Envelopes of conditional probabilities extending a strategy and a prior probability

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Abstract

Any assessment formed by a strategy and a prior probability is a coherent conditional probability and can be extended, generally not in a unique way, to a full conditional probability. The corresponding class of all extensions is studied and a closed form expression for its envelopes is provided. Subclasses of extensions meeting further analytical properties are considered by imposing conglomerability and a conditional version of conglomerability, respectively. Then, the envelopes of extensions satisfying these conditions are characterized.

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1. Introduction

In the seminal paper [31] by Dubins, the notion of *strategy* σ together with the ensuing concepts of *conglomerability* and *disintegrability* with respect to a *(finitely additive) prior probability* π are presented and it is proved that the assessment $\{\pi, \sigma\}$ can always be extended, generally not in a unique way, to a full conditional probability. The extension of an assessment $\{\pi, \sigma\}$ is particularly meaningful in statistics [3, 10, 16, 36, 38, 46, 48, 59], in limit theorems, stochastic processes and their applications [2, 30, 31, 32, 41, 45, 56].

An open problem in this context is to characterize the whole class of full conditional probabilities extending an assessment $\{\pi, \sigma\}$, so, a first aim of this paper is to provide a closed form expression for the envelopes of such class.

Generally, this class of extensions can contain full conditional probabilities failing conglomerability (see, e.g., [1, 36, 41, 50, 56]). Conglomerability is a regularity condition often required in applications, since non-conglomerable extensions of $\{\pi, \sigma\}$ can show a pathological behaviour as they could not be approximated in the total variation norm by conglomerable ones [31]. Thus, there

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is an advantage in restricting to extensions meeting this property. Moreover, as is well-known, conglomerability reduces to disintegrability when σ is integrable with respect to π (see [3]).

In this paper, the class of conglomerable full conditional probabilities extending $\{\pi, \sigma\}$ is considered and a closed form expression for the envelopes of such class is provided.

Essentially, the conglomerability requirement reduces the class of joint probabilities consistent with $\{\pi, \sigma\}$, the latter being restrictions of conglomerable full conditional probabilities extending $\{\pi, \sigma\}$. Moreover, every conglomerable full conditional probability extending $\{\pi, \sigma\}$ satisfies a constraint also on all conditional events $F|K$'s with positive conglomerable joint probability of K. Nevertheless, conglomerability does not affect the conditional probability on those $F|K$'s whose conditioning event K has null conglomerable joint probability (see Example 7). The lower envelope of the class of conglomerable full conditional probability extensions inherits this constraint just on those conditional events $F|K$'s with positive lower conglomerable joint probability of K (as shown in Theorems 4 and 5).

Thus, a conditional version of conglomerability is introduced in order to reinforce the conglomerability constraint on those conditional events $F|K$'s whose conditioning event K has null conglomerable joint probability. This is reached by requiring conglomerability to hold with respect to a full conditional prior probability extending π. Hence, the class of conditionally conglomerable full conditional probabilities extending $\{\pi, \sigma\}$ is considered and its envelopes are characterized.

The lower envelope of such class reveals to be a totally monotone capacity on a specific subfamily of conditional events (see Corollary 2). As a consequence, this allows to compute (as a Choquet integral) the corresponding lower conditional prevision on a suitable class of conditional bounded random quantities. However, the lower envelope of conditionally conglomerable extensions is generally not 2-monotone, as shown in Example 4.

The paper is organized as follows. In Section 2 we recall some preliminaries on coherent conditional previsions, n-monotone capacities and Choquet integration. In Section 3 different notions of conglomerability for previsions and probabilities are introduced and their role in Kolmogorov's and Walley's theories is highlighted. Section 4 copes with characterizing the envelopes of the following classes of full conditional probabilities extending $\{\pi, \sigma\}$: *(i)* the whole class of extensions; *(ii)* the subclass of extensions meeting conglomerability. Finally, in Section 5 we introduce conditional conglomerability, which is a reinforcement of Dubins' conglomerability for probabilities. Furthermore, we provide a closed form expression for the envelopes of the extensions of $\{\pi, \sigma\}$ meeting this property.

2. Preliminaries

Throughout this work Ω denotes a non-empty set, whose subsets are considered as events. For any set of events $\mathcal{G} = \{E_i\}_{i \in I}$, denote with $\langle \mathcal{G} \rangle$, $\langle \mathcal{G} \rangle^\sigma$ and

$\langle \mathcal{G} \rangle^*$, respectively, the minimal algebra, σ-algebra and complete atomic algebra of subsets of Ω containing $\mathcal{G}$.

Let $X : \Omega \to \mathbb{R}$ be a *random quantity* and denote with $\mathbb{L}(\Omega)$ the linear space of all bounded random quantities. Given an algebra $\mathcal{A}$ of subsets of Ω, let $\mathbb{I}(\Omega, \mathcal{A})$ be the set of indicators $\mathbf{1}_E$'s of events $E \in \mathcal{A}$. The uniform norm closure of the linear space spanned by $\mathbb{I}(\Omega, \mathcal{A})$, is the linear subspace $\mathbb{L}(\Omega, \mathcal{A})$ of $\mathbb{L}(\Omega)$, consisting of all $\mathcal{A}$-*continuous* functions [58] (see also [6, 34] for an equivalent definition).

Definition 1. *The function $X : \Omega \to \mathbb{R}$ is said* $\mathcal{A}$-**continuous** *if it is bounded and for every $t \in \mathbb{R}$ and $\epsilon > 0$ there exists $A \in \mathcal{A}$ such that*

$$(X \geq t) \supseteq A \supseteq (X \geq t + \epsilon),$$

where $(X \geq t) = \{\omega \in \Omega \ : \ X(\omega) \geq t\}$.

The notion of $\mathcal{A}$-continuity coincides with the notion of measurability required in [51], moreover, every $\mathcal{A}$-continuous function is Stieltjes integrable [6] with respect to every finitely additive probability on $\mathcal{A}$. In particular, if $\mathcal{A}$ is a σ-algebra, then the class of $\mathcal{A}$-continuous functions exactly coincides with the class of bounded $\mathcal{A}$-measurable functions (see Theorem 2.2 in [23]).

A *conditional random quantity* is a pair (X, H), denoted as $X|H$, where X is a random quantity and $H \neq \emptyset$ is a subset of Ω. A conditional event $E|H$ is identified with $\mathbf{1}_E|H$.

Let $\mathcal{H} \subseteq \mathcal{A}^0$ be a set closed under finite unions, where $\mathcal{A}^0 = \mathcal{A} \setminus \{\emptyset\}$.

Definition 2. *A* **conditional prevision** $\mathbf{P}(\cdot|\cdot)$ *defined on $\mathbb{L}(\Omega, \mathcal{A}) \times \mathcal{H}$ is a real function satisfying the following conditions:*

(P1) $\mathbf{P}(\cdot|H)$ *is a linear functional on $\mathbb{L}(\Omega, \mathcal{A})$, for every $H \in \mathcal{H}$;*

(P2) $\inf_{\omega \in H} X(\omega) \leq \mathbf{P}(X|H) \leq \sup_{\omega \in H} X(\omega)$, *for every $X \in \mathbb{L}(\Omega, \mathcal{A})$ and $H \in \mathcal{H}$;*

(P3) $\mathbf{P}(X\mathbf{1}_K|H) = \mathbf{P}(X|H \cap K) \cdot \mathbf{P}(\mathbf{1}_K|H)$, *for every $H, H \cap K \in \mathcal{H}$ and $X, \mathbf{1}_K \in \mathbb{L}(\Omega, \mathcal{A})$.*

In particular, if $\Omega \in \mathcal{H}$ the function $\mathbf{P}(\cdot) = \mathbf{P}(\cdot|\Omega)$ is simply called a *prevision*: if $\mathcal{H}$ reduces to $\{\Omega\}$ then **(P3)** is vacuously satisfied. By following the terminology of [31], $\mathbf{P}(\cdot|\cdot)$ is said a *full* conditional prevision on $\mathcal{A}$ if it is defined on $\mathbb{L}(\Omega, \mathcal{A}) \times \mathcal{A}^0$.

The following axioms for a *conditional probability* (in the sense of de Finetti-Dubins [26, 31], see also [22, 49]) can be deduced from **(P1)**–**(P3)** restricting the domain of $\mathbf{P}(\cdot|\cdot)$ to $\mathbb{I}(\Omega, \mathcal{A}) \times \mathcal{H}$ and considering it as a function $P(\cdot|\cdot)$ on $\mathcal{A} \times \mathcal{H}$:

(C1) $P(E|H) = P(E \cap H|H)$, for every $E \in \mathcal{A}$ and $H \in \mathcal{H}$;

(C2) $P(\cdot|H)$ is a finitely additive probability on $\mathcal{A}$, for every $H \in \mathcal{H}$;

(C3) $P(E \cap F|H) = P(E|H) \cdot P(F|E \cap H)$, for every $H, E \cap H \in \mathcal{H}$ and $E, F \in \mathcal{A}$.

Recall that a conditional probability in the sense of Rényi [49] is obtained requiring countable additivity instead of finite additivity in **(C2)**.

A conditional probability $P(\cdot|\cdot)$ is said *full on* $\mathcal{A}$ (or a *f.c.p. on* $\mathcal{A}$ for short) when $\mathcal{H} = \mathcal{A}^0$.

It is well-known (see, e.g., [47]) that every conditional prevision $\mathbf{P}(\cdot|\cdot)$ defined on $\mathbb{L}(\Omega, \mathcal{A}) \times \mathcal{H}$ is completely characterized by its restriction on $\mathbb{I}(\Omega, \mathcal{A}) \times \mathcal{H}$ as

$$\mathbf{P}(X|H) = \int X(\omega) P(\mathrm{d}\omega|H), \tag{1}$$

where the right-side integral is of Stieltjes type [6].

2.1. Coherent (lower) conditional previsions in de Finetti-Williams' theory

In [37, 40] a betting scheme notion of *coherence* for a conditional prevision assessment $\mathbf{P}(\cdot|\cdot)$ defined on a set $\mathbb{G} \subseteq \mathbb{L}(\Omega) \times \wp(\Omega)^0$ of conditional random quantities has been introduced (for an equivalent formulation see also [47, 64]):

Definition 3. *Let $\mathbb{G} = \{X_i|H_i\}_{i \in I}$ be a set of conditional random quantities. A function $\mathbf{P} : \mathbb{G} \to \mathbb{R}$ is a* **coherent conditional prevision** *if and only if, for every $n \in \mathbb{N}$, every $X_{i_1}|H_{i_1}, \ldots, X_{i_n}|H_{i_n} \in \mathbb{G}$ and every real numbers $s_1, \ldots, s_n$, the random gain*

$$G = \sum_{j=1}^{n} s_j \mathbf{1}_{H_{i_j}} (X_{i_j} - P(X_{i_j}|H_{i_j})),$$

satisfies the following inequalities

$$\inf_{\omega \in H_0^0} G(\omega) \leq 0 \leq \sup_{\omega \in H_0^0} G(\omega),$$

where $H_0^0 = \bigcup_{j=1}^{n} H_{i_j}$.

If $\mathbb{G}$ is a set of conditional events, then $\mathbf{P}(\cdot|\cdot)$ is simply denoted as $P(\cdot|\cdot)$ and is said a *coherent conditional probability.*

Every coherent conditional prevision can be extended to every superset of conditional random quantities by the following Theorem 1, proved in [47, 64], which is the conditional version of the *fundamental theorem for previsions* [26].

Theorem 1. *Let $\mathbb{G}$ and $\mathbb{G}'$ be arbitrary sets of conditional random quantities with $\mathbb{G} \subset \mathbb{G}'$ and $\mathbf{P} : \mathbb{G} \to \mathbb{R}$. Then, there exists a coherent conditional prevision $\tilde{\mathbf{P}}(\cdot|\cdot)$ on $\mathbb{G}'$ such that $\tilde{\mathbf{P}}_{|\mathbb{G}} = \mathbf{P}$ if and only if $\mathbf{P}$ is a coherent conditional prevision on $\mathbb{G}$. Moreover, if $\mathbb{G}' = \mathbb{G} \cup \{X|H\}$ the coherent values for the conditional prevision of $X|H$ range in a closed interval $\mathbb{I}_{X|H} = [\underline{\mathbf{P}}(X|H), \overline{\mathbf{P}}(X|H)]$.*

As proven in [47, 64], a $\mathbf{P}$ on $\mathbb{G}$ is a coherent conditional prevision if and only if it can be extended to a conditional prevision (see Definition 2) on a superset

$\mathbb{G}' = \mathbb{L}(\Omega, \mathcal{A}) \times \mathcal{H}$, where $\mathcal{A}$ is an algebra and $\mathcal{H} \subseteq \mathcal{A}^0$ is closed under finite unions.

