the n considered inputs. The main difference of our proposal with respect to [21] is that our weights vary across different α levels and is influenced by the different arities of agreeing subsets of the input.

It is well known that the arithmetic mean, no matter whether crisp or fuzzy, is characterized by a convex combination of the extremes of the α-cuts with uniform weights $\frac{1}{n}$.

The $SMART$ operators generalize the fuzzy mean by tuning such coefficients in order to obtain a specific aimed behavior of the merging: towards the more external values for the *SMART-or* average (in line with the canonical max t-conorm if applied "vertically") and towards the more inner ones for the *SMART-and* average (in line with the canonical min t-norm if applied "vertically").

2.2 SMART-or disagreement based average

As we have already mentioned, the *SMART-or* operator weights the outer extremes of any α-cut more than the inner ones. Since the same procedure is applied for any $\alpha \in [0,1]$, we omit the superscript α whenever not strictly necessary.

In Fig. 2, we visually anticipate the *SMART-or* operator; the aforementioned deformation, with respect to the fuzzy arithmetic mean, towards the max connective is evident from the picture. Different π_f^j, with $f, j = 1, \ldots, 3$, stemming from α-cuts of three membership functions are depicted in the zoomed part of the graph.

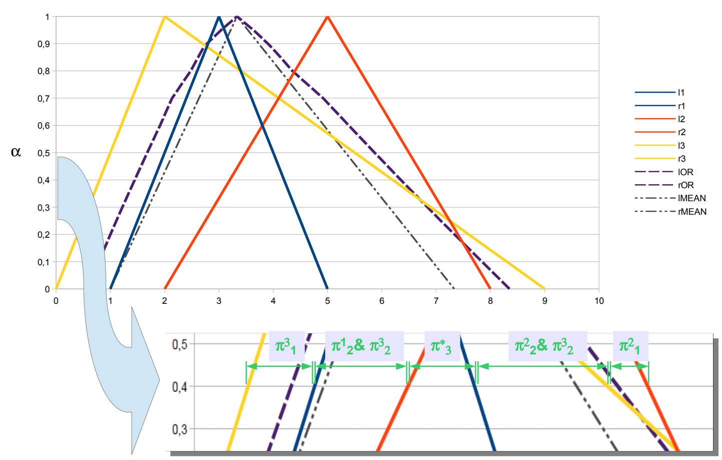

Figure 2: *SMART-or* (dashed line) among 3 fuzzy numbers (solid lines) compared to the fuzzy arithmetic mean (dashed-dotted line). The zoom shows the relaxed intersections computed through adapted Marzullo's algorithm.

In order define the operator formally we need to order the left extremes of the n α-cuts to be merged, for each α, as well as the rights extremes. For example, in Figure 2 the order of the left extremes according to their indices is $\{3, 1, 2\}$ while the order of the right extremes is $\{1, 2, 3\}$ with n=3 fuzzy numbers to be merged.

For the general case, let $O_l \subset \{1, \ldots, n\}$ be the set of indices of the first $n-1$ left extremes and $O_r \subset \{1, \ldots, n\}$ be the set of indices of the last $n-1$ right extremes of the n input α-cuts (from left to right in the horizontal axis)[1]. For the SMART-or operator, the extremes of the α-cuts

$$[(\mu 1 \veebar \ldots \veebar \mu n)_l^\alpha, (\mu 1 \veebar \ldots \veebar \mu n)_r^\alpha] \tag{4}$$

are computed as convex combinations of the original ones with weights

$$\frac{1}{n}(1 + \epsilon_j) \qquad j \in O_l \text{ or } j \in O_r\ . \tag{5}$$

[1]In the previous example we would have $O_l = \{3, 1\}$ and $O_r = \{2, 3\}$

with

$$\epsilon_j = \begin{cases} \frac{\sum_{f=1}^{n} \frac{1}{f} \pi_f^j}{\Delta} & \text{if } \Delta \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad , \quad j \in J_l \text{ or } j \in J_r, \tag{6}$$

and $\Delta = \max\{\mu i_r^\alpha\}_{i=1}^n - \min\{\mu i_l^\alpha\}_{i=1}^n$. Equations (5, 6) display the weighted contributions of the $n-1$ more relevant extremes, i.e. for the aggregation of the left extremes those with index in O_l, while those in O_r for the aggregation of the right ones. The n-th coefficients, associated to the inner extremes in $\{1,\ldots,n\} \setminus O_l$ and in $\{1,\ldots,n\} \setminus O_r$, are simply given by

$$\frac{1}{n}(1 - \sum_{j \in O_*} \epsilon_j) \qquad O_* = O_l,\, O_r \text{ respectively.} \tag{7}$$

Note that the division by Δ in (6) makes the contribution of each j-th α-cut relative with respect to the length of closed envelope of the α-cuts and it coincides with the α-cut of the usual fuzzy disjunction membership $\bigvee_{i=1}^n \mu i : \mathbb{R} \to [0,1]$ with $\bigvee_{i=1}^n \mu i(x) = \max\{\mu i(x) : i = 1,\ldots,n\}$, for all $x \in \mathbb{R}$. Moreover, whenever the original memberships have at least two distinct cores[2], the terms ϵ_j tend to zero whenever α tends to one so that the upper side of the merged membership converge to the core of the fuzzy mean.

Let us also stress that, since each weight π_f^j is shared exactly by f α-cuts, then we have $\sum_{j=1}^{n-1} \sum_{f=1}^{n} \frac{1}{f} \pi_f^j \leq \Delta$ so that $0 \leq \sum_{j=1}^{n-1} \epsilon_j \leq 1$; hence, the coefficients introduced in (5, 7) are suitable to define a convex combination.

From Fig.2 it is clear that the *SMART-or* α-cuts, except for the core associated to $\alpha = 1$, evidence wider intervals than the fuzzy mean, and extremes are stretched toward those of the the max connective, when computed vertically. Hence, taking into account the strength of the disagreements among the different α-cuts leads to a more vague estimation with respect to the fuzzy mean.

2.3 SMART-and average

Let $I_l \subset \{1,\ldots,n\}$ be the set of indices of the last $n-1$ left extremes and $I_r \subset \{1,\ldots,n\}$ be the set of indices of the first $n-1$ right extremes (thought in ascending ordering) of the n α-cuts in input[3]. The construction of the SMART-and operator $\bar{\wedge}$ follows the same basic rule of the SMART-or one, but needs a more articulated formulation. Indeed, the α-cuts of the merging operator

$$[(\mu 1 \,\bar{\wedge} \ldots \bar{\wedge}\, \mu n)_l^\alpha, (\mu 1 \,\bar{\wedge} \ldots \bar{\wedge}\, \mu n)_r^\alpha] \;, \tag{8}$$

are computed differently whether α is below or above a fixed value $h \in (0,1)$, defined as the higher level of not empty intersection among all the n α-cuts of the original fuzzy numbers (see Fig.1 for the case n=2).

[2]The cores of a fuzzy number are those corresponding to membership value 1 (the maximum).

[3]In the previous example we would have $I_l = \{1,2\}$ and $I_r = \{1,2\}$

If $\alpha \leq h$, the extremes of the α-cuts are obtained as convex combinations of the original ones with coefficients

$$\frac{1}{n}(1+\gamma_j) \qquad j \in I_l \text{ or } I_r \;, \tag{9}$$

where the quantities

$$\gamma_j = \frac{\sum_{f=1}^{n} \frac{1}{n+1-f} \pi_f^j}{\sum_{k=1}^{n} \sum_{f=1}^{n} \frac{1}{n+1-f} \pi_f^k} \tag{10}$$

reflect the weighted normalized contribution of the $n-1$ "more relevant" extremes, i.e. those in I_l for the left extremes and those in I_r for the the right ones.