The interval $\mathbb{I}_{X|H}$ in Theorem 1 can be computed in terms of finite subfamilies of $\mathbb{G}$ as $\bigcap \left\{ \mathbb{I}^{\mathbb{F}}_{X|H} : \mathbb{F} \subseteq \mathbb{G}, \operatorname{card} \mathbb{F} < \aleph_0 \right\}$, where the closed interval $\mathbb{I}^{\mathbb{F}}_{X|H} = [\underline{\mathbf{P}}^{\mathbb{F}}(X|H), \overline{\mathbf{P}}^{\mathbb{F}}(X|H)]$ is obtained extending $\mathbf{P}_{|\mathbb{F}}$ on $\mathbb{F} \cup \{X|H\}$ (see, e.g., [60]). Thus, it holds

$$\begin{aligned}
\underline{\mathbf{P}}(X|H) &= \sup \left\{ \underline{\mathbf{P}}^{\mathbb{F}}(X|H) : \mathbb{F} \subseteq \mathbb{G}, \operatorname{card} \mathbb{F} < \aleph_0 \right\}, \\
\overline{\mathbf{P}}(X|H) &= \inf \left\{ \overline{\mathbf{P}}^{\mathbb{F}}(X|H) : \mathbb{F} \subseteq \mathbb{G}, \operatorname{card} \mathbb{F} < \aleph_0 \right\}.
\end{aligned}$$

For the explicit computation in terms of linear programming of the bounds of $\mathbb{I}^{\mathbb{F}}_{E|H}$ in the case where $\mathbb{G}'$ reduces to a set of conditional events see, for instance, [7, 12, 19, 63].

The class $\mathcal{P} = \{\tilde{\mathbf{P}}(\cdot|\cdot)\}$ of coherent extensions of a coherent conditional prevision in Theorem 1 is a non-empty compact subset of $\mathbb{R}^{\mathbb{G}'}$ endowed with the product topology and determines the lower and upper envelopes $\underline{\mathbf{P}} = \min \mathcal{P}$ and $\overline{\mathbf{P}} = \max \mathcal{P}$, which are said *coherent lower* and *upper conditional previsions*.

In general, coherent lower and upper conditional previsions can be defined without starting from a coherent conditional prevision [64]:

Definition 4. *A function $\underline{\mathbf{P}}(\cdot|\cdot)$ [$\overline{\mathbf{P}}(\cdot|\cdot)$] on a set $\mathbb{G} \subseteq \mathbb{L}(\Omega) \times \wp(\Omega)^0$ of conditional random quantities is said a* **coherent lower conditional prevision** *[***coherent upper conditional prevision***] if there exists a class $\mathcal{P} = \{\tilde{\mathbf{P}}(\cdot|\cdot)\}$ of coherent conditional previsions on $\mathbb{G}$ such that $\underline{\mathbf{P}} = \inf \mathcal{P}$ [$\overline{\mathbf{P}} = \sup \mathcal{P}$].*

If $\mathbb{G} = \mathbb{I}(\Omega, \mathcal{A}) \times \mathcal{H}$, the functions $\underline{\mathbf{P}}(\cdot|\cdot)$ and $\overline{\mathbf{P}}(\cdot|\cdot)$ are denoted as $\underline{P}(\cdot|\cdot)$ and $\overline{P}(\cdot|\cdot)$ and are simply said *lower* and *upper conditional probabilities*, moreover, if $\Omega \in \mathcal{H}$, then the functions $\underline{P}(\cdot) = \underline{P}(\cdot|\Omega)$ and $\overline{P}(\cdot) = \overline{P}(\cdot|\Omega)$ are simply called *lower* and *upper probabilities*.

2.2. n-monotone capacities and Choquet integration

A *(normalized) n-monotone capacity*, for $n \geq 2$, on an algebra $\mathcal{A}$ (see, e.g., [13]) is a function $\varphi : \mathcal{A} \to [0,1]$ such that $\varphi(\emptyset) = 0$, $\varphi(\Omega) = 1$, and every $A_1, \ldots, A_n \in \mathcal{A}$,

$$\varphi\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \varphi\left(\bigcap_{i \in I} A_i\right).$$

In particular, φ is *totally monotone* if it is n-monotone for every $n \geq 2$.

Every n-monotone capacity induces a *core* [51], i.e., a non-empty closed convex set of finitely additive probabilities on $\mathcal{A}$

$$\mathcal{P}_{\varphi} = \{\tilde{\pi} : \tilde{\pi} \text{ is a finitely additive probability on } \mathcal{A}, \varphi \leq \tilde{\pi}\}, \tag{2}$$

such that $\varphi = \min \mathcal{P}_{\varphi}$.

The *inner measure* φ_* on the complete atomic algebra $\langle\mathcal{A}\rangle^*$ induced by a n-monotone capacity φ on $\mathcal{A}$ is defined for every $E \in \langle\mathcal{A}\rangle^*$ as

$$\varphi_*(E) = \sup\{\varphi(B) \,:\, B \subseteq E, B \in \mathcal{A}\}.$$

Notice that, $\langle\mathcal{A}\rangle^* \subseteq \wp(\Omega)$, where the inclusion is possibly strict, thus φ_* on $\langle\mathcal{A}\rangle^*$ is actually the restriction of the inner measure induced by φ on the whole $\wp(\Omega)$. In [11, 13, 24, 61] it is proved that if φ is n-monotone, then also φ_* is, so the inner measure induced by a finitely additive probability is always totally monotone.

Let $\mathcal{L} = \{H_i\}_{i\in I}$ be a *partition* of Ω and $\mathcal{A}_\mathcal{L}$ an algebra such that $\langle\mathcal{L}\rangle \subseteq \mathcal{A}_\mathcal{L} \subseteq \langle\mathcal{L}\rangle^*$. Obviously, since both $\langle\mathcal{L}\rangle$ and $\langle\mathcal{L}\rangle^*$ are atomic with set of atoms $\mathcal{L}$, the same holds for $\mathcal{A}_\mathcal{L}$. Moreover, we always have $\langle\mathcal{A}_\mathcal{L}\rangle^* = \langle\mathcal{L}\rangle^*$, while $\mathcal{A}_\mathcal{L} = \langle\mathcal{L}\rangle = \langle\mathcal{L}\rangle^*$ whenever $\mathcal{L}$ is finite.

In the rest of the paper we will be mainly concerned with the integration of real-valued functions defined on $\mathcal{L}$. We recall that every function $X : \mathcal{L} \to \mathbb{R}$ can be identified with the random quantity $X : \Omega \to \mathbb{R}$ such that $X(\omega) = X(H_i)$ for every $\omega \in H_i \in \mathcal{L}$, and vice versa. Thus, for $t \in \mathbb{R}$, we have

$$(X \geq t) = \bigcup\{H_i \in \mathcal{L} \,:\, X(H_i) \geq t\} = \{\omega \in \Omega \,:\, X(\omega) \geq t\},$$

and this, in turn, allows to define $\mathcal{A}_\mathcal{L}$-continuity for $X : \mathcal{L} \to \mathbb{R}$ as in Definition 1.

By Definition 1 it immediately follows that every bounded function $X : \mathcal{L} \to \mathbb{R}$ is $\langle\mathcal{L}\rangle^*$-continuous as, for every $t \in \mathbb{R}$, $(X \geq t) \in \langle\mathcal{L}\rangle^*$ since $\langle\mathcal{L}\rangle^*$ is closed under arbitrary unions.

Given a n-monotone capacity φ on $\mathcal{A}_\mathcal{L}$ with associated inner measure φ_* on $\langle\mathcal{L}\rangle^*$, the *Choquet integral* of an $\mathcal{A}_\mathcal{L}$-continuous $X : \mathcal{L} \to \mathbb{R}$ (see, e.g., [27]) with respect to φ is defined as

$$\oint X(H_i)\varphi(\mathrm{d}H_i) = \int_{-\infty}^{0} [\varphi_*(X \geq t) - 1]\mathrm{d}t + \int_{0}^{+\infty} \varphi_*(X \geq t)\mathrm{d}t, \tag{3}$$

where the integrals on the right side are usual Riemann integrals. It follows that $\oint X(\omega)\varphi(\mathrm{d}\omega) = \oint X(H_i)\varphi(\mathrm{d}H_i)$, thus we can simply write $\oint X\mathrm{d}\varphi$.

Recall that if φ is finitely additive, then $\oint X\mathrm{d}\varphi = \int X\mathrm{d}\varphi$, where the latter denotes a Stieltjes integral. Moreover, for any n-monotone φ it holds (see, e.g., [51])

$$\oint X(H_i)\varphi(\mathrm{d}H_i) = \min\left\{\int X(H_i)\tilde{\pi}(\mathrm{d}H_i) \,:\, \tilde{\pi} \in \mathcal{P}_\varphi\right\}.$$

Remark 1. *Since every bounded function $X : \mathcal{L} \to \mathbb{R}$ is $\langle\mathcal{L}\rangle^*$-continuous, given a n-monotone capacity $\varphi : \mathcal{A}_\mathcal{L} \to [0, 1]$ and taking its inner measure φ_* on $\langle\mathcal{L}\rangle^*$, then $\oint X(H_i)\varphi_*(\mathrm{d}H_i)$ can always be computed but, generally, $\oint X(H_i)\varphi(\mathrm{d}H_i)$ could not be computed if X is not $\mathcal{A}_\mathcal{L}$-continuous. In particular, if φ is finitely additive we can always compute the Choquet integral $\oint X(H_i)\varphi_*(\mathrm{d}H_i)$ but possibly not the Stieltjes integral $\int X(H_i)\varphi(\mathrm{d}H_i)$. We stress that, since $X : \mathcal{L} \to \mathbb{R}$ is $\langle\mathcal{L}\rangle^*$-continuous, then it is sufficient to consider the inner measure φ_* only on $\langle\mathcal{L}\rangle^*$ (instead of on the whole $\wp(\Omega)$).*

3. Conglomerability and disintegrability

Consider a partition $\mathcal{L} = \{H_i\}_{i\in I}$ of Ω, which is assumed to be fixed throughout the rest of this section. In the seminal paper by Dubins [31], a *strategy* is introduced as a function $\kappa : \mathbb{L}(\Omega) \times \mathcal{L} \to \mathbb{R}$ satisfying the following conditions, for every $H_i \in \mathcal{L}$:

(S1) $\kappa(\mathbf{1}_{H_i}|H_i) = 1$;

(S2) $\kappa(\cdot|H_i)$ is a prevision on $\mathbb{L}(\Omega)$.

In the same paper, Dubins considers a prevision $\mathbf{P}(\cdot)$ on $\mathbb{L}(\Omega)$ and defines it to be $\mathcal{L}$*-conglomerable* with respect to a strategy κ on $\mathbb{L}(\Omega) \times \mathcal{L}$ if, for every $X \in \mathbb{L}(\Omega)$,

$$\kappa(X|H_i) \geq 0 \text{ for every } H_i \in \mathcal{L} \Longrightarrow \mathbf{P}(X) \geq 0. \tag{4}$$

Theorem 1 in [31] states that $\mathcal{L}$-conglomerability for $\mathbf{P}(\cdot)$ on $\mathbb{L}(\Omega)$ with respect to a strategy κ on $\mathbb{L}(\Omega) \times \mathcal{L}$ is equivalent to its $\mathcal{L}$*-disintegrability* with respect to κ, i.e., $\mathbf{P}(\cdot)$ satisfies condition (4) for every $X \in \mathbb{L}(\Omega)$, if and only if it also satisfies

$$\mathbf{P}(X) = \int \kappa(X|H_i)\pi(\mathrm{d}H_i), \tag{5}$$

where the finitely additive probability $\pi : \langle\mathcal{L}\rangle^* \to [0,1]$ is defined, for every $E \in \langle\mathcal{L}\rangle^*$, as $\pi(E) = \mathbf{P}(\mathbf{1}_E)$.

Remark 2. *For every fixed $H_i \in \mathcal{L}$, the function $\kappa(\cdot|H_i)$ is a prevision on $\mathbb{L}(\Omega)$, therefore it can be evaluated on every bounded random quantity $X : \Omega \to \mathbb{R}$. On the other hand, for every fixed $X \in \mathbb{L}(\Omega)$, $\kappa(X|\cdot)$ is plainly a $\langle\mathcal{L}\rangle^*$-continuous function defined on $\mathcal{L}$, so it can be integrated (in the Sietljes or, equivalently, in the Choquet sense) with respect to any finitely additive probability π on $\langle\mathcal{L}\rangle^*$.*

In particular, in the proof of Theorem 1 in [31] (see also [4, 5]), condition (4) is shown to be equivalent, for every $X \in \mathbb{L}(\Omega)$, to the following condition

$$\inf_{H_i\in\mathcal{L}} \kappa(X|H_i) \leq \mathbf{P}(X) \leq \sup_{H_i\in\mathcal{L}} \kappa(X|H_i), \tag{6}$$

which, in turn, is equivalent, for every $X \in \mathbb{L}(\Omega)$ and $B \in \langle\mathcal{L}\rangle^*$, to

$$\pi(B) \inf_{H_i\subseteq B} \kappa(X|H_i) \leq \mathbf{P}(X\mathbf{1}_B) \leq \pi(B) \sup_{H_i\subseteq B} \kappa(X|H_i). \tag{7}$$

Let us stress that the equivalence between conditions (4), (5), (6) and (7) essentially relies on the fact that $\mathbf{P}(\cdot)$ is defined on the set $\mathbb{L}(\Omega)$ of all bounded random quantities. Indeed, when $\mathbf{P}(\cdot)$ is a coherent prevision defined on a subset $\mathbb{K} \subseteq \mathbb{L}(\Omega)$, $\mathcal{L}$-conglomerability could not imply $\mathcal{L}$-disintegrability as κ could not be integrable with respect to π, see [4, 5].

Remark 3. *Condition (7) is actually stronger than (4) and (6) (see Example 3.3 in [4]) when $\mathbf{P}$ is defined on a proper subset $\mathbb{K} \subset \mathbb{L}(\Omega)$. Then, condition (7) should be taken as the definition of $\mathcal{L}$-conglomerability (in particular when $\mathbb{K} = \mathbb{I}(\Omega, \mathcal{A})$) if one wishes equivalence (under integrability of κ with respect to π) with $\mathcal{L}$-disintegrability.*

The properties of $\mathcal{L}$-conglomerability and $\mathcal{L}$-disintegrability are particularly meaningful for probabilities due to their important role in Bayesian statistics. Actually, Bayesian statistics mainly refers (see, e.g., [3]) to strategies as functions defined on $\mathcal{A} \times \mathcal{L}$ instead of on $\mathbb{L}(\Omega) \times \mathcal{L}$, where $\mathcal{A}$ is an algebra containing $\mathcal{L}$. For this, we restrict to these functions in the following subsection.

3.1. Conglomerability: a comparison of de Finetti and Dubins notions

Let $\mathcal{A}$ be an algebra containing a partition $\mathcal{L}$ and denote $\mathcal{A}_{\mathcal{L}} = \mathcal{A} \cap \langle \mathcal{L} \rangle^*$.

Starting from the the original work by Dubins [31], the term strategy is used interchangeably both referring to previsions and to probabilities relying on the context for a distinction. Here we use two different symbols κ and σ to avoid any misunderstanding.

In this case a *strategy* is any map $\sigma : \mathcal{A} \times \mathcal{L} \to [0,1]$ satisfying the following conditions, for every $H_i \in \mathcal{L}$:

(S1') $\sigma(H_i|H_i) = 1$;

(S2') $\sigma(\cdot|H_i)$ is a finitely additive probability on $\mathcal{A}$.

Conditions **(S1')** and **(S2')** imply, for every $F|H_i \in \mathcal{A} \times \mathcal{L}$,

$$\sigma(F|H_i) = \sigma(F \cap H_i|H_i).$$

Previously, $\mathcal{L}$-conglomerability has been introduced for previsions, even though historically the concept of $\mathcal{L}$-conglomerability was originally introduced by de Finetti for probabilities [25].

A finitely additive probability $P(\cdot)$ on $\mathcal{A}$ is *dF-$\mathcal{L}$-conglomerable* with respect to a strategy σ on $\mathcal{A} \times \mathcal{L}$ if the global assessment $\{P, \sigma\}$ is coherent and for every $F \in \mathcal{A}$,

$$\inf_{H_i \in \mathcal{L}} \sigma(F|H_i) \leq P(F) \leq \sup_{H_i \in \mathcal{L}} \sigma(F|H_i). \tag{8}$$

As already acknowledged by Dubins in [31], dF-$\mathcal{L}$-conglomerability is weaker than Dubins' notion of $\mathcal{L}$-conglomerability for previsions, which, in the case of probability, can be reformulated as follows, taking into account Remark 3.

A finitely additive probability $P(\cdot)$ on $\mathcal{A}$ is *$\mathcal{L}$-conglomerable* with respect to a strategy σ on $\mathcal{A} \times \mathcal{L}$ if, for every $F \in \mathcal{A}$ and $B \in \mathcal{A}_{\mathcal{L}}$,

$$\pi(B) \inf_{H_i \subseteq B} \sigma(F|H_i) \leq P(F \cap B) \leq \pi(B) \sup_{H_i \subseteq B} \sigma(F|H_i), \tag{9}$$

where $\pi = P_{|\mathcal{A}_{\mathcal{L}}}$. Notice that, if the probability $P(\cdot)$ and the strategy σ satisfy (9), then $\{P, \sigma\}$ is automatically coherent by Corollary 2.6 in [5].

As follows by Theorem 1.6 in [3], if $\sigma(F|\cdot)$ is integrable with respect to π, for every $F \in \mathcal{A}$, then $\mathcal{L}$-conglomerability with respect to σ reduces to $\mathcal{L}$-*disintegrability* with respect to σ, i.e., for every $F \in \mathcal{A}$ it holds

$$P(F) = \int \sigma(F|H_i)\pi(\mathrm{d}H_i). \tag{10}$$

In other terms, the quoted theorem establishes that $\mathcal{L}$-disintegrability of $P(\cdot)$ on $\mathcal{A}$ with respect to a strategy σ on $\mathcal{A} \times \mathcal{L}$ is equivalent to its $\mathcal{L}$-conglomerability with respect to σ plus the integrability of $\sigma(F|\cdot)$, for every $F \in \mathcal{A}$, with respect to π.

It is well-known that a probability $P(\cdot)$ on $\mathcal{A}$ can fail dF-$\mathcal{L}$-conglomerability (see [25]) and so $\mathcal{L}$-conglomerability, nevertheless, there are examples showing that a $P(\cdot)$ can be dF-$\mathcal{L}$-conglomerable but not $\mathcal{L}$-conglomerable [4]. The issue of non-dF-$\mathcal{L}$-conglomerability is studied in depth in [50, 53], where it is stated that a merely finitely additive probability admits a countable partition $\mathcal{L}$ where dF-$\mathcal{L}$-conglomerability fails.

3.2. Disintegrability and Kolmogorovian conditioning

In order to highlight the relationship between Definition 3 and the Kolmogorovian notion of conditioning we recall the following definition due to [8].

Definition 5. *Let $\mathcal{B}$ and $\mathcal{D}$ be σ-algebras of subsets of Ω with $\mathcal{D} \subseteq \mathcal{B}$ and $P : \mathcal{B} \to [0,1]$ a countably additive probability. A* **conditional (probability) distribution** *given $\mathcal{D}$ for P is a function $R : \Omega \times \mathcal{B} \to \mathbb{R}$ satisfying the following conditions:*

(R1) *$R(\cdot, B)$ is $\mathcal{D}$-measurable, for every $B \in \mathcal{B}$;*

(R2) *$P(B \cap D) = \int R(\omega, B)\mathbf{1}_D(\omega)P(\mathrm{d}\omega)$, for every $B \in \mathcal{B}$ and $D \in \mathcal{D}$ (where the integral is of Lebesgue type).*

A conditional distribution R given $\mathcal{D}$ for P is said **regular** *if*

(R3) *$R(\omega, \cdot)$ is a countably additive probability on $\mathcal{B}$, for every $\omega \in \Omega$.*

A regular conditional distribution R given $\mathcal{D}$ for P is said **proper** *if*

(R4) *$R(\omega, D) = 1$, for every $\omega \in D$ with $D \in \mathcal{D}$.*

In the definition due to Kolmogorov [39] a *conditional probability distribution* is introduced via Radon-Nikodym derivatives as a function $R(\cdot, \cdot)$ satisfying conditions **(R1)**–**(R2)**, while **(R3)** is proven to hold only almost surely. It is well-known that for particular choices of $\mathcal{B}$ and $\mathcal{D}$ it can happen that no function $R(\cdot, \cdot)$ satisfying **(R1)**–**(R3)** can exist [28], or, even if there are functions satisfying **(R1)**–**(R3)**, it can be that none of them satisfies **(R4)** [8, 9].

Theorem 1 in [8] states that if the σ-algebra $\mathcal{B}$ is countably generated (i.e., it holds $\mathcal{B} = \langle \mathcal{G} \rangle^\sigma$ where $\mathcal{G} \subseteq \mathcal{B}$ is countable) and the sub-σ-algebra $\mathcal{D}$ is not countably generated, then no proper regular conditional distribution given $\mathcal{D}$ for P can exist. The result is independent of the choice of P. In particular, if Ω is a Borel subset of $\mathbb{R}$ and $\mathcal{B}$ is the corresponding Borel σ-algebra, then $\mathcal{B}$ is countably generated, while, Corollary 4.5.10 in [55] implies that every proper sub-σ-algebra $\mathcal{D}$ of $\mathcal{B}$ containing the singletons is not countably generated. Thus, for such Ω, $\mathcal{B}$ and $\mathcal{D}$, Theorem 1 in [8] implies that no proper regular conditional distribution given $\mathcal{D}$ for P can exist (the case of $\Omega = [0, 1]$ and P equal to the Lebesgue measure is studied in Theorem 1 in [29]).

Then, the notion of conditioning due to de Finetti-Dubins and that due to Kolmogorov are not directly comparable in general. In order to make a comparison we need to consider a function $R(\cdot, \cdot)$ satisfying **(R1)**–**(R4)** and take an atomic sub-σ-algebra $\mathcal{D} \subseteq \mathcal{B}$ with set of atoms forming a partition $\mathcal{L}$. Such a function $R(\cdot, \cdot)$ (if it exists) is consistent in the Kolmogorovian setting and gives also rise, together with P, to a coherent conditional probability [4, 5]. Indeed, conditions **(R1)**–**(R4)** imply that, for every $B \in \mathcal{B}$, $R(\cdot, B)$ is constant on the elements of $\mathcal{L}$, moreover, setting, for every $B|H_i \in \mathcal{B} \times \mathcal{L}$, $\sigma(B|H_i) = R(\omega, B)$ for $\omega \in H_i$, we get a strategy σ on $\mathcal{B} \times \mathcal{L}$ and the assessment $\{P, \sigma\}$ is a coherent conditional probability.

Remark 4. *Under the above requirements, condition* **(R2)** *reduces to impose the $\mathcal{L}$-disintegrability of P, i.e., for every $B \in \mathcal{B}$, $P(B)$ can be recovered "averaging" $R(\cdot, B)$ with respect to $P_{|\mathcal{D}}$. This highlights that disintegrability is fundamental in order to create a bridge between the two notions of conditioning.*

Example 1. *Let $\mathbb{N} = \{1, 2, \ldots\}$, $\Omega = (0, 1]$ and consider the partition $\mathcal{L} = \left\{H_i = \left(\frac{1}{2^i}, \frac{1}{2^{i-1}}\right]\right\}_{i \in \mathbb{N}}$. Take $\mathcal{B}$ equal to the Borel σ-algebra on Ω and $\mathcal{D} = \langle \mathcal{L} \rangle^\sigma$, thus $\mathcal{D}$ is an atomic sub-σ-algebra of $\mathcal{B}$ and has set of atoms $\mathcal{L}$. Let P be the Lebesgue measure on $\mathcal{B}$ and define $R : \Omega \times \mathcal{B} \to \mathbb{R}$, for every $B \in \mathcal{B}$, as*

$$R(\omega, B) = \frac{P(B \cap H_i)}{P(H_i)}, \quad \text{for every } \omega \in H_i \in \mathcal{L},$$

which is well-defined since, for every $i \in \mathbb{N}$, $P(H_i) = \frac{1}{2^i}$. We need to verify that R is a proper regular conditional distribution given $\mathcal{D}$ for P.