Note that the factor $\frac{1}{n+1-f}$ for each addend in the numerator of γ_j is proportional to the number of overlaps (representing the agreement). Moreover, whenever all the n α-cuts coincide, the γ_j-s can be consistently set to zero.

Again, the n-th coefficients, i.e. those associated to the outer left extreme in $\{1,\ldots,n\} \setminus I_l$ and to the outer right extreme in $\{1,\ldots,n\} \setminus I_r$, are simply given by

$$\frac{1}{n}(1-\sum_{j=I_*}\gamma_j) \qquad I_* = I_l,\, I_r \text{ respectively.} \tag{11}$$

Since the denominator in (10) is simply a normalizing constant, it comes to the fore that $\sum_{j=1}^{n-1}\gamma_j \leq 1$ so that (9, 11) define proper convex combinations of the left and right extremes, respectively.

When $\alpha > h$, the formal definition is more subtle since many subgroups of intersections involving two or more memberships can be identified. Assume we have k subgroups of indexes $J_l \subset \{1,\ldots,n\}$, each with cardinality n_l, $l = 1,\ldots,k$; since subgroups may share some elements, we have $\sum_{l=1}^{k} n_l \geq n$. The logic underlying this approach is to compute the *SMART-and* operator $\bar{\wedge}$ first, within each subgroup J_l, $l = 1,\ldots,k$, obtaining k intermediate α-levels, and to merge them in a second step, by applying the *SMART-or* operator $\underline{\vee}$.

Formally, for each $j \in J_l$, we compute

$$\overline{\gamma}_j = \frac{\sum_{f=1}^{n_l} \frac{1}{n_l+1-f} \pi_f^j}{\sum_{k\in J_l} \sum_{f=1}^{n} \frac{1}{n+1-f} \pi_f^k} \tag{12}$$

and we define the convex combination of the extremes of the α-cuts in J_l with coefficients

$$\frac{1}{n_l}\left((1+\overline{\gamma}_1)\ldots,(1+\overline{\gamma}_{n_l-1}),(1-\sum_{j=1}^{n_l-1}\overline{\gamma}_j)\right) \tag{13}$$

where the order is from the inner to the outer. Once we obtain these k *SMART-and*s of subgroups, characterized by the extremes

$$\overline{\mu}i_l \quad \text{and} \quad \overline{\mu}i_r \quad , \quad i=1,\ldots,k, \tag{14}$$

we compute the new relaxed intersection coefficients $\overline{\pi}^l_f,\ l, f = 1, \ldots, k$, obtaining new weights

$$\overline{\epsilon}_i = \frac{\sum_{f=1}^k \frac{1}{f}\overline{\pi}^i_f}{\overline{\Delta}}, \qquad \text{with } \overline{\Delta} = \max\{\overline{\mu}i_r\}_{i=1}^k - \min\{\overline{\mu}i_l\}_{i=1}^k, \tag{15}$$

that can be plugged in the coefficients for the convex combination to take into account the cardinality of each contribution:

$$\frac{1}{\sum_{i=1}^k n_i}\left(n_1(1+\overline{\epsilon}_1), \ldots, n_{k-1}(1+\overline{\epsilon}_{k-1}), (n_k - \sum_{l=1}^{k-1} n_l\overline{\epsilon}_l)\right). \tag{16}$$

Note that in this case the order is from the outer to the inner, the latter being the "most relevant" in this case.

The above steps need to be iterated for increasing values of α since partial intersections change whenever one of indexes $\overline{\pi}^j_f$ vanish. In addition, to guarantee that the output of this merge is a fuzzy number, the α-cuts obtained through the described aggregation procedure must be "glued" to those of the levels below; this can be obtained by a proper translation and deformation of the extremes.

Due to the necessary discretization of the α levels to be considered n practical applications (see [5]), we can formulate the transformation by referring to two consecutive values α_m and α_{m+1} so that the transformed α_{m+1}-cut will have extremes computed recursively by:

$$\widehat{\mu}_l^{\alpha_{m+1}} = \widehat{\mu}_l^{\alpha_m} + \varrho^{\alpha_m}|\overline{\mu}_l^{\alpha_{m+1}} - \overline{\mu}_l^{\alpha_m}| \tag{17}$$

$$\widehat{\mu}_r^{\alpha_{m+1}} = \widehat{\mu}_r^{\alpha_m} - \varrho^{\alpha_m}|\overline{\mu}_r^{\alpha_{m+1}} - \overline{\mu}_r^{\alpha_m}| \tag{18}$$

with

$$\varrho^{\alpha_m} = \frac{\widehat{\mu}_r^{\alpha_m} - \widehat{\mu}_l^{\alpha_m}}{\overline{\mu}_r^{\alpha_m} - \overline{\mu}_l^{\alpha_m}} \tag{19}$$

and where the "overlined" extremes are those obtained by the subgroup merging after the two-step procedure and the "hatted" ones are those finally obtained by the "gluing" transformation at the specified levels.

In Fig. 3 an example of *SMART-and* is plotted, obtained by applying the whole procedure; from the picture it is possible to appreciate the aforementioned deformation with respect to the fuzzy mean of the *SMART-and* average towards the min, when computed vertically. Note that the convex combinations vary with the level α according to changes of partial overlaps; in the picture this aspect is emphasized by the rough discretization adopted for the α levels; when a finer mesh is chosen, as is is done in the empirical application, the "gluing" process produces smoother memberships.

It is clear from Fig.3 that taking into account the strength of the agreements among the different α-cuts leads to a narrower membership, and hence a less vague estimation, with respect to the fuzzy mean.

2.4 Properties of the SMART average operators

We provide here some basic properties of the two average operators previously introduced. Averaging operators are investigated from fuzzy set-theoretic points of view in several papers such as, among others, [17, 18].

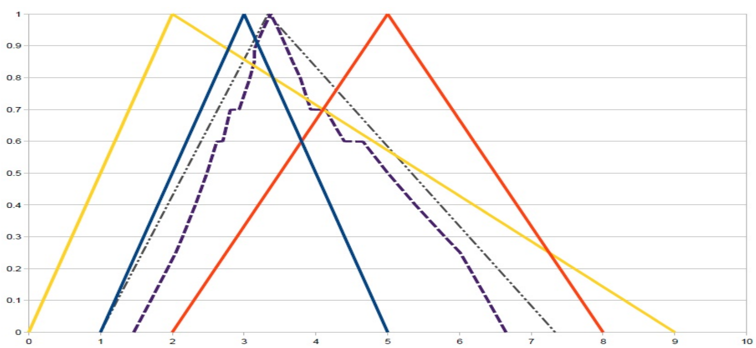

Figure 3: *SMART-and* (dashed line) among 3 fuzzy numbers (solid lines) compared to the fuzzy arithmetic mean (dashed-dotted line).

Let Ψ be the set of fuzzy numbers. Standard properties for an average operator $M : \Psi^n \longrightarrow \Psi$ when applied to n fuzzy numbers $A_1, A_2, \ldots, A_n$ are

- $M(A_1, A_1) = A_1$ (idempotency).
- $\min(A_1, A_2, \ldots, A_n) \leq M(A_1, A_2, \ldots, A_n) \leq \max(A_1, A_2, \ldots, A_n)$ (internality);
- $M(A_1, A_2) = M(A_2, A_1)$ (simmetry);
- given a specific order relation $\preceq$ between fuzzy numbers, if $A_2 \preceq A_2'$ then $M(A_1, A_2) \preceq M(A_1, A_2')$ (monotonicity);

For what concerns *SMART-or* and *SMART-and* operators, we can show that they are properly defined since their outcome lays inside the set of fuzzy numbers. Precisely, the following result holds:

Proposition 1 *The SMART-or* $\veebar$ *and SMART-and* $\barwedge$ *operators defined through (4-19) are fuzzy average operators from* Ψ^n *to* Ψ.