Condition **(R1)**. *For every $B \in \mathcal{B}$, $R(\cdot, B)$ is bounded and constant on the elements of $\mathcal{L}$, so it can be identified with a function on $\mathcal{L}$. Moreover, being $\mathcal{D}$ a σ-algebra generated by a countable partition $\mathcal{L}$ it holds $\mathcal{D} = \langle \mathcal{L} \rangle^\sigma = \langle \mathcal{L} \rangle^*$, thus $R(\cdot, B)$ is trivially $\mathcal{D}$-measurable as $(R(\cdot, B) \geq t) \in \langle \mathcal{L} \rangle^*$, for every $t \in \mathbb{R}$.*

Condition **(R2)**. *For every $B \in \mathcal{B}$ and $D \in \mathcal{D}$, it holds $D = \bigcup_{i \in I} H_i$ with $I \subseteq \mathbb{N}$, thus*

$$\begin{aligned} P(B \cap D) &= P\left(B \cap \bigcup_{i \in I} H_i\right) = P\left(\bigcup_{i \in I} (B \cap H_i)\right) = \sum_{i \in I} P(B \cap H_i) \\ &= \sum_{i \in I} \left(\int R(\omega, B) \mathbf{1}_{H_i}(\omega) P(\mathrm{d}\omega)\right) = \int R(\omega, B) \mathbf{1}_D(\omega) P(\mathrm{d}\omega). \end{aligned}$$

Condition **(R3)**. *For every $\omega \in \Omega$ there exists a unique $H_i \in \mathcal{L}$ such that $\omega \in H_i$, so we have that $R(\omega, \cdot) = \frac{P(\cdot \cap H_i)}{P(H_i)}$ which is countably additive since P is countably additive.*

Condition **(R4)**. *For every $\omega \in \mathcal{D}$ with $D \in \mathcal{D}$ there exists a unique $H_i \in \mathcal{L}$ such that $\omega \in H_i$ which is such that $H_i \subseteq D$, and this implies*

$$R(\omega, D) = \frac{P(D \cap H_i)}{P(H_i)} = \frac{P(H_i)}{P(H_i)} = 1.$$

Since R is a proper regular conditional distribution given $\mathcal{D}$ for P, then setting, for every $B|H_i \in \mathcal{B}\times\mathcal{L}$, $\sigma(B|H_i) = R(\omega, B)$ for $\omega \in H_i$, we get a strategy σ on $\mathcal{B} \times \mathcal{L}$ and the assessment $\{P, \sigma\}$ is a coherent conditional probability. Moreover, P can be recovered through $\mathcal{L}$-disintegrability using the restriction $\pi = P_{|\mathcal{D}}$ as, for every $B \in \mathcal{B}$, it holds

$$P(B) = \int R(\omega, B)\pi(\mathrm{d}\omega) = \int \sigma(B|H_i)\pi(\mathrm{d}H_i).$$

3.3. Walley's coherent conditional lower previsions

The notion of conditioning for lower previsions due to Walley relies on a partition $\mathcal{L} = \{H_i\}_{i\in I}$ of Ω, which is assumed to be fixed.

A *lower conditional prevision* in the sense of Walley [62] is a set of real functions $\{\underline{\kappa}(\cdot|H_i) : H_i \in \mathcal{L}\}$ each one defined on $\mathbb{L}(\Omega)$ that can be globally regarded as a function $\underline{\kappa}(\cdot|\cdot)$ on $\mathbb{L}(\Omega) \times \mathcal{L}$. The function $\underline{\kappa}(\cdot|\cdot)$ is said *separately coherent* if for every $H_i \in \mathcal{L}$:

(W1) $\underline{\kappa}(\mathbf{1}_{H_i}|H_i) = 1$;

(W2) $\underline{\kappa}(\cdot|H_i)$ is a lower prevision on $\mathbb{L}(\Omega)$.

It can be easily seen that, if $\underline{\kappa}(\cdot|H_i)$ reduces to a prevision for every $H_i \in \mathcal{L}$, then a separately coherent lower conditional prevision exactly coincides with the notion of strategy introduced in the beginning of Section 3. In what follows, $\underline{\kappa}$ is always assumed to be separately coherent.

Consider now a lower prevision $\underline{\mathbf{P}}(\cdot)$ on $\mathbb{L}(\Omega)$. Walley defines the pair $\{\underline{\mathbf{P}}, \underline{\kappa}\}$ to be *W-coherent*[1] if and only if both the following conditions hold

(GBR) $\underline{\mathbf{P}}(\mathbf{1}_{H_i}(X - \underline{\kappa}(X|H_i))) = 0$, for every $X \in \mathbb{L}(\Omega)$ and $H_i \in \mathcal{L}$;

(CNG) $\underline{\mathbf{P}}\left(\sum_{H_i\in\mathcal{L}} (\mathbf{1}_{H_i}(X - \underline{\kappa}(X|H_i)))\right) \geq 0.$

The condition **(GBR)** is said *Generalized Bayesian Rule*, while condition **(CNG)** is a form of $\mathcal{L}$-conglomerability for lower conditional previsions.

In the particular case $\underline{\kappa}(\cdot|H_i)$ reduces to a prevision for every $H_i \in \mathcal{L}$, and so we write κ in place of $\underline{\kappa}$, then the pair $\{\underline{\mathbf{P}}, \kappa\}$ is W-coherent [62] if and only if, for every $X \in \mathbb{L}(\Omega)$,

$$\underline{\mathbf{P}}(X) = \underline{\mathbf{P}}(\kappa(X|\mathcal{L})), \tag{11}$$

where $\kappa(X|\mathcal{L})$ is the random quantity defined as $\kappa(X|\mathcal{L})(\omega) = \kappa(X|H_i)$ for every $\omega \in H_i \in \mathcal{L}$.

[1] Here, W-coherence stands for Walley coherence and not for Williams coherence as in other papers on the topic.

Moreover, if also $\underline{\mathbf{P}}(\cdot)$ is a prevision on $\mathbb{L}(\Omega)$, and so we write $\mathbf{P}$ in place of $\underline{\mathbf{P}}$, then the pair $\{\mathbf{P}, \kappa\}$ is W-coherent if and only if, denoting with π the restriction of $\mathbf{P}$ on $\langle\mathcal{L}\rangle^*$,

$$\mathbf{P}(X) = \mathbf{P}(\kappa(X|\mathcal{L})) = \int \kappa(X|\mathcal{L})(\omega)\pi(\mathrm{d}\omega) = \int \kappa(X|H_i)\pi(\mathrm{d}H_i), \tag{12}$$

that is if and only if $\mathbf{P}(\cdot)$ is $\mathcal{L}$-disintegrable (or, equivalently, $\mathcal{L}$-conglomerable) in the sense of Dubins with respect to κ.

For further recent references concerning W-coherence and related topics involving conglomerability, see [29, 42, 43, 44].

4. Coherent extensions of a strategy and a prior probability

The notion of conglomerability taken into account from now on is always the one of Dubins restricted to events, according to formula (9).

Let $\mathcal{L} = \{H_i\}_{i\in I}$ and $\mathcal{E} = \{E_j\}_{j\in J}$ be two partitions of Ω, with I, J arbitrary index sets, and consider the algebras $\mathcal{A}_\mathcal{L}$ and $\mathcal{A}_\mathcal{E}$ such that $\langle\mathcal{L}\rangle \subseteq \mathcal{A}_\mathcal{L} \subseteq \langle\mathcal{L}\rangle^*$ and $\langle\mathcal{E}\rangle \subseteq \mathcal{A}_\mathcal{E} \subseteq \langle\mathcal{E}\rangle^*$. Consider an algebra $\mathcal{A}$ such that $\langle\mathcal{A}_\mathcal{L}\cup\mathcal{A}_\mathcal{E}\rangle \subseteq \mathcal{A} \subseteq \langle\mathcal{A}_\mathcal{L}\cup\mathcal{A}_\mathcal{E}\rangle^*$. The partitions $\mathcal{L}$ and $\mathcal{E}$ play the roles of the sets of mutually exclusive and exhaustive “hypotheses” and “evidences” in the Bayesian jargon, respectively.

The generality of the above formulation is twofold: *(i)* it is possible to take σ-algebras in a way to cover classical Bayesian analysis situations without losing the mathematical generality needed, for example, in game theory or economical applications; *(ii)* it allows to consider complete atomic algebras in a way to remove measurability restrictions. Notice that the events of the partitions $\mathcal{L}$ and $\mathcal{E}$ (and so the corresponding algebras) can be linked by logical relations.

In the standard Bayesian setting, a prior probability π is assessed on the algebra $\mathcal{A}_\mathcal{L}$ and a strategy σ is given on $\mathcal{A}\times\mathcal{L}$, whose restriction $\lambda = \sigma_{|\mathcal{A}_\mathcal{E}\times\mathcal{L}}$, is usually referred to as *statistical model* [3, 57]. Being the restriction of a strategy, λ is such that, for every $H_i \in \mathcal{L}$:

(L1) $\lambda(B|H_i) = 0$ if $B \cap H_i = \emptyset$ and $\lambda(B|H_i) = 1$ if $B \cap H_i = H_i$, for every $B \in \mathcal{A}_\mathcal{E}$;

(L2) $\lambda(\cdot|H_i)$ is a finitely additive probability on $\mathcal{A}_\mathcal{E}$.

The previous properties **(L1)** and **(L2)** are an immediate inheritance of the definition of strategy and, actually, completely characterize a statistical model λ on $\mathcal{A}_\mathcal{E}\times\mathcal{L}$, in the sense that they guarantee its extendibility to a strategy on $\mathcal{A}\times\mathcal{L}$.

In general, given a statistical model λ there can exist possibly infinite strategies extending it to $\mathcal{A}\times\mathcal{L}$. Nevertheless, in the case $\mathcal{A} = \langle\mathcal{A}_\mathcal{L}\cup\mathcal{A}_\mathcal{E}\rangle$, a statistical model λ extends uniquely to a strategy σ on $\mathcal{A}\times\mathcal{L}$ as proven in the following proposition.

Proposition 1. *Let λ be a statistical model on $\mathcal{A}_\mathcal{E}\times\mathcal{L}$ and $\mathcal{A} = \langle\mathcal{A}_\mathcal{L}\cup\mathcal{A}_\mathcal{E}\rangle$, then there exists a unique strategy σ on $\mathcal{A}\times\mathcal{L}$ such that $\sigma_{|\mathcal{A}_\mathcal{E}\times\mathcal{L}} = \lambda$.*

Proof. Every $F \in \mathcal{A}$ is such that $F = \bigcup_{s=1}^{m} \bigcap_{t=1}^{n_s} A_{s_t}$, with $A_{s_t} \in \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}}$. For $H_i \in \mathcal{L}$ it holds

$$F \cap H_i = \left(\bigcup_{s=1}^{m} \bigcap_{t=1}^{n_s} A_{s_t} \right) \cap H_i = \bigcup_{s=1}^{m} \left(\left(\bigcap_{t=1}^{n_s} A_{s_t} \right) \cap H_i \right).$$

Define the index set $S = \{s \in \{1, \ldots, m\} : (\bigcap_{t=1}^{n_s} A_{s_t}) \cap H_i \neq \emptyset\}$, and for each $s \in S$ define the index set $T_s = \{t \in \{1, \ldots, n_s\} : A_{s_t} \in \mathcal{A}_{\mathcal{E}}\}$. This implies that the event $F_{H_i} = \bigcup_{s \in S} \bigcap_{t \in T_s} A_{s_t}$ belongs to $\mathcal{A}_{\mathcal{E}}$ and is such that $F \cap H_i = F_{H_i} \cap H_i$, where $F_{H_i} = \emptyset$ if $S = \emptyset$ and $\bigcap_{t \in T_s} A_{s_t} = \emptyset$ if $T_s = \emptyset$.

Let σ be a strategy extending on $\mathcal{A} \times \mathcal{L}$ the statistical model λ defined on $\mathcal{A}_{\mathcal{E}} \times \mathcal{L}$. For $F|H_i \in \mathcal{A} \times \mathcal{L}$ it must be $\sigma(F|H_i) = \sigma(F \cap H_i|H_i) = \sigma(F_{H_i} \cap H_i|H_i) = \sigma(F_{H_i}|H_i) = \lambda(F_{H_i}|H_i)$, i.e., σ is uniquely determined by λ. □

A finitely additive probability $\tilde{P} : \mathcal{A} \to [0, 1]$ is said a *joint probability consistent with* $\{\pi, \sigma\}$ if $\tilde{P}_{|\mathcal{A}_{\mathcal{L}}} = \pi$ and $\{\tilde{P}, \sigma\}$ is a coherent conditional probability on $\mathcal{G}' = \mathcal{A} \times (\{\Omega\} \cup \mathcal{L})$. The joint probability on $\mathcal{A}$ consistent with $\{\pi, \sigma\}$ is generally not unique, so, a first aim is to characterize the whole set of consistent joint probabilities $\mathcal{P}^{\mathbf{j}} = \{\tilde{P}(\cdot)\}$, which is a non-empty convex compact subset of $[0, 1]^{\mathcal{A}}$ endowed with the product topology, whose envelopes are $\underline{P}^{\mathbf{j}} = \min \mathcal{P}^{\mathbf{j}}$ and $\overline{P}^{\mathbf{j}} = \max \mathcal{P}^{\mathbf{j}}$.

Among the joint probabilities in $\mathcal{P}^{\mathbf{j}}$ we can focus on those meeting some analytical properties such as $\mathcal{L}$-conglomerability and $\mathcal{L}$-disintegrability with respect to σ. When the function $\sigma(F|\cdot)$ is Stieltjes integrable with respect to π, for every $F \in \mathcal{A}$ (see [3, 31, 48]), the function $P^{\mathbf{jd}}$ defined, for every $F \in \mathcal{A}$, as

$$P^{\mathbf{jd}}(F) = \int \sigma(F|H_i)\pi(\mathrm{d}H_i),$$

belongs to $\mathcal{P}^{\mathbf{j}}$ and is $\mathcal{L}$-disintegrable (with respect to σ). However, as claimed in [3], this is just one of the possible joint probabilities on $\mathcal{A}$ consistent with $\{\pi, \sigma\}$. We refer to this particular element of $\mathcal{P}^{\mathbf{j}}$ as *$\mathcal{L}$-disintegrable joint probability.*

Denote with $\mathcal{P}^{\mathbf{jc}} \subseteq \mathcal{P}^{\mathbf{j}}$ the subset of $\mathcal{L}$-conglomerable (with respect to σ) joint probabilities on $\mathcal{A}$ consistent with $\{\pi, \sigma\}$, whose topological structure is investigated in the following result.