Proof. In order to prove that our operators are well defined, i.e. that the result of the proposed n-ary operators are fuzzy numbers, we have to prove nestedness, compactness and closure of α cuts and uniqueness of the core values for the operators' output. Compactness and closure of the resulting α-cuts comes directly from their definitions (4) and (8).

For the *SMART-or* average $\veebar$, nestedness of the α-cuts of the output is guaranteed by nestedness of those of the inputs, since each α-cut is a closed interval with extremes that are convex combinations of those in input. Moreover there is a unique core value with membership equal to one and it coincides, as already noted in the description of the weights ϵ_j, with the arithmetic mean of the cores in input. The same considerations are valid for the α-cuts of *SMART-and* average $\barwedge$ below the level h of not empty intersection. Besides, the "gluing" steps (17-19) defined when α is above level h, have been designed exactly to ensure that output α-cuts are actually nested. As for the unique core value, since coefficients (10) or (15) vanish at level α=1, the core of the output is either the core of the fuzzy arithmetic mean or a shifted value of this core, where the shift is caused by the "gluing" steps previously mentioned. □

For what concerns standard properties, we can state the following:

Proposition 2 *The SMART-or* $\veebar$ *and the SMART-and* $\barwedge$ *average operators satisfy idempotency, internality and simmetry. The SMART-or operator* $\veebar$ *is monotone with respect to the simple ordering between core values (i.e.* $A_1 \preceq A_2 \Leftrightarrow core(A_1) \leq core(A_2)$ *) while the SMART-and* $\barwedge$ *is not.*

Proof: Idempotency and internality of both the operators are straightforward consequences of the fact that α-cuts of the output fuzzy number are convex combinations of the α-cuts of the inputs, and the "gluing" steps (17-19) for $\barwedge$ are designed to make the α-cuts belong to the envelope of those in input. Simmetry is a direct consequence of the reordering of the extremes of the α-cuts of the fuzzy numbers to be merged. Monotonicity of $\veebar$ with respect to the simple ordering between core values derives again from the fact that the core of the output fuzzy number coincides with the core of the fuzzy arithmetic mean that is monotone. The same is not valid in general for $\barwedge$ since the aforementioned "gluing" steps that could produce a shift of the core value with respect to that of the fuzzy mean, the size of the shift depending on the grade of disagreement among the input fuzzy numbers. Hence it is possible that two n-tuples of fuzzy numbers with the same core values, but with different intersections among the α-cuts, produce different core values of the output, with evident violation of the monotonicity property. □

We remark that the two $SMART$ average operators are obviously not associative, since the levels of agreement or disagreement are significantly influenced by the specific subgroups of the input fuzzy numbers. Nevertheless, the lack of associativity is quite common in other merging operators (both connectives and averages) and it is widely accepted to have non-associative operators, especially when they are applied in a dynamical setting where it is reasonable to assume that the result is influenced by the order of arrival of the new information.

In order to disentangle the different behaviour of the proposed $SMART$ operators when applied to the same fuzzy numbers in input and how they differ from the fuzzy mean, which is our benchmark operator, we plot in Figures 4 and 5 the fuzzy memberships obtained by merging two fuzzy inputs, respectively within two opposite scenarios: in the former the two inputs significantly overlap indicating a strong agreement and a low disagreement levels; in the latter the two inputs are disjoint, hence they fully disagree, but their fuzzy mean is the same of the former scenario.

In Figure 4 it is possible to appreciate how the proposed operators depart from the fuzzy mean in opposite directions: the strong agreement is taken into account by the *SMART-and* operator and leads to a tightening of the fuzzy membership with respect to the mean. Conversely, the *SMART-or* operator takes into account the partial disagreement, resulting in a wider membership with respect to the fuzzy mean.

It is also worth noticing that the left branches of the two operators are closer to the fuzzy average than the right branches. This is emphasized in Figure 6 where it is evident that the aforementioned directions are towards

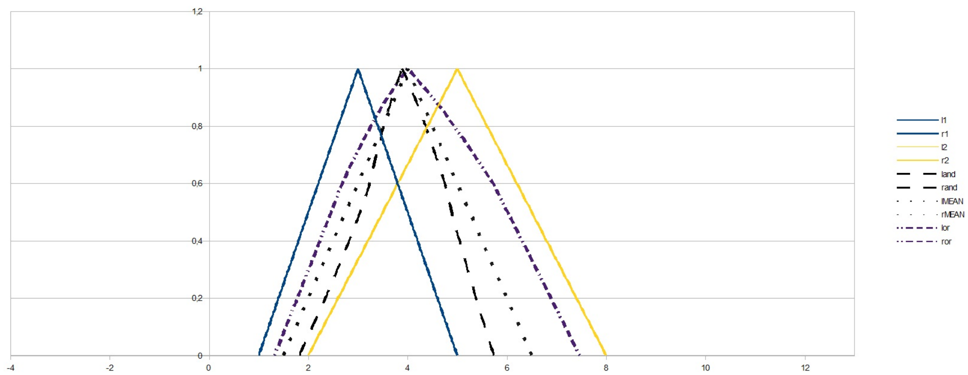

Figure 4: Comparison of the fuzzy mean (dotted) and the *SMART-and* (dashed), *SMART-or* (dashed-dotted) outputs with partially overlapping fuzzy inputs.

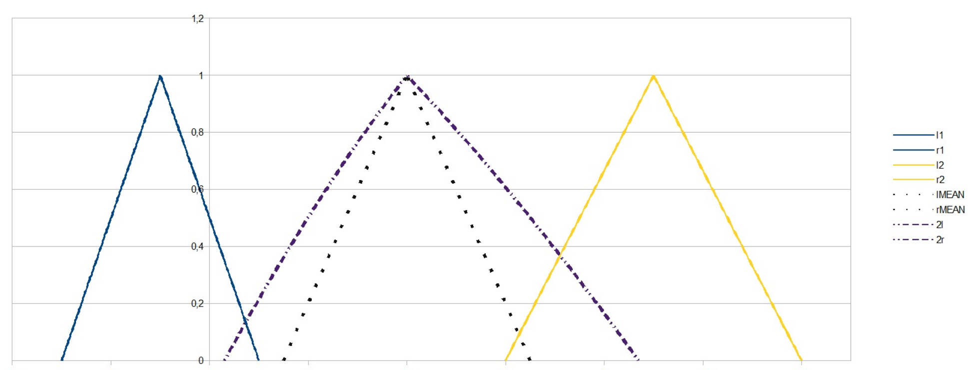

Figure 5: Comparison of the fuzzy mean (dotted) and the coinciding *SMART-and*, *SMART-or* (dashed-dotted) outputs with fully disagreeing fuzzy inputs.