Theorem 2. *The set $\mathcal{P}^{\mathbf{jc}}$ is a non-empty convex compact subset of $[0, 1]^{\mathcal{A}}$ endowed with the product topology.*

Proof. We first prove that $\mathcal{P}^{\mathbf{jc}}$ is not empty. For every $F \in \mathcal{A}$, $\sigma(F|\cdot)$ is trivially a $\langle\mathcal{L}\rangle^*$-continuous function on $\mathcal{L}$. Let $\mathcal{P}_{\pi_*}$ be the core of the inner measure π_* induced by π on $\langle\mathcal{L}\rangle^*$, defined as in (2). For every $\tilde{\pi} \in \mathcal{P}_{\pi_*}$, define the function $\tilde{P}$ setting for every $F \in \mathcal{A}$

$$\tilde{P}(F) = \int \sigma(F|H_i)\tilde{\pi}(\mathrm{d}H_i),$$

which is a finitely additive joint probability on $\mathcal{A}$ consistent with $\{\pi, \sigma\}$. Let $\mathcal{B} = \langle \mathcal{A} \cup \langle\mathcal{L}\rangle^* \rangle$ and ρ be any strategy on $\mathcal{B} \times \mathcal{L}$ extending σ. For every $\tilde{\pi} \in \mathcal{P}_{\varphi_*}$, the assessment $\{\tilde{\pi}, \rho\}$ is coherent and $\rho(F|\cdot)$ is trivially $\langle\mathcal{L}\rangle^*$-continuous, for every $F \in \mathcal{B}$. So, the $\mathcal{L}$-disintegrable joint probability $\tilde{P}^{\mathbf{jd}}$ on $\mathcal{B}$ consistent with $\{\tilde{\pi}, \rho\}$ is an extension of $\tilde{P}$ and is also $\mathcal{L}$-conglomerable. In turn, this implies $\tilde{P}$ belongs to $\mathcal{P}^{\mathbf{jc}}$ and so $\mathcal{P}^{\mathbf{jc}}$ is not empty.

To prove $\mathcal{P}^{\mathbf{jc}}$ is compact, it is sufficient to consider a net $(\tilde{P}_\alpha)_\alpha$ in $\mathcal{P}^{\mathbf{jc}}$ converging pointwise to $\tilde{P}$. The compactness of $\mathcal{P}^{\mathbf{j}}$ implies that $\tilde{P}$ is an element of $\mathcal{P}^{\mathbf{j}}$, moreover, since the pointwise limits of nets preserve non-strict inequalities and both π and σ are fixed, it follows that, for every $F \in \mathcal{A}$ and $B \in \mathcal{A}_\mathcal{L}$,

$$\pi(B) \inf_{H_i \subseteq B} \sigma(F|H_i) \leq \tilde{P}(F \cap B) \leq \pi(B) \sup_{H_i \subseteq B} \sigma(F|H_i),$$

which implies that $\tilde{P}$ is also an element of $\mathcal{P}^{\mathbf{jc}}$ and the claim follows. Convexity of $\mathcal{P}^{\mathbf{jc}}$ is trivial. $\square$

Let $\underline{P}^{\mathbf{j}} = \min \mathcal{P}^{\mathbf{j}}$, $\overline{P}^{\mathbf{j}} = \max \mathcal{P}^{\mathbf{j}}$, $\underline{P}^{\mathbf{jc}} = \min \mathcal{P}^{\mathbf{jc}}$ and $\overline{P}^{\mathbf{jc}} = \max \mathcal{P}^{\mathbf{jc}}$, be the envelopes of the sets $\mathcal{P}^{\mathbf{j}}$ and $\mathcal{P}^{\mathbf{jc}}$, respectively. Notice that when the function $\sigma(F|\cdot)$ is Stieltjes integrable with respect to π, for every $F \in \mathcal{A}$, the class $\mathcal{P}^{\mathbf{jc}}$ collapses in the singleton $\{P^{\mathbf{jd}}\}$ and so we have $\underline{P}^{\mathbf{jc}} = \overline{P}^{\mathbf{jc}} = P^{\mathbf{jd}}$.

The following theorem provides a characterization of the lower envelopes $\underline{P}^{\mathbf{j}}$ and $\underline{P}^{\mathbf{jc}}$.

Theorem 3. *For any finitely additive prior probability π on $\mathcal{A}_\mathcal{L}$ and strategy σ on $\mathcal{A} \times \mathcal{L}$, the following statements hold:*

(i) the lower envelope $\underline{P}^{\mathbf{j}}$ is such that, for every $F \in \mathcal{A}$, it holds

$$\underline{P}^{\mathbf{j}}(F) = \sup_{\mathcal{L}^\mathcal{F} \subseteq \mathcal{A}_\mathcal{L}} \left\{ \sum_{h=1}^n \sigma(F|H_{i_h})\pi(H_{i_h}) + \sum_{B_k \subseteq F} \pi(B_k) \right\},$$

where $\mathcal{L}^\mathcal{F} = \{H_{i_h}\}_{h=1}^n \cup \{B_k\}_{k=1}^t \subseteq \mathcal{A}_\mathcal{L}$ is a finite partition of Ω;

(ii) the lower envelope $\underline{P}^{\mathbf{jc}}$ is such that, for every $F \in \mathcal{A}$, it holds

$$\underline{P}^{\mathbf{jc}}(F) = \oint \sigma(F|H_i)\pi_*(\mathrm{d}H_i)$$

where π_* is the inner measure induced by π of $\langle\mathcal{L}\rangle^*$.

Proof. Condition *(i)*. The proof is trivial if $\mathcal{L}$ is finite. Thus suppose $\operatorname{card} \mathcal{L} \geq \aleph_0$ and let $\mathcal{G} = (\mathcal{A}_\mathcal{L} \times \{\Omega\}) \cup (\mathcal{A} \times \mathcal{L})$. By Theorem 1, for every $F \in \mathcal{A}$, the interval of coherent extensions $\mathbb{I}_F = [\underline{P}^{\mathbf{j}}(F), \overline{P}^{\mathbf{j}}(F)]$ can be computed in terms of finite subfamilies of $\mathcal{G}$. Since for every $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{G}$ with $\operatorname{card} \mathcal{F}_2 < \aleph_0$ one has $\underline{P}^{\mathbf{j}^{\mathcal{F}_1}}(F) \leq \underline{P}^{\mathbf{j}^{\mathcal{F}_2}}(F)$, we can restrict to finite subfamilies of $\mathcal{G}$ containing a set of the form $(\mathcal{L}^\mathcal{F} \times \{\Omega\}) \cup (\{F\} \times \{H_{i_h}\}_{h=1}^n)$, where $\mathcal{L}^\mathcal{F} = \{H_{i_h}\}_{h=1}^n \cup \{B_k\}_{k=1}^t$ is a finite partition of Ω contained in $\mathcal{A}_\mathcal{L}$. Indeed, every finite subfamily can be suitably enlarged in order to contain a set of this form. For such a set $\mathcal{F}$ we have $\underline{P}^{\mathbf{j}^\mathcal{F}}(F) = \sum_{h=1}^n \sigma(F|H_{i_h})\pi(H_{i_h}) + \sum_{B_k \subseteq F} \pi(B_k)$ and so the thesis follows.

Condition *(ii)*. For every $F \in \mathcal{A}$, $\sigma(F|\cdot)$ is trivially a $\langle\mathcal{L}\rangle^*$-continuous function on $\mathcal{L}$. Let $\mathcal{P}_{\pi_*}$ be the core of the inner measure π_* induced by π on $\langle\mathcal{L}\rangle^*$, defined as in (2). By the proof of Theorem 2 and Proposition 3 in [51], for every $F \in \mathcal{A}$, the lower envelope of the set $\mathcal{P}^{\mathbf{jc}}$ is given by

$$\underline{P}^{\mathbf{jc}}(F) = \min\left\{\int \sigma(F|H_i)\tilde{\pi}(\mathrm{d}H_i) \,:\, \tilde{\pi} \in \mathcal{P}_{\pi_*}\right\} = \oint \sigma(F|H_i)\pi_*(\mathrm{d}H_i).$$

□

If $\mathcal{L}$ is countable and π is countably additive on $\mathcal{A}_{\mathcal{L}}$, then for every $F \in \mathcal{A}$ it holds $\underline{P}^{\mathbf{j}}(F) = \underline{P}^{\mathbf{jc}}(F) = P^{\mathbf{jd}}(F) = \sum_{i=1}^{\infty} \sigma(F|H_i)\pi(H_i)$, i.e., $\underline{P}^{\mathbf{j}}$ is a finitely additive probability on $\mathcal{A}$, moreover, if $\sigma(\cdot|H_i)$ is countably additive on $\mathcal{A}$ for every $H_i \in \mathcal{L}$, then $\underline{P}^{\mathbf{j}}$ is countably additive. On the contrary, if $\mathrm{card}\,\mathcal{L} > \aleph_0$, then the countable additivity of π does not imply the unicity of the joint probability in $\mathcal{P}^{\mathbf{j}}$ as showed by the following example. The same example also shows that the lower bounds $\underline{P}^{\mathbf{j}}$ and $\underline{P}^{\mathbf{jc}}$ do not generally coincide.

Example 2. *Let $\Omega = [0,1] \times \{1,2\}$ and define $H_i = \{i\} \times \{1,2\}$, for $i \in [0,1]$, and $E_j = [0,1] \times \{j\}$, for $j = 1,2$. Thus it holds $H_i \cap E_j \neq \emptyset$, for every i, j. Let $\mathcal{L} = \{H_i\}_{i\in[0,1]}$, $\mathcal{E} = \{E_1, E_2\}$, and take $\mathcal{A}_{\mathcal{L}}$ isomorphic to the Borel σ-algebra on $[0,1]$, $\mathcal{A}_{\mathcal{E}} = \langle\mathcal{E}\rangle$, $\mathcal{A} = \langle\mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}}\rangle$. Let π on $\mathcal{A}_{\mathcal{L}}$ be the Lebesgue measure on $[0,1]$.*

By the well-known construction of a non-measurable subset of $[0,1]$ due to Vitali, for every $\epsilon \in (0,1)$, it is possible to find a non-measurable subset of $[0,1]$ with inner Lebesgue measure 0 and outer Lebesgue measure ϵ. Let V_ϵ be an element of $\langle\mathcal{L}\rangle^*$ isomorphic to the above Vitali set, for which one has $\pi_*(V_\epsilon^c) = 1-\epsilon$. The corresponding indicator $\mathbf{1}_{V_\epsilon^c}$ can be identified with a function on $\mathcal{L}$ with values in $\{0,1\}$ which is $\langle\mathcal{L}\rangle^*$-continuous.

Consider the statistical model λ on $\mathcal{A}_{\mathcal{E}} \times \mathcal{L}$ such that, for $i \in [0,1]$, $\lambda(\cdot|H_i)$ is a probability on $\mathcal{A}_{\mathcal{E}}$ such that

$$\lambda(E_1|H_i) = \mathbf{1}_{V_\epsilon^c}(H_i) \quad \textit{and} \quad \lambda(E_2|H_i) = 1 - \lambda(E_1|H_i).$$

By Proposition 1 the statistical model λ extends uniquely to a strategy σ on $\mathcal{A} \times \mathcal{L}$.

For every finite partition $\mathcal{L}^{\mathcal{F}} = \{H_{i_h}\}_{h=1}^n \cup \{B_k\}_{k=1}^t$ contained in $\mathcal{A}_{\mathcal{L}}$ it holds $\sigma(E_j|H_{i_h})\pi(H_{i_h}) = 0$, for $h = 1,\ldots,n$. Furthermore, for $k = 1,\ldots,t$ and $j = 1,2$, it holds $\emptyset \neq E_j \cap B_k \neq E_j$, thus by Theorem 3 we have $\underline{P}^{\mathbf{j}}(E_j) = 0$ and $\overline{P}^{\mathbf{j}}(E_j) = 1$ for $j = 1,2$.