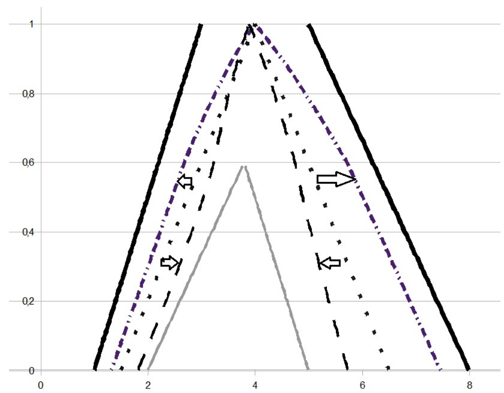

Figure 6: Deformations directions of the fuzzy mean (dotted) for the *SMART-and* $\bar{\wedge}$ (dashed) towards bounds of the usual conjunction (solid-gray) and for *SMART-or* $\veebar$ (dashed-dotted) towards bounds of the usual disjunction (solid-black).

the bounds of the α-cuts of the usual fuzzy disjunction for the *SMART-or* $\veebar$ and towards the bounds of the α-cuts of the usual fuzzy conjunction for the *SMART-and* $\bar{\wedge}$, respectively (hence our choice for names and symbols of the two new operators).

In Figure 5 the two proposed operators coincide; indeed, the level of agreement vanishes at any α-cuts and the *SMART-and* operator collapses into the *SMART-or* by construction.

Differently from the fuzzy mean, our *SMART-and* and *SMART-or* are sensitive to agreement or disagreement among the inputs, as desired. Such difference is highlighted in Figure 7 where we jointly plot the fuzzy output memberships for the two scenarios.

3 A practical application of the SMART operators to option pricing

As an illustrative example we apply the generalized SMART operators described in previous sections in the option pricing framework of Cox, Ross, Rubinstein [12], (CRR henceforth) where the dynamics of the risky asset is modeled by a binomial recombining tree with N periods ($N = 6$ in our numerical exercise). The crucial point is to obtain a fuzzy number for the volatility parameter σ which determines the up and down factors in the binomial tree; as remarked also in [37], we can infer about the volatility through different estimators. Since we aim at eliciting the volatility parameter of the $S\&P500$ index, we build the membership elicitation for σ on both the historical volatility estimator $\widehat{\sigma}$ and the estimator $\nu = \mathrm{VIX}/100$ (based on the VIX Index that represents the one-month ahead integrated volatility implied by option prices on the $S\&P500$ index). The elicitation

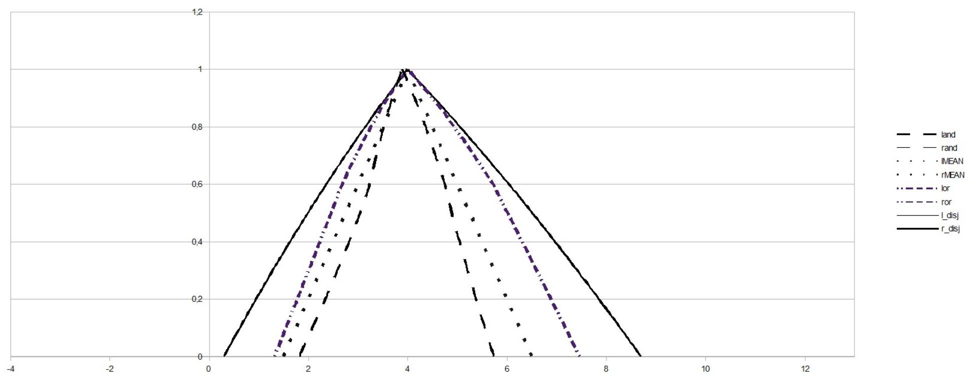

Figure 7: Comparison of the coinciding fuzzy mean (dotted) of the two previous scenarios and the *SMART-and* (dashed), *SMART-or* (dashed-dotted) outputs with partially overlapping fuzzy inputs and the coinciding *SMART-and*, *SMART-or* (full) outputs with fully disagreeing fuzzy inputs.

procedure we adopt here follows the idea we developed in [5]; aggregations of different memberships are performed with the generalized SMART operators introduced in the previous section.

For the sake of completeness, after a brief recall of financial derivatives basics (see [22]), we report in the following the main steps to elicit fuzzy volatility and to compute corresponding fuzzy option prices.

3.1 Stock volatility and crisp option pricing

One of the main streams of research in financial mathematics is the evaluation/pricing of derivative assets. A derivative contract is a financial asset the price of which is a function of time and of the price of so called underlying assets; these are basic assets (stocks, commodities, currencies) and their prices are determined by the market according to some supply-demand equilibrium. More precisely, let us denote with $\mathbf{S} = (S^1, S^2, \dots, S^k)$ a basket of such simple assets; given a time horizon T, a T-derivative on the basket $\mathbf{S}$ is defined as a contract which pays Π_T at time T where Π_T is a known function of $\mathbf{S}_T$ (or possibly of the whole history of basket prices until time T).

Classical examples of T-derivatives are European CALL and PUT options which are contracts written on a single underlying stock S. The payoff of a CALL option is given by

$$C_T = \max(S_T - K, 0) \tag{20}$$

and that of a PUT option is

$$P_T = \max(K - S_T, 0), \tag{21}$$

where K is the so-called strike price of the option and T its expiration date.

From a financial viewpoint a CALL (PUT) option is a contract which gives the right, but not the obligation, of buying (selling) the underlying stock at time T with price K. Notice that K and T are fixed in the contract at the time it is issued.

From the early seventies much research has been devoted to the computation of a fair value/price at time $t < T$ of derivative contracts, say Π_t. The basic assumption is that, for $t \in [0,T]$, $\Pi_t = \Pi(t, \mathbf{S}_t)$ where $\mathbf{S}_t$ is the price at time t of the underlying basket and where the function Π satisfies some mathematical regularity conditions.

Pioneering approaches for the evaluation of CALL and PUT options are the ones of [3, 12]. Both models rely on assumptions on the dynamics of the price process $(S_t, t \geq 0)$ of the underlying asset (typically a risky stock); the former assumes a continuous time dynamics described by a Geometric Brownian Motion i.e.

$$dS_t = \mu S_t dt + \sigma S_t dZ_t \tag{22}$$

where $(Z_t, t \geq 0)$ is a standard Brownian motion and μ, σ are model parameters representing the continuously compounded mean return and the volatility of the underlying stock. The latter suggest a discrete time process, with time step δ; at time $t = n\delta$, for $n = 1, 2, \ldots, N$ the stock S_t is a discrete random variable with $n+1$ outcomes described by $\left(S_0 u^j d^{(n-j)}, j = 0, 1, \ldots, n\right)$ with a Binomial distribution $\mathcal{B}(n,p)(j), j = 0, 1, 2, \ldots, n$. The parameters of the model are u, d representing the increase and decrease price factor at each step and the increase probability p at each step. For details on the two models the reader is referred to the original paper or the comprehensive book of [22]. If we further assume the availability of a risk-free asset with fixed continuously compounded rate r (e.g. a money market account), a closed formula can be derived for the price at time $t < T$ of European CALLs; the price of European PUTs are then derived by applying simple arguments under the "no-arbitrage" assumption.

In the case of Black and Scholes model [3] we have that

$$C(S_t) = \Phi(d_1) S_t - \Phi(d_2) K\, e^{-r(T-t)} \tag{23}$$

with

$$d_{1,2} = \frac{\ln(\frac{S_t}{K}) + (r \pm \frac{\sigma^2}{2}(T-t))}{\sigma\sqrt{T-t}}$$

and where $\Phi(.)$ is the cumulative distribution function of a standard normal random variable; for the CRR model [12], if $T = N\delta$, we have

$$C(S_t) = \overline{\mathrm{Bin}}_{n,q^*}(m) S_t - \overline{\mathrm{Bin}}_{n,q}(m) K\, e^{-r(T-t)} \tag{24}$$

with $m = \min\left\{j | S_t u^j d^{n-j} > K\right\}$, $q = \frac{e^{r\delta} - d}{u-d}$, $q^* = e^{-r\delta} q$ and where $\overline{\mathrm{Bin}}_{n,q}$ is the survival function of a binomial random variable with parameters n and q.

For the two above models to be consistent (the discrete one converging to the continuous one as $\delta \to 0$) parameters are set such that $u = e^{\sigma\sqrt{\delta}}$,

$d = e^{-\sigma\sqrt{\delta}}$. Hence, the estimation of the volatility parameter σ is a crucial issue in both cases.

As already noticed, much research has been devoted to the evaluation of derivative contracts in more general settings than the ones above described; so far, one main direction is towards the so called Stochastic Volatility models, where the volatility of stock price changes is described itself by a stochastic process, possibly correlated with the stock price process. These models are able to reproduce many empirical facts in the stock and/or in the derivative markets, such as the leptokurtosis of the stock log-returns and the so called smile curve of the option implied volatility when plotted against the option strike price (see e.g. [11] for details).

Among stochastic volatility models the most well known are the Heston model [20] in continuous time and the GARCH(1,1) [4] in discrete time for which a quasi closed price formula is suggested in [13]. As an alternative to the stochastic volatility approach, in [5] the volatility is modeled as a fuzzy number. In this framework, once the membership function for the fuzzy volatility is known, it is possible to derive the membership function of the fuzzy option price by applying Zadeh's extension principle [44, 29].

Differently from [29], we didn't assign in [5] a specific "a priori" membership function to the fuzzy volatility but we rather estimated (elicited) the membership function for the volatility parameter. For each of the considered sources of information and estimators, we obtained unimodal membership functions with nested α-cuts identified by intervals $[\mu_l^\alpha, \mu_r^\alpha]$ in the extended reals $\widetilde{\mathbb{R}}$ and we computed fuzzy option prices once again via the extension principle. To make this paper self contained, a brief description of the eliciting procedure is given below.

3.2 Membership elicitation

This subsection recalls the outcomes in [5] in order to assess a membership function for the volatility parameter σ. This is done by looking at membership functions as coherent conditional probabilities assessments and by applying generalized Bayes rule. More precisely, we deal with some vague statement about σ e.g. "σ is around a specific value", and we have to rank the willingness of an agent to claim it, conditionally to the true unknown value of the parameter. We look for

$$\mu_{H_s}(x) = P(\sigma \text{ ``is claimed to be a value around '' } \overline{\sigma}_s | \sigma = x), \qquad (25)$$

where each value of $\overline{\sigma}_s$ is representing a market scenario H_s, $s = 1, \ldots, n_s$, while x is the exact value attained by the parameter σ. Given that σ is not observable, beliefs on the parameter must be based on some estimators. Thus, what we actually elicit is

$$\widetilde{\mu}_{H_s}(\hat{\theta}) = P(H_s | \text{Info}^{\hat{\theta}}), \qquad (26)$$

where $\text{Info}^{\hat{\theta}}$ represents information obtained from an estimator $\hat{\theta}$ of σ. Let us assume, for computational purposes, that any empirical or simulated distribution for $\widehat{\theta}$ is discretized into n_b classes $\hat{\theta}_b$, named "bins". With a little abuse we set $\hat{\theta}_b \equiv (\hat{\theta} \in \hat{\theta}_b)$ so that values for the *pseudo*-memberships

(26) are computed "bin by bin" with $\text{Info}^{\hat{\theta}} \equiv \hat{\theta}_b$. The input domain $\mathcal{E}$ is given by $\left\{H_s, \hat{\theta}_b|H_s\right\}_{s=1,\dots,n_s; b=1,\dots,n_b}$ i.e. by the unconditional scenarios ranges and by the realization of the estimator inside one of its possible bins, conditioned to different market scenarios.

As a first step we need the *pseudo*-memberships (26); they can be derived either via Bayes rule, if scenarios H_s, $s = 1, \dots, n_s$, come from a partition, or via *coherent extension*, if scenarios come from partial knowledge (i.e. they overlap, or they do not cover all the possibilities or there is some logical constraint among them and possible values of the parameter). In any case, the value of (26) relies on *likelihoods* $P(\hat{\theta}_b|H_s)$ and *priors* $P(H_s)$. Once likelihoods $P(\hat{\theta}_b|H_s)$ are obtained, it is possible to infer on the probabilities $P(\hat{\theta}_b)$. If we have full information, i.e. the scenarios form a partition and there is not any logical constraint among the H_s and the $\hat{\theta}_b$, then the available assessment $P(\cdot|\cdot)$ on $\mathcal{E}$ is surely coherent [38, Prop.1]. In this case we obtain the $P(\hat{\theta}_b)$ through the usual disintegration formula

$$P(\hat{\theta}_b) = \sum_{s=1}^{n_s} P(\hat{\theta}_b|H_s)P(H_s), \tag{27}$$

and the *pseudo*-membership $P(H_s|\hat{\theta}_b)$ by Bayes rule

$$P(H_s|\hat{\theta}_b) = \frac{P(\hat{\theta}_b|H_s)P(H_s)}{P(\hat{\theta}_b)}. \tag{28}$$

If information is partial, once overall conditional coherence of the assessment $P(\cdot|\cdot)$ on $\mathcal{E}$ has been ensured, we can extend it by the procedures detailed in [7, 9] to $H_s|\hat{\theta}_b$, obtaining coherent intervals

$$[P_*(H_s|\hat{\theta}_b), P^*(H_s|\hat{\theta}_b)]. \tag{29}$$

Given the incompatibility of the various bins $\hat{\theta}_b$, Theorem 2 in [10] guarantees the coherence of any value inside the intervals (29) . Hence, we obtain a set of plausible *pseudo*-memberships instead of a single one, so that we have to deal with interval type-2 [25] *pseudo*-memberships.

The further step is to consider the current (observed) value $\widehat{\theta}_{obs}$ of the parameter estimator. On the base of the bin $\hat{\theta}_{\overline{b}}$ including $\widehat{\theta}_{obs}$ we can select most probable scenarios $H_{\overline{s}}$ by maximizing $P(H_s|\hat{\theta}_{\overline{b}})$. Ties among type-1 or interval incomparability among type-2 *pseudo*-memberships can induce more than one plausible scenario; in this case, any of such scenarios can be a valid candidate so we need to take the disjunction of them.

At this point we elicit the searched membership $\mu_{\tilde{\sigma}}(x)$ by transforming the simulating distributions $\pi_{\overline{s}}$ associated with the selected scenario(s) through a probability-possibility transformation among those proposed in [14]; in particular we use one transformation induced by confidence intervals around the median.

It is worth noticing that the elicitation procedure for this membership is briefly summarized for the reader's convenience but it is beyond the scope of the present contribution to give further details.

Here, the focus is on the merging procedure among n memberships, whatever their origin.

Going back to the exercise in [5], we recall that the *priors* are obtained by "experts evaluations about scenarios" based on historical data, whereas *likelihoods* are derived through simulation. More precisely, we assume that, by randomly generating values of σ from a specific distribution π_s for each scenario H_s, we are able to obtain the distribution for $\hat{\theta}$ conditioned to $H_s, s = 1, 2, ..., n_s$. The latter step relies on the availability of an explicit relation between σ and $\hat{\theta}$.

As already mentioned, we can infer about the volatility through different estimators. The most common is the sample standard deviation of asset log-returns, so called historical volatility; in this case indirect information is based on a sample of stock prices. Alternatively, we may profit, when available, from the value of a volatility index obtained by considering as input a set of suitably selected traded Call and Put options thus relying on a sample of derivatives prices.