On the converse, Theorem 3 implies

$$\begin{aligned} \underline{P}^{\mathbf{jc}}(E_1) &= \oint \sigma(E_1|H_i)\pi_*(\mathrm{d}H_i) = \oint \mathbf{1}_{V_\epsilon^c}(H_i)\pi_*(\mathrm{d}H_i) \\ &= \pi_*(V_\epsilon^c) = 1-\epsilon > 0 = \underline{P}^{\mathbf{j}}(E_1). \end{aligned}$$

Except for the trivial case where $\sigma(F|\cdot)$ is Stieltjes integrable with respect to π, for every $F \in \mathcal{A}$, it is well-known (see, e.g., [25, 50]) that there are joint probabilities consistent with $\{\pi, \sigma\}$ that are not dF-$\mathcal{L}$-conglomerable, thus they are neither $\mathcal{L}$-conglomerable: this implies that the inclusion $\mathcal{P}^{\mathbf{jc}} \subseteq \mathcal{P}^{\mathbf{j}}$ can be strict. When $\mathcal{L}$ is finite obviously $\mathcal{P}^{\mathbf{jc}} = \mathcal{P}^{\mathbf{j}}$. The following example shows that under particular choices of π and σ, and the related algebras, it can happen $\mathcal{P}^{\mathbf{jc}} = \mathcal{P}^{\mathbf{j}}$ even for an infinite $\mathcal{L}$ and a finitely additive, but not countably additive, prior probability π.

Example 3. *Let $\Omega = \mathbb{N} \times \{1, 2\}$, $\mathcal{L} = \{H_i\}_{i\in\mathbb{N}}$ and $\mathcal{E} = \{E_1, E_2\}$ with $H_i = \{i\} \times \{1, 2\}$ and $E_j = \mathbb{N} \times \{j\}$, for every i, j. Thus it holds $H_i \cap E_j \neq \emptyset$, for every i, j. Take $\mathcal{A}_{\mathcal{L}} = \langle\mathcal{L}\rangle$, $\mathcal{A}_{\mathcal{E}} = \langle\mathcal{E}\rangle$ and $\mathcal{A} = \langle\mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}}\rangle$, thus $\mathcal{A}_{\mathcal{L}}$ is isomorphic to the algebra of finite-cofinite subsets of $\mathbb{N}$. Consider the finitely additive prior probability defined for $K \in \mathcal{A}_{\mathcal{L}}$ as*

$$\pi(K) = \begin{cases} 0 & \text{if } K = \bigcup_{i\in I_K} H_i \text{ and } \operatorname{card} I_K < \aleph_0, \\ 1 & \text{otherwise,} \end{cases}$$

and the statistical model on $\mathcal{A}_{\mathcal{E}} \times \mathcal{L}$ singled out for $i \in \mathbb{N}$ by

$$\lambda(E_1|H_i) = \begin{cases} 1 & \text{if } i \text{ is even,} \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \quad \lambda(E_2|H_i) = 1 - \lambda(E_1|H_i),$$

which extends uniquely to a strategy σ on $\mathcal{A} \times \mathcal{L}$ by Proposition 1.

Notice that $\sigma(E_1|H_i) = \mathbf{1}_A(H_i)$ and $\sigma(E_2|H_i) = \mathbf{1}_{A^c}(H_i)$ with $A = \bigcup_{i\in\mathbb{N}} H_{2i}$, thus none of them is Stieltjes integrable with respect to π which is defined on $\mathcal{A}_{\mathcal{L}}$. However, since A belongs to $\langle\mathcal{L}\rangle^*$ the corresponding indicator $\mathbf{1}_A$ can be identified with a function on $\mathcal{L}$ with values in $\{0, 1\}$ which is $\langle\mathcal{L}\rangle^*$-continuous.

It holds

$$\underline{P}^{\mathbf{jc}}(E_1) = \oint \mathbf{1}_A(H_i)\pi_*(\mathrm{d}H_i) = \pi_*(A) = 0,$$

and an analogous computation shows $\underline{P}^{\mathbf{jc}}(E_2) = 0$, thus $\overline{P}^{\mathbf{jc}}(E_1) = 1 - \underline{P}^{\mathbf{jc}}(E_2) = 1$. In turn, since $\underline{P}^{\mathbf{j}} \leq \underline{P}^{\mathbf{jc}} \leq \overline{P}^{\mathbf{jc}} \leq \overline{P}^{\mathbf{j}}$, it holds $\underline{P}^{\mathbf{j}}(E_j) = 0$ and $\overline{P}^{\mathbf{j}}(E_j) = 1$, for $j = 1, 2$, so we obtain the same bounds for E_j, for $j = 1, 2$, determined by the whole set of joint probabilities consistent with $\{\pi, \sigma\}$.

Actually, simple computations show that every joint probability in $\mathcal{P}^{\mathbf{j}}$ is $\mathcal{L}$-conglomerable, i.e., $\mathcal{P}^{\mathbf{jc}} = \mathcal{P}^{\mathbf{j}}$, so the envelopes (trivially) coincide on the whole $\mathcal{A}$. To see this, let $\tilde{P} \in \mathcal{P}^{\mathbf{j}}$. We need to show that for every $F \in \mathcal{A}$ and every $B \in \mathcal{A}_{\mathcal{L}}$, $\tilde{P}$ satisfies condition (9).

If $B = \bigcup_{i\in I_B} H_i$ with $\operatorname{card} I_B < \aleph_0$, then $\pi(B) = 0$ which implies $\tilde{P}(F \cap B) = 0$ since $\tilde{P}$ extends π, thus (9) holds.

Hence, suppose $B = \bigcup_{i\in I_B} H_i$ with $\operatorname{card} I_B = \aleph_0$, which implies $\pi(B) = 1$, from which condition (9) reduces to $\inf\limits_{H_i \subseteq B} \sigma(F|H_i) \leq \tilde{P}(F \cap B) \leq \sup\limits_{H_i \subseteq B} \sigma(F|H_i)$.

Case (a). If $F \in \mathcal{A}_{\mathcal{L}}$ with $F = \bigcup_{i\in I_F} H_i$ we distinguish two sub-cases.

Case (a.1). If $\operatorname{card} I_F < \aleph_0$, then $\tilde{P}(F \cap B) = \pi(F \cap B) = 0 = \inf\limits_{H_i \subseteq B} \sigma(F|H_i)$, since there exists $i \in I_B \setminus I_F$ such that $\sigma(F|H_i) = \sigma(F \cap H_i|H_i) = \sigma(\emptyset|H_i) = 0$, thus (9) holds.

Case (a.2). If $\operatorname{card} I_F = \aleph_0$*, then* $\tilde{P}(F\cap B) = \pi(F\cap B) = 1 = \sup_{H_i \subseteq B} \sigma(F|H_i)$*, since there exists* $i \in I_B \cap I_F$ *such that* $\sigma(F|H_i) = \sigma(F\cap H_i|H_i) = \sigma(H_i|H_i) = 1$*, thus (9) holds.*

Case (b). If $F \in \mathcal{A}_{\mathcal{E}} \setminus \mathcal{A}_{\mathcal{L}}$*, then the conclusion follows by the fact that* $\underline{P}^{\mathbf{j}}(E_j) = \underline{P}^{\mathbf{jc}}(E_j)$ *and* $\overline{P}^{\mathbf{j}}(E_j) = \overline{P}^{\mathbf{jc}}(E_j)$*, for* $j = 1, 2$*.*

Case (c). If $F \in \mathcal{A} \setminus (\mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}})$*, since* $\mathcal{A} = \langle \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}} \rangle$*, then by the proof of Proposition 1, for every* $i \in \mathbb{N}$*, there exists* $F_{H_i} \in \mathcal{A}_{\mathcal{E}}$ *such that* $F \cap H_i = F_{H_i} \cap H_i$*. This implies that* $\sigma(F|H_i) = \lambda(F_{H_i}|H_i)$*, for* $i \in \mathbb{N}$*, and so* $\sigma(F|\cdot)$ *ranges in* $\{0, 1\}$*. Moreover, we have* $F \cap B = \bigcup_{i\in I_B}(F_{H_i} \cap H_i)$*. Since* $\inf_{H_i\subseteq B} \sigma(F|H_i) \leq \sup_{H_i\subseteq B} \sigma(F|H_i)$*, we distinguish the following three sub-cases.*

Case (c.1). If $\inf_{H_i\subseteq B} \sigma(F|H_i) = 0$ *and* $\sup_{H_i\subseteq B} \sigma(F|H_i) = 1$ *then the conclusion is trivial.*

Case (c.2). If $\inf_{H_i\subseteq B} \sigma(F|H_i) = \sup_{H_i\subseteq B} \sigma(F|H_i) = 1$*, then for every even* $i \in I_B$ *it holds* $E_1 \subseteq F_{H_i}$*, and for every odd* $i \in I_B$ *it holds* $E_2 \subseteq F_{H_i}$*. Let* $I_B^1 = \{i \in I_B : E_1 = F_{H_i}\}$ *and* $I_B^2 = \{i \in I_B : E_2 = F_{H_i}\}$*. Since* $\mathcal{A}_{\mathcal{L}} = \langle \mathcal{L} \rangle$*,* $\mathcal{A}_{\mathcal{E}} = \langle \mathcal{E} \rangle$ *and* $\mathcal{A} = \langle \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}} \rangle = \langle \mathcal{L} \cup \mathcal{E} \rangle$*, then* $\operatorname{card} I_B^1 < \aleph_0$ *and* $\operatorname{card} I_B^2 < \aleph_0$*, so,* $\operatorname{card}(I_B \setminus (I_B^1 \cup I_B^2)) = \aleph_0$ *and, for* $i \in I_B \setminus (I_B^1 \cup I_B^2)$*,* $F_{H_i} = \Omega$*. Finally, since* $F \cap B = \bigcup_{i\in I_B}(F_{H_i} \cap H_i) = \bigcup_{i\in I_B^1\cup I_B^2}(F_{H_i} \cap H_i) \cup \bigcup_{i\in I_B\setminus(I_B^1\cup I_B^2)} H_i$ *we have* $1 = \pi\left(\bigcup_{i\in I_B\setminus(I_B^1\cup I_B^2)} H_i\right) \leq \tilde{P}(F \cap B)$ *and the conclusion follows.*

Case (c.3). If $\inf_{H_i\subseteq B} \sigma(F|H_i) = \sup_{H_i\subseteq B} \sigma(F|H_i) = 0$*, then for every even* $i \in I_B$ *it holds* $F_{H_i} \subseteq E_2$*, and for every odd* $i \in I_B$ *it holds* $F_{H_i} \subseteq E_1$*. Let* $I_B^1 = \{i \in I_B : F_{H_i} = E_2\}$ *and* $I_B^2 = \{i \in I_B : F_{H_i} = E_1\}$*. Since* $\mathcal{A}_{\mathcal{L}} = \langle \mathcal{L} \rangle$*,* $\mathcal{A}_{\mathcal{E}} = \langle \mathcal{E} \rangle$ *and* $\mathcal{A} = \langle \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}} \rangle = \langle \mathcal{L} \cup \mathcal{E} \rangle$*, then* $\operatorname{card} I_B^1 < \aleph_0$ *and* $\operatorname{card} I_B^2 < \aleph_0$*, so,* $\operatorname{card}(I_B \setminus (I_B^1 \cup I_B^2)) = \aleph_0$ *and, for* $i \in I_B \setminus (I_B^1 \cup I_B^2)$*,* $F_{H_i} = \emptyset$*. Finally, since* $F \cap B = \bigcup_{i\in I_B}(F_{H_i} \cap H_i) = \bigcup_{i\in I_B^1\cup I_B^2}(F_{H_i} \cap H_i)$ *we have* $0 = \pi\left(\bigcup_{i\in I_B^1\cup I_B^2} H_i\right) \geq \tilde{P}(F \cap B)$ *and the conclusion follows.*

Let us stress that $\underline{P}^{\mathbf{jc}}$ is generally not 2-monotone, as shown in the following example.

Example 4. *Let* $\Omega = \mathbb{N} \times \{1, 2, 3, 4\}$*,* $\mathcal{L} = \{H_i\}_{i\in\mathbb{N}}$ *and* $\mathcal{E} = \{E_1, E_2, E_3, E_4\}$ *with* $H_i = \{i\} \times \{1, 2, 3, 4\}$ *and* $E_j = \mathbb{N} \times \{j\}$*, for every* i, j*. This implies* $H_i \cap E_j \neq \emptyset$*, for every* i, j*. Consider* $\mathcal{A}_{\mathcal{L}} = \langle \mathcal{L} \rangle$*,* $\mathcal{A}_{\mathcal{E}} = \langle \mathcal{E} \rangle$*,* $\mathcal{A} = \langle \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}} \rangle$ *and take the statistical model on* $\mathcal{A}_{\mathcal{E}} \times \mathcal{L}$ *such that for* i *odd*

$$\lambda(E_1|H_i) = \lambda(E_2|H_i) = \lambda(E_3|H_i) = \frac{1}{6} \quad \textit{and} \quad \lambda(E_4|H_i) = \frac{1}{2},$$

and for i *even*

$$\lambda(E_1|H_i) = \lambda(E_3|H_i) = \frac{1}{2} \quad \textit{and} \quad \lambda(E_2|H_j) = \lambda(E_4|H_i) = 0,$$

which extends uniquely to a strategy σ on $\mathcal{A} \times \mathcal{L}$ by Proposition 1. Consider the events $A = E_1 \cup E_2$ and $B = E_2 \cup E_3$.