Since we aim at eliciting the volatility parameter of the $S\&P500$ index, we based the membership elicitation for σ on both the historical volatility estimator $\widehat{\sigma}$ and the estimator ν = VIX/100, based on the VIX Index. The elicitation procedure we adopt follows the idea we developed in [5]; however, any different transformation, or better a coherent extension of the probabilities assessed in the previous steps, could be adopted to obtain membership functions for σ.

Note that the value of the VIX is released by the Chicago Board Options Exchange and its computation is based on a sample of prices of Call and Put options written on the $S\&P500$ Index. For the aim of this paper we just rely on the fact that the VIX Index is considered an estimate of the volatility of the $S\&P500$ expected for the near future; in [40] the interested reader may find the motivation for the VIX to be a volatility estimate as well as details on its practical computation. We can summarize the main steps of our procedure in the following items (for further details refer to the cited papers):

1. The available time series are considered in order to elicit a pseudo-membership for each estimator;
2. on the test date the most probable scenarios are selected according to the current value of the estimator and depending on the pseudo-memberships obtained in step 1;
3. a probability-possibility transformations is applied to the simulating distributions (Uniform, LogNormal, Gamma) corresponding to the selected scenario(s);

3a. whenever there are more than one membership associated to a simulating model, they are merged via our ***SMART-or*** operator;

4. the memberships, stemming from the different simulation models, are merged into a single membership through our ***SMART-or***.
5. steps 1-4 are performed for both $\widehat{\sigma}$ and ν leading to two different fuzzy numbers;
6. the memberships associated to the two different estimators are merged via our ***SMART-and*** to obtain a single fuzzy number for the volatility σ.

The above steps are repeated by applying the fuzzy mean in place of *SMART-or* and *SMART-and*, specifically for steps 3 to 6. This allow to obtain a single alternative representation of the fuzzy volatility parameter.

3.3 Fuzzy option pricing

Once we have obtained a single aggregated membership function for $\tilde{\sigma}$, the fuzzy volatility parameter, on a specific test-date, it is possible to price options quoted that day by a straightforward extension of standard CRR model in [12] to a fuzzy multi-period binomial model. This will be done for the two alternative fuzzy numbers stemming from the application of our operators or from the application of the fuzzy mean.

Our explicit numerical evaluation of each α-cut of the fuzzy number for $\tilde{\sigma}$ allows us to take advantage of others contributions available in literature for each step of the pricing procedure. In particular:
- from $\tilde{\sigma}$ to the the fuzzy "UP" and "DOWN" jump factors
(Zadeh's extension principle [44])

$$[\underline{u}^{\alpha}, \overline{u}^{\alpha}] = [e^{\underline{\sigma}^{\alpha}\sqrt{\Delta t}}, e^{\overline{\sigma}^{\alpha}\sqrt{\Delta t}}] \qquad [\underline{d}^{\alpha}, \overline{d}^{\alpha}] = [e^{-\overline{\sigma}^{\alpha}\sqrt{\Delta t}}, e^{-\underline{\sigma}^{\alpha}\sqrt{\Delta t}}]\ ; \tag{30}$$

- from $\tilde{u}$ and $\tilde{d}$ to the fuzzy risk neutral probabilities *(Muzzioli & Torricelli [36])*

$$[\underline{p}_u^{\alpha}, \overline{p}_u^{\alpha}] = \left[\frac{e^{r\Delta t} - \overline{d}^{\alpha}}{\overline{u}^{\alpha} - \overline{d}^{\alpha}}, \frac{e^{r\Delta t} - \underline{d}^{\alpha}}{\underline{u}^{\alpha} - \underline{d}^{\alpha}}\right] [\underline{p}_d^{\alpha}, \overline{p}_d^{\alpha}] = \left[\frac{\underline{u}^{\alpha} - e^{r\Delta t}}{\underline{u}^{\alpha} - \underline{d}^{\alpha}}, \frac{\overline{u}^{\alpha} - e^{r\Delta t}}{\overline{u}^{\alpha} - \overline{d}^{\alpha}}\right]\ ; \tag{31}$$

- from $\tilde{p}_u$ and $\tilde{p}_d$ to option price (e.g. call) *(Li & Han [32])*

$$[\underline{C}_0^{\alpha}, \overline{C}_0^{\alpha}] = e^{-rN\Delta t}\left[\sum_{i=0}^{N}(\underline{p}_u^{\alpha})^i(\underline{p}_d^{\alpha})^{N-i}\underline{C}_{N,i}^{\alpha}, \sum_{i=0}^{N}(\overline{p}_u^{\alpha})^i(\overline{p}_d^{\alpha})^{N-i}\overline{C}_{N,i}^{\alpha}\right] \tag{32}$$

with

$$[\underline{C}_{N,i}^{\alpha}, \overline{C}_{N,i}^{\alpha}] = \left[\max(S_0(\underline{u}^{\alpha})^i(\underline{d}^{\alpha})^{N-i} - K, 0), \max(S_0(\overline{u}^{\alpha})^i(\overline{d}^{\alpha})^{N-i} - K, 0)\right]. \tag{33}$$

Fuzzy option prices relative to the two approaches are compared to market bid-ask prices for the quoted options on a test date, in order to assess the overall pricing performance; the comparison is based on the computation of a proper similarity index (see, e.g., [38]).

3.4 Numerical results

In the numerical application the elicitation step n.1 is performed on the $S\&P500$ daily returns from January 1960 to September 2016 for $\widehat{\sigma}$ and on daily observations of the VIX index from January 1990 to September 2016 for $\widehat{\nu}$. The considered test-date for evaluating pricing performance is October 5, 2016; the current scenario on this date is the "low volatility" scenario for all simulating models and both estimators. After obtaining the three memberships associated to the simulating distributions (Uniform, Log-Normal, Gamma) corresponding to the current "low volatility" scenario, they are merged (step n.4) either with the $\veebar$ operator, for both the historical volatility and the VIX based estimators (step n.5). The outcomes corresponding the two estimators are fused via the $\bar{\wedge}$ (step n. 6), obtaining the fuzzy representation $\tilde{\sigma} = \mu_{\widehat{\sigma}_{obs}} \bar{\wedge} \mu_{\nu_{obs}}$ for the volatility parameter σ. Steps n.4 to n.6 are then applied by replacing the standard fuzzy arithmetic mean to our SMART *SMART-or*/*SMART-and* operators. The memberships derived are plotted in Fig. 8.

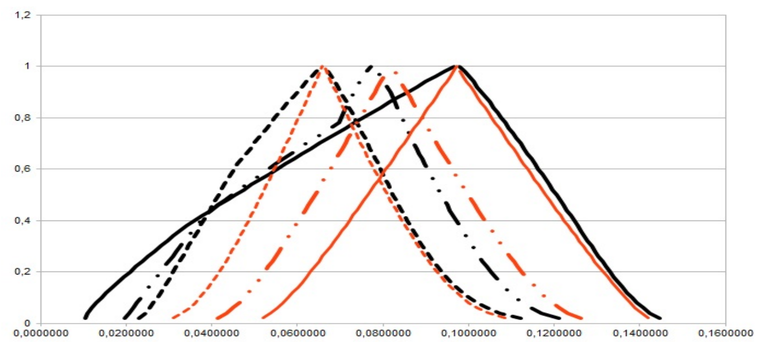

Figure 8: The merging results: *SMART-or* of the fuzzy numbers stemming from different models for $\widehat{\sigma}$ (dashed lines on the left) and for ν (solid lines on the right) and their final *SMART-and* (dashed-dotted lines on the center), by applying our $\veebar$ and $\bar{\wedge}$ (black) or the fuzzy mean (gray).