Taking the finitely additive prior probability defined for $K \in \mathcal{A}_{\mathcal{L}}$ as

$$\pi(K) = \begin{cases} 0 & \text{if } K = \bigcup_{i \in I_K} H_i \text{ and } \operatorname{card} I_K < \aleph_0, \\ 1 & \text{otherwise,} \end{cases}$$

it is easily seen that $\underline{P}^{\mathbf{jc}}(A) = \underline{P}^{\mathbf{jc}}(B) = \frac{1}{3}$, $\underline{P}^{\mathbf{jc}}(A \cup B) = \frac{1}{2}$ and $\underline{P}^{\mathbf{jc}}(A \cap B) = 0$, thus $\underline{P}^{\mathbf{jc}}(A \cup B) < \underline{P}^{\mathbf{jc}}(A) + \underline{P}^{\mathbf{jc}}(B) - \underline{P}^{\mathbf{jc}}(A \cap B)$, so 2-monotonicity fails.

Remark 5. *The previous results, in particular Theorem 3, are related to the literature on Walley's lower (conditional) previsions and to the ensuing notion of conglomerability (condition* **(CNG)***) and W-coherence recalled in Subsection 3.3 (see [62]).*

Consider the pair $\{\underline{P}^{\mathbf{jc}}, \sigma\}$, where $\underline{P}^{\mathbf{jc}}$ is the lower $\mathcal{L}$-conglomerable joint probability characterized in Theorem 3, corresponding to the prior probability π and the strategy σ. Since σ is coherent (in the sense of de Finetti-Williams) then it can be extended, generally not in a unique way, to a strategy κ on $\mathbb{L}(\Omega) \times \mathcal{L}$ by Theorem 1. Moreover, considering the inner measure π_* induced by π on $\langle \mathcal{L} \rangle^*$ and defining, for every $X \in \mathbb{L}(\Omega)$,

$$\underline{\mathbf{P}}^{\mathbf{jc}}(X) = \oint \kappa(X|H_i)\pi_*(\mathrm{d}H_i),$$

we get a lower prevision on $\mathbb{L}(\Omega)$ extending $\underline{P}^{\mathbf{jc}}$. In particular, for every $X \in \mathbb{L}(\Omega)$, defining $\kappa(X|\mathcal{L})(\omega) = \kappa(X|H_i)$ for every $\omega \in H_i \in \mathcal{L}$, we get a separately coherent conditional prevision in the sense of Walley.

Simple computations show that, for every $X \in \mathbb{L}(\Omega)$,

$$\underline{\mathbf{P}}^{\mathbf{jc}}(X) = \oint \kappa(X|H_i)\pi_*(\mathrm{d}H_i) = \oint \kappa(\kappa(X|\mathcal{L})|H_i)\pi_*(\mathrm{d}H_i) = \underline{\mathbf{P}}^{\mathbf{jc}}(\kappa(X|\mathcal{L})),$$

thus $\{\underline{\mathbf{P}}^{\mathbf{jc}}, \kappa\}$ is W-coherent (see Section 6.5.5 in [62]). Hence, the pair $\{\underline{P}^{\mathbf{jc}}, \sigma\}$ reveals to be a restriction of a W-coherent pair $\{\underline{\mathbf{P}}^{\mathbf{jc}}, \kappa\}$. This highlights a connection with Walley's theory. Let us stress that the previous construction holds for every strategy κ extending σ on $\mathbb{L}(\Omega) \times \mathcal{L}$.

For the construction above it is crucial to select a strategy κ, i.e., a linear separately coherent lower conditional prevision extending σ (in the sense of Walley). On the other hand, in order to deal with Walley's marginal extension problem of $\{\pi, \sigma\}$ (see Section 6.7 of [62]) one needs to look for a minimal separately coherent lower conditional prevision $\underline{\kappa}$ on $\mathbb{L}(\Omega) \times \mathcal{L}$ extending σ and a minimal lower prevision $\underline{\mathbf{R}}(\cdot)$ on $\mathbb{L}(\Omega)$ extending π such that $\{\underline{\mathbf{R}}, \underline{\kappa}\}$ is W-coherent. Thus, $\{\underline{\mathbf{P}}^{\mathbf{jc}}, \kappa\}$ is not necessarily a solution of Walley's marginal extension problem of $\{\pi, \sigma\}$.

Notice that in our setting we are taking a lower envelope of $\mathcal{L}$-conglomerable "precise" models and the conglomerability of the resulting lower envelope follows since κ is a fixed linear separately coherent lower conditional prevision. Hence, the approach differs from the one proposed in [62] where, working with a separately coherent lower conditional prevision $\underline{\kappa}$, a suitable conglomerability condition is asked directly on the lower envelope.

4.1. Full conditional probability extensions

Consider the set

$$\mathcal{Q} = \{\tilde{Q} \,:\, \tilde{Q} \text{ is a f.c.p. on } \mathcal{A} \text{ extending } \{\pi,\sigma\}\},$$

which is a non-empty compact subset of $[0,1]^{\mathcal{A}\times\mathcal{A}}$ endowed with the product topology, whose lower and upper envelopes are $\underline{Q} = \min \mathcal{Q}$ and $\overline{Q} = \max \mathcal{Q}$. Note that it holds $\underline{P}^{\mathbf{j}} = \underline{Q}_{|\mathcal{A}\times\{\Omega\}}$ and $\overline{P}^{\mathbf{j}} = \overline{Q}_{|\mathcal{A}\times\{\Omega\}}$.

We provide a characterization of $\underline{Q}$ relying on $\underline{P}^{\mathbf{j}}$, $\overline{P}^{\mathbf{j}}$ and the functions $L^{\mathbf{j}}$ and $U^{\mathbf{j}}$ defined for $F \in \mathcal{A}$ and $K \in \mathcal{A}^0$ as

$$\begin{aligned} L^{\mathbf{j}}(F,K) &= \min\left\{\tilde{P}(F\cap K) \,:\, \tilde{P} \in \mathcal{P}^{\mathbf{j}}, \tilde{P}(F^c \cap K) = \overline{P}^{\mathbf{j}}(F^c\cap K)\right\}, \\ U^{\mathbf{j}}(F,K) &= \max\left\{\tilde{P}(F\cap K) \,:\, \tilde{P} \in \mathcal{P}^{\mathbf{j}}, \tilde{P}(F^c \cap K) = \underline{P}^{\mathbf{j}}(F^c\cap K)\right\}, \end{aligned}$$

for which it holds $\underline{P}^{\mathbf{j}}(F\cap K) \le L^{\mathbf{j}}(F,K) \le U^{\mathbf{j}}(F,K) \le \overline{P}^{\mathbf{j}}(F\cap K)$.

The following theorem generalizes Theorem 7 in [16], in which $\mathcal{E}$ is assumed to be finite and $\mathcal{A} = \langle \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}}\rangle$.

Theorem 4. *The lower envelope $\underline{Q}(\cdot|\cdot)$ is such that, for every $F|K \in \mathcal{A}\times\mathcal{A}^0$, $\underline{Q}(F|K) = 1$ when $F\cap K = K$, and if $F \cap K \ne K$, then:*

(i) if $\underline{P}^{\mathbf{j}}(K) > 0$, then

$$\underline{Q}(F|K) = \min\left\{\frac{\underline{P}^{\mathbf{j}}(F\cap K)}{\underline{P}^{\mathbf{j}}(F\cap K) + U^{\mathbf{j}}(F^c,K)}, \frac{L^{\mathbf{j}}(F,K)}{L^{\mathbf{j}}(F,K) + \overline{P}^{\mathbf{j}}(F^c\cap K)}\right\};$$

(ii) if $\underline{P}^{\mathbf{j}}(K) = 0$, then

$$\underline{Q}(F|K) = \begin{cases} \displaystyle\min_{i \in I_1^{F|K}} \frac{\sigma(F\cap K|H_i)}{\sigma(K|H_i)} & \text{if } I_1^{F|K} \ne \emptyset = I_2^{F|K},\ \mathrm{card}\, I_1^{F|K} < \aleph_0 \\ & \text{and } \sigma(K|H_i) > 0 \text{ for all } i \in I_1^{F|K}, \\ 0 & \text{otherwise,} \end{cases}$$

where $I_1^{F|K} = \{i \in I \,:\, H_i \cap F\cap K \ne \emptyset \ne H_i \cap F^c \cap K\}$ and $I_2^{F|K} = \{i \in I \,:\, H_i \cap F \cap K = \emptyset \ne H_i \cap F^c \cap K\}$.

Proof. The statement is trivial if $F\cap K = K$ since in this case $\tilde{Q}(F|K) = 1$ for every $\tilde{Q} \in \mathcal{Q}$, for this suppose $F\cap K \ne K$.

To prove condition *(i)*, suppose $\underline{P}^{\mathbf{j}}(K) > 0$, which implies $\tilde{P}(K) > 0$ for every $\tilde{P} \in \mathcal{P}^{\mathbf{j}}$, and so $\underline{Q}(F|K) = \min\left\{\frac{\tilde{P}(F\cap K)}{\tilde{P}(F\cap K)+\tilde{P}(F^c\cap K)} \,:\, \tilde{P} \in \mathcal{P}^{\mathbf{j}}\right\}$. The conclusion follows since the real function $\frac{x}{x+y}$ is increasing in x and decreasing in y, so the minimum is attained in correspondence of $\frac{\underline{P}^{\mathbf{j}}(F\cap K)}{\underline{P}^{\mathbf{j}}(F\cap K)+U^{\mathbf{j}}(F^c,K)}$ or $\frac{L^{\mathbf{j}}(F,K)}{L^{\mathbf{j}}(F,K)+\overline{P}^{\mathbf{j}}(F^c\cap K)}$.

To prove condition *(ii)*, let $\mathcal{G} = (\mathcal{A}_{\mathcal{L}} \times \{\Omega\}) \cup (\mathcal{A} \times \mathcal{L})$ and assume $\underline{P}^{\mathbf{j}}(K) = 0$. By Theorem 1, for every $F|K \in \mathcal{A} \times \mathcal{A}^0$, the interval of coherent extensions $\mathbb{I}_{F|K} = [\underline{Q}(F|K), \overline{Q}(F|K)]$ can be computed in terms of finite subfamilies of $\mathcal{G}$.

Since for every $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{G}$ and $\operatorname{card} \mathcal{F}_2 < \aleph_0$ one has $\underline{Q}^{\mathcal{F}_1}(F|K) \leq \underline{Q}^{\mathcal{F}_2}(F|K)$, for arbitrary $I'_k \subseteq I_k^{F|K}$ with $\operatorname{card} I'_k < \aleph_0$, $k = 1, 2$, and $I' = I'_1 \cup I'_2$, we can restrict to finite subfamilies containing $(\mathcal{L}^{\mathcal{F}} \times \{\Omega\}) \cup (\mathcal{C}_{\{F,K\}} \times \{H_i\}_{i \in I'})$, where $\mathcal{L}^{\mathcal{F}} = \{H_i\}_{i \in I'} \cup \{B_k\}_{k=1}^{t}$ is a finite partition of Ω contained in $\mathcal{A}_{\mathcal{L}}$, and $\mathcal{C}_{\{F,K\}} = \{A_h\}_{h=1}^{m}$, with $m \leq 4$, is the set of atoms of the algebra generated by $\{F, K\}$. Indeed, every finite subfamily can be suitably enlarged in order to contain a set of this form.

For such a finite subfamily $\mathcal{F}$, let $\mathcal{C}_{\mathcal{F}} = \{C_1, \ldots, C_m\}$ be the set of atoms of the algebra generated by $\mathcal{L}^{\mathcal{F}} \cup \mathcal{C}_{\{F,K\}}$.

Let $\mathcal{C}_1 = \{C_r \in \mathcal{C}_{\mathcal{F}} : \underline{P}^{\mathbf{j}}(C_r) = 0\}$. As described in [12] (see also [20]) the lower bound $\underline{Q}^{\mathcal{F}}(F|K)$ can be explicitly computed by solving the optimization problem with non-negative unknowns x_r^1 for $C_r \in \mathcal{C}_1$,

$$\text{minimize} \left[\sum_{C_r \subseteq F \cap K} x_r^1 \right]$$

$$\begin{cases} x_r^1 = \sigma(A_h|H_i) \cdot \left(\displaystyle\sum_{C_s \subseteq H_i} x_s^1 \right) & \text{if } \sigma(K|H_i) > 0 \text{ and } \pi(H_i) = 0 \\ & \text{and } i \in I' \text{ and } C_r = A_h \cap H_i \in \mathcal{C}_1, \\ \displaystyle\sum_{C_r \subseteq K} x_r^1 = 1. & \end{cases}$$

Denote with ξ^1, whose r-th component is ξ_r^1, a solution of the previous system.