Fuzzy option prices for option traded on October, 5, 2016, are computed by applying the formulas outlined in previous section.

In Fig. 9 we plot the fuzzy price obtained by applying either our SMART merging operators (solid) or the fuzzy mean (dashed) for two examples of options; the bid-ask interval is also included in the picture. We point out that in both examples our procedure gives narrower memberships; this feature is common to all option prices. In the example in the right picture we also note that the core value is closer to the mid-point of the bid-ask.

In Fig. 10 we plot the similarity values for the two approaches; the number of times that our approach overcome the fuzzy mean is not significantly above 50%; though, whenever the fuzzy mean is better, the two values are indeed very similar. In Fig. 11 we plot the histogram of the difference between the similarity value obtained with our approach and with the fuzzy mean; if this difference is positive we have a better performance of our *SMART-or*/*SMART-and* in describing option prices. It is evident from the picture that this difference is often null and that in the other cases there is an asymmetric distribution of the difference towards positive values. Summary statistics in Table 2 confirm this evidence.

A deeper investigation of the results also highlights that the two approaches are good enough whenever the core of the fuzzy prices are within the bid-ask range and in that cases the SMART merging approach is definitely better since it usually produces less vague values. Conversely, the fuzzy arithmetic mean seems to be better when both similarities are indeed very low; and this happens mostly when both memberships are "far" from

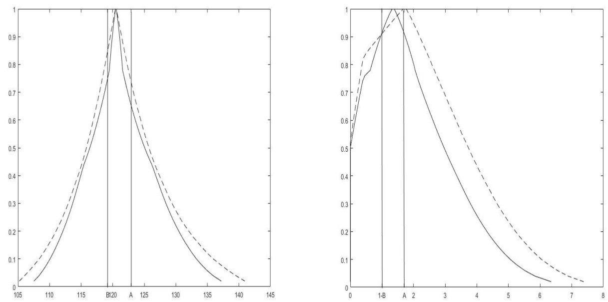

Figure 9: Market bid-ask (crisp interval), "smart" fuzzy prices (solid), "arithmetic mean" fuzzy prices (dashed) for two examples of CALL options traded on October 5, 2016 with expiration in one month and strike price K=2040 (left panel) and K=2210 (right panel). The $S\&P500$ value is $S_0 = 2159.7$.

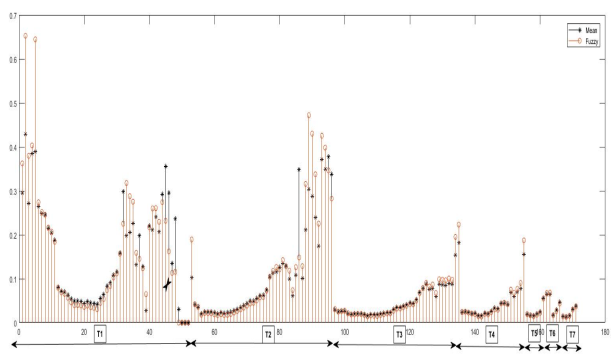

Figure 10: Similarities between fuzzy SMART (solid circle) or fuzzy mean (dashed plus) against market bid-ask option prices.

Table 2: Summary statistics for similarity differences between SMART and fuzzy mean w.r.t. bid-ask market prices.

Mean	StD	Median	Q1	Q3	10th perc.	90th perc.	Skewness
0.0058	0.0433	0	-0.0032	0.0076	-0.0094	0.0416	1.3362

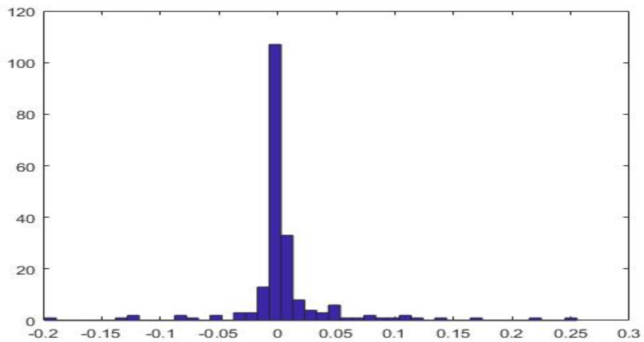

Figure 11: Histogram of the difference between the similarity for SMART and average fuzzy merging.

the bid-ask spread, since the one obtained through fuzzy mean is usually larger and hence more likely to overlap the bid-ask interval. An example is given in Fig. 12.

We believe that this feature is due to the pricing model itself (the core of both the fuzzy prices are usually outside the bid-ask range within these cases) and not to the particular aggregation method used for the elicited (estimated) fuzzy volatility. In addition, the aim of this contribution is not that of convincing the reader about a new model for option pricing; rather the present section was aimed at giving a practical example for the application of our merging operators in the financial world where several sources of data are available for the same claim. Hence, we don't discuss further this issue.

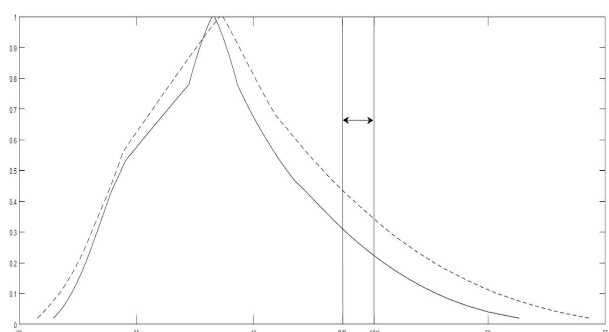

Figure 12: A case of SMART (solid) and fuzzy mean (dashed) option prices with low similarity w.r.t. market bid-ask (crisp interval).

4 Conclusion

The problem of merging information stemming from several sources is crucial in recent times where both size and variety of available data is hugely increasing. To address this issue we propose SMART fuzzy operators to aggregate sources of information and models when estimation for some (fuzzy) parameter is the final aim of the merging. Note that our proposal come as alternative to the use of the fuzzy mean, commonly adopted to merge fuzzy estimates of the same quantity, and are defined through a slight modification of the fuzzy mean itself, similar to generalizations like Fuzzy Ordinal Weighted Average (FOWA, see e.g. [31]). However, the operators we introduce can be applied in a disjunctive, when sources of information are alternative one another, or in a conjunctive way, when they give concomitant information. This is achieved through a proper choice of the weights which, notably, is endogenous to the input fuzzy numbers. Specifically, the definition of the proposed operators is based on an adapted Marzullo's algorithm to properly measure the weight assigned to overlapping intervals in each of the α-cuts of the memberships to be merged. The whole methodology is illustrated within the problem of estimating stock volatility in the US financial markets, usually represented by volatility of the S&P 500 Market Index, on a fixed test date. Many source of information described by different data are available to this scope. Traditionally, volatility on a certain date is measured as the so called historical volatility which is nothing but the sample standard deviation of past daily observations of logarithmic returns of the Index itself; the VIX Index is provided by the Chicago Board Option Exchange as an alternative measure for the volatility of the S&P 500 Market Index, based on the price of a basket of financial Call and Put Options on the Index, traded on the same Exchange, see [40]. The two memberships obtained for the historical volatility and the VIX based estimators, according to the elicitation procedure defined in [5] by merging the simulating distributions (Uniform, Log-Normal, Gamma) with the *SMART-or* $\veebar$ operator, are then merged via the *SMART-and* $\barwedge$ operator, obtaining the fuzzy representation $\tilde{\sigma} = \mu_{\widehat{\sigma}_{obs}} \barwedge \mu_{\nu_{obs}}$ for the volatility parameter σ. The same is done by applying the arithmetic fuzzy mean operator whenever merging is due and the outcomes are summed up in Fig. 8. Since a true membership for the volatility, to be compared with our estimates, is not available, in order to have a tangible comparison of the different merging procedure we should shift to observable quantities, such as market prices for options on the S&P market Index, whose prices notoriously depend on the underlying volatility, see [22]. Hence, we compute fuzzy option prices corresponding to a chosen model, once the fuzzy volatility parameter is elicited, and compare them to the corresponding Market Bid and Ask price intervals. More precisely, fuzzy option prices are computed within the fuzzy generalization of the Cox Ross Rubinstein model by building on existing results such as *(Muzzioli & Torricelli [36])* and *(Li & Han [32])* as well as on the extension principle of [44]). Model prices are computed for all options traded on the test date and are compared to the corresponding market Bid and Ask price intervals. The results of our numerical exercise are promising: in particular, whenever closeness is measured by fuzzy similarity as defined in [38] we find that our proposed merging procedure produces fuzzy option prices which are closer to Bid-Ask market prices with respect to the ones obtained by applying the fuzzy mean averaging operator. Hence, taking into consideration agreement and disagreement among input fuzzy estimates may indeed be useful for practical applications, of which option pricing is just an illustrative example.

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A Merging operators between fuzzy numbers

A merging or aggregation operator for fuzzy numbers is intended as an operator which applied to an input of n fuzzy numbers, with $n \geq 2$, delivers a single fuzzy number as an output.

Two approaches can be distinguished for merging/aggregating fuzzy numbers.

One seeks to merge fuzzy numbers related to preferences or evaluations stemmed from different experts or information sources in one single fuzzy number which should subsume the fuzzy number related to a representative expert or information source and can have several reasonable properties like those reported in [8].

The other interprets fuzzy numbers as estimates stemming from different experts or sources of information and aims at aggregating those estimates to a single fuzzy number representing a compromise of original inputs, as done e.g. in [1, 21, 35]. The α-cuts corresponding to the same level of plausibility for the n input fuzzy numbers are merged in order to obtain the α-cut of the output fuzzy number.

In order to make the difference between the two approaches crystal clear from a computational point of view, other that only motivational, we remark that the former combines fuzzy numbers with set-theoretically operations, such as t-norms or t-conorms, applied pointwisely on their membership functions (i.e. vertically, on [0, 1]) , while the latter applies algebraic operations "on" fuzzy numbers (addition, multiplication or averages) which are defined on the α-cuts (i.e. horizzontally, on $\mathbb{R}$).

A.1 Merging connectives

The classical t-norm fuzzy conjunction and t-conorm disjunction fall within the first approach as well as some of their generalizations like, e.g., overlap and grouping functions (see [2]). For the sake of clarity we briefly recall their definition in what follows

Definition: A triangular norm (t-norm shortly) is a function $T : [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing and admits 1 as neutral element.

The simplest example of a t-norm id the min function i.e. $T(a,b) = \min(a,b)$. If A and B are two fuzzy numbers represented by fuzzy memberships $\mu_A(x), \mu_B(x) : \mathbb{R} \to [0,1]$ the fuzzy number C merging A, B through T is defined through the fuzzy membership $\mu_C(x) : \mathbb{R} \to [0,1]$ where $\mu_C(x) := T(\mu_A(x), \mu_B(x))$ suitable scaled to fulfill fuzzy memberships properties. Depending on the choice of the t-norm function it is possible to obtain different fuzzy-theoretic intersection operators.

Definition: A triangular conorm (t-conorm or s-norm shortly) is a function $S : [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing and admits 0 as neutral element.

The simplest example of a t-conorm id the max function i.e. $S(a,b) = \max(a,b)$. If A and B are two fuzzy numbers represented by fuzzy memberships $\mu_A(x), \mu_B(x) : \mathbb{R} \to [0,1]$ the fuzzy number D merging A, B

through S is defined through the fuzzy membership $\mu_D(x) : \mathbb{R} \to [0,1]$ where $\mu_D(x) := S(\mu_A(x), \mu_B(x))$ suitable scaled to achieve normality if required. Depending on the choice of the t-conorm function it is possible to obtain different fuzzy-theoretic union operators.

A.2 Merging averages

We recall here main examples of merging averages by following the approach in [35]. The simplest example is the fuzzy weighted average (FWA) which, when weights are crisp numbers, is defined as follows

Definition: Let Ψ be the family of fuzzy numbers. A fuzzy weighted average operator is a mapping $FWA : \Psi^n \longrightarrow \Psi$, associated to a vector $w = (w_1, w_2, \ldots, w_n)$ of non negative weights with $\sum_{i=1}^n w_i = 1$, such that

$$FWA(A_1, A_2, \ldots, A_n) = \sum_{i=1}^n w_i A_i, \tag{34}$$

where $A_i, i = 1, 2, \ldots, n$ are the fuzzy numbers to merge, such that it results as the fuzzy number with α-cuts:

$$[\sum_{i=1}^n w_i \mu^\alpha_{A_{il}}, \sum_{i=1}^n w_i \mu^\alpha_{A_{ir}}]. \tag{35}$$

In the special case of equal weights $w_i = \frac{1}{n}$, FWA reduces to the simple fuzzy arithmetic mean, i.e. the fuzzy number with α-cuts:

$$[\frac{1}{n} \sum_{i=1}^n \mu^\alpha_{A_{il}}, \frac{1}{n} \sum_{i=1}^n \mu^\alpha_{A_{ir}}]. \tag{36}$$

The above definition may be generalized to the case where weights are also described by fuzzy numbers introducing anyhow further suddle technicalities (one for all: we could louse the convexity of the combination); since we do adopt such generalization, we do not recall it here.

Instead, it is worth to mention a modification of the FWA in order to consider a predetermined order of the input fuzzy numbers: the fuzzy ordered weighted averaging operator, $FOWA$, which is the extension of the ordered weighted operator for crisp numbers, introduced in [41].

Definition: Let Ψ be the family of fuzzy numbers. A fuzzy ordered weighted average operator is a mapping $FOWA : \Psi^n \longrightarrow \Psi$, associated to a vector $w = (w_1, w_2, \ldots, w_n)$ of non negative weights with $\sum_{i=1}^n w_i = 1$, such that

$$FOWA(A_1, A_2, \ldots, A_n) = \sum_{i=1}^n w_i B_i, \tag{37}$$

where $A_i, i = 1, 2, \ldots, n$ are the fuzzy numbers to average and B_i is the i-th largest of the input fuzzy numbers.

As pointed out in [35], the reordering of the fuzzy numbers introduces a further complexity since there are many alternative valid proposals that might lead to different results. As recommended in the same paper [35], the simplest criteria is to order fuzzy numbers according to the order of their core values, i.e. the α-cuts for $\alpha = 1$.

Several families of $FOWA$ operators can be classified according to specific choices of the weights; an interesting example is the S-FOWA family of operators based on [42] where the weights are proper modifications of those of the fuzzy arithmetic mean. In particular $w_1 = \frac{1}{n}(1-(\alpha+\beta))+\alpha$, $w_n = \frac{1}{n}(1-(\alpha+\beta))+\beta$ and $w_j = \frac{1}{n}(1-(\alpha+\beta))$, for $j = 2, \ldots, n-1$. This choice of the weights is in the same spirit of our deformations of the fuzzy arithmetic mean; however, parameters α and β are left to an exogenous decision and remain constant for all $\alpha-$cuts, contrarily to ours.