If $I_2^{F|K} \neq \emptyset$, we can restrict to finite subfamilies having $I'_2 \neq \emptyset$. In this case, the previous system has always a solution such that $\sum_{C_r \subseteq F \cap K} \xi_r^1 = 0$ and $\sum_{C_r \subseteq F^c \cap K} \xi_r^1 = 1$, which implies $\underline{Q}^{\mathcal{F}}(F|K) = 0$. Since every finite subfamily can be suitably enlarged to a finite subfamily having $I'_2 \neq \emptyset$, then $\underline{Q}(F|K) = 0$.

If $I_1^{F|K} = \emptyset$, in order to be $F \cap K \neq K$, it must be $I_2^{F|K} \neq \emptyset$ so we fall in the previous case. Hence, assume $I_1^{F|K} \neq \emptyset = I_2^{F|K}$.

If $I_1^{F|K} \neq \emptyset$ and $\operatorname{card} I_1^{F|K} \geq \aleph_0$, we can restrict to finite subfamilies having $I'_1 \neq \emptyset$, for which the previous system has always a solution such that $\sum_{C_r \subseteq F \cap K} \xi_r^1 = 0$ and $\sum_{C_r \subseteq F^c \cap K} \xi_r^1 = 1$, which implies $\underline{Q}^{\mathcal{F}}(F|K) = 0$. Since every finite subfamily can be suitably enlarged to a finite subfamily having $I'_1 \neq \emptyset$, then $\underline{Q}(F|K) = 0$.

Finally, if $I_1^{F|K} \neq \emptyset$ and $\operatorname{card} I_1^{F|K} < \aleph_0$, we can restrict to finite subfamilies having $I'_1 = I_1^{F|K}$ for which the minimum of the previous optimization problem is easily seen to be 0 if there is $i \in I'_1$ such that $\sigma(K|H_i) = 0$. Otherwise, the minimum is achieved in correspondence of those solutions such that $\sum_{C_r \subseteq K \cap H_i} \xi_r^1 = 1$ for $i \in I_1^{F|K}$, that implies $\sum_{C_r \subseteq F \cap K \cap H_i} \xi_r^1 = \frac{\sigma(F \cap K|H_i)}{\sigma(K|H_i)}$ for $i \in I_1^{F|K}$, and then $\underline{Q}^{\mathcal{F}}(F|K) = \min_{i \in I_1^{F|K}} \frac{\sigma(F \cap K|H_i)}{\sigma(K|H_i)}$. Since every finite subfamily can be suitably enlarged to a finite subfamily having $I_1' = I_1^{F|K}$, then $\underline{Q}(F|K) = \min_{i \in I_1^{F|K}} \frac{\sigma(F \cap K|H_i)}{\sigma(K|H_i)}$ if $\sigma(K|H_i) > 0$ for all $i \in I_1^{F|K}$ and 0 otherwise. $\square$

The following example, inspired by Example 2.1 in [46], shows an application of the previous theorem related to Bayesian inference.

Example 5. *Consider a finite population of unknown size and let Θ be the random relative frequency of a characteristic under study. For the range of Θ it is "natural" to assume $\mathbf{\Theta} = [0, 1] \cap \mathbb{Q}$, so let $\mathcal{L} = \{H_\theta = (\Theta = \theta) : \theta \in \mathbf{\Theta}\}$, and $\mathcal{A}_{\mathcal{L}} = \langle \mathcal{L} \rangle^*$.*

Assign a uniform distribution to Θ, specifying a prior probability π on $\mathcal{A}_{\mathcal{L}}$ such that $\pi(\Theta \in [0, \theta] \cap \mathbf{\Theta}) = \theta$, for $\theta \in \mathbf{\Theta}$. The probability π is only finitely additive since, for $\theta \in \mathbf{\Theta}$, $(\Theta = \theta) = \bigcap_{n \in \mathbb{N}} \left(\Theta \in \left(\theta - \frac{1}{n}, \theta\right] \cap \mathbf{\Theta}\right)$, thus $\pi(\Theta = \theta) = \lim_{n \in \mathbb{N}} \pi \left(\Theta \in \left(\theta - \frac{1}{n}, \theta\right] \cap \mathbf{\Theta}\right) = 0$.

Draw a sample with replacement of size n from the population, and let X be the number of individuals showing the characteristic under study, whose range is $\mathbf{X} = \{0, \ldots, n\}$. Let $\mathcal{E} = \{E_x = (X = x) : x \in \mathbf{X}\}$ for which it holds $H_0 \subseteq E_0$, $H_1 \subseteq E_n$, and $H_\theta \cap E_x \neq \emptyset$ for $\theta \in \mathbf{\Theta} \setminus \{0, 1\}$ and $x \in \mathbf{X}$. Take $\mathcal{A}_{\mathcal{E}} = \langle \mathcal{E} \rangle$ and let λ be the statistical model on $\mathcal{A}_{\mathcal{E}} \times \mathcal{L}$ such that $\lambda(X = x|\Theta = 0) = 1$ for $x = 0$ and 0 otherwise, $\lambda(X = x|\Theta = 1) = 1$ for $x = n$ and 0 otherwise, and for $\theta \in \mathbf{\Theta} \setminus \{0, 1\}$ and $x \in \mathbf{X}$,

$$\lambda(X = x|\Theta = \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x},$$

which uniquely extends to a strategy σ on $\mathcal{A} \times \mathcal{L}$, where $\mathcal{A} = \langle \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}} \rangle$, by Proposition 1.

Then, consider the conditional event $A|B$ with $A = (X = n)$ and $B = \left(\Theta \in \left\{\frac{i}{10} : i = 1, \ldots, 9\right\}\right)$. We have $\underline{P}^{\mathbf{j}}(B) = \pi(B) = 0$, $I_2^{A|B} = I_2^{A^c|B} = \emptyset$, and $I_1^{A|B} = I_1^{A^c|B} = \left\{\frac{i}{10} : i = 1, \ldots, 9\right\}$. Since it holds $\sigma\left(B|\Theta = \frac{i}{10}\right) = 1$, for $i = 1, \ldots, 9$, it follows $\underline{Q}(A|B) = \left(\frac{1}{10}\right)^n$, $\underline{Q}(A^c|B) = 1 - \left(\frac{9}{10}\right)^n$, and so $\overline{Q}(A|B) = \left(\frac{9}{10}\right)^n$, which implies that the coherent probability values of $A|B$ range in $\left[\left(\frac{1}{10}\right)^n, \left(\frac{9}{10}\right)^n\right]$.

Consider $A|C$ with $C = \left(\Theta \in \left\{\frac{1}{i} + \frac{1}{3} : i \geq 2\right\}\right)$. We have $\underline{P}^{\mathbf{j}}(C) = \pi(C) = 0$, $I_2^{A|C} = I_2^{A^c|C} = \emptyset$, and $I_1^{A|C} = I_1^{A^c|C} = \left\{\frac{1}{i} + \frac{1}{3} : i \geq 2\right\}$, so it holds $\underline{Q}(A|C) = \underline{Q}(A^c|C) = 0$ which implies that the coherent probability values of $A|C$ range in $[0, 1]$.

Take $D|E$ with $D = \left(X = 1, \Theta \notin \left\{\frac{1}{2}, \frac{1}{3}\right\}\right)$ and $E = \left(X = 1, \Theta \notin \left\{\frac{1}{2}, \frac{1}{3}\right\}\right) \cup \left(X = 2, \Theta \notin \left\{\frac{1}{2}\right\}\right)$. By Theorem 3 it follows $\underline{P}^{\mathbf{j}}(E) = 0$, moreover, since $I_2^{D^c|E} = \emptyset$, $I_2^{D|E} = \left\{\frac{1}{3}\right\}$ and $I_1^{D|E} = I_1^{D^c|E} = \mathbf{\Theta} \setminus \left\{0, \frac{1}{2}, \frac{1}{3}, 1\right\}$, it holds $\underline{Q}(D|E) =$

$Q(D^c|E) = 0$ *which implies that the coherent probability values of* $D|E$ *range in* $[0, 1]$.

For every $\tilde{P} \in \mathcal{P}^{\mathbf{jc}}$, the assessment $\{\tilde{P}, \sigma\}$ is coherent, thus it can be extended to a full conditional probability on $\mathcal{A}$ which is called $\mathcal{L}$-*conglomerable* full conditional probability on $\mathcal{A}$:

Definition 6. *A full conditional probability* $\tilde{Q}(\cdot|\cdot)$ *on* $\mathcal{A}$ *extending* $\{\pi, \sigma\}$ *is* **$\mathcal{L}$-conglomerable** *if, for every* $F \in \mathcal{A}$ *and every* $B \in \mathcal{A}_{\mathcal{L}}$*, it holds*

$$\pi(B) \inf_{H_i \subseteq B} \sigma(F|H_i) \leq \tilde{Q}(F \cap B|\Omega) \leq \pi(B) \sup_{H_i \subseteq B} \sigma(F|H_i). \tag{13}$$

In the case $\sigma(F|\cdot)$ is Stieltjes integrable with respect to π, for every $F \in \mathcal{A}$, then a $\mathcal{L}$-conglomerable $\tilde{Q}(\cdot|\cdot)$ can be expressed, for every $F \in \mathcal{A}$, as

$$\tilde{Q}(F|\Omega) = \int \sigma(F|H_i)\pi(\mathrm{d}H_i), \tag{14}$$

and is said $\mathcal{L}$-*disintegrable*. In particular, this happens if $\sigma(F|\cdot)$ is $\mathcal{A}_{\mathcal{L}}$-continuous for every $F \in \mathcal{A}$.

Consider the set

$$\mathcal{Q}^{\mathbf{c}} = \{\tilde{Q} \,:\, \tilde{Q} \text{ is a } \mathcal{L}\text{-conglomerable f.c.p. extending } \{\pi, \sigma\}\},$$

whose topological structure is an immediate consequence of coherence and Theorem 2.

Corollary 1. *The set* $\mathcal{Q}^{\mathbf{c}}$ *is a non-empty compact subset of* $[0, 1]^{\mathcal{A} \times \mathcal{A}^0}$ *endowed with the product topology.*

Let $\underline{Q}^{\mathbf{c}} = \min \mathcal{Q}^{\mathbf{c}}$ and $\overline{Q}^{\mathbf{c}} = \max \mathcal{Q}^{\mathbf{c}}$ be the envelopes of the set $\mathcal{Q}^{\mathbf{c}}$. Notice that $\underline{P}^{\mathbf{jc}} = \underline{Q}^{\mathbf{c}}_{|\mathcal{A} \times \{\Omega\}}$ and $\overline{P}^{\mathbf{jc}} = \overline{Q}^{\mathbf{c}}_{|\mathcal{A} \times \{\Omega\}}$.

The next result provides a characterization of $\underline{Q}^{\mathbf{c}}$ relying on $\underline{P}^{\mathbf{jc}}$, $\overline{P}^{\mathbf{jc}}$ and the functions $L^{\mathbf{jc}}$ and $U^{\mathbf{jc}}$ defined for $F \in \mathcal{A}$ and $K \in \mathcal{A}^0$ as

$$\begin{aligned}
L^{\mathbf{jc}}(F, K) &= \min\left\{\tilde{P}(F \cap K) \,:\, \tilde{P} \in \mathcal{P}^{\mathbf{jc}}, \tilde{P}(F^c \cap K) = \overline{P}^{\mathbf{jc}}(F^c \cap K)\right\}, \\
U^{\mathbf{jc}}(F, K) &= \max\left\{\tilde{P}(F \cap K) \,:\, \tilde{P} \in \mathcal{P}^{\mathbf{jc}}, \tilde{P}(F^c \cap K) = \underline{P}^{\mathbf{jc}}(F^c \cap K)\right\},
\end{aligned}$$

for which it holds $\underline{P}^{\mathbf{jc}}(F \cap K) \leq L^{\mathbf{jc}}(F, K) \leq U^{\mathbf{jc}}(F, K) \leq \overline{P}^{\mathbf{jc}}(F \cap K)$.

Theorem 5. *The lower envelope* $\underline{Q}^{\mathbf{c}}(\cdot|\cdot)$ *is such that, for every* $F|K \in \mathcal{A} \times \mathcal{A}^0$, $\underline{Q}^{\mathbf{c}}(F|K) = 1$ *when* $F \cap K = K$*, and if* $F \cap K \neq K$*, then:*

(i) if $\underline{P}^{\mathbf{jc}}(K) > 0$*, then*

$$\underline{Q}^{\mathbf{c}}(F|K) = \min\left\{\frac{\underline{P}^{\mathbf{jc}}(F \cap K)}{\underline{P}^{\mathbf{jc}}(F \cap K) + U^{\mathbf{jc}}(F^c, K)}, \frac{L^{\mathbf{jc}}(F, K)}{L^{\mathbf{jc}}(F, K) + \overline{P}^{\mathbf{jc}}(F^c \cap K)}\right\};$$

(ii) if $\underline{P}^{\mathbf{jc}}(K) = 0$, then $\underline{Q}^{\mathbf{c}}(F|K) = \underline{Q}(F|K)$ as defined in condition (ii) of Theorem 4.

Proof. If $F \cap K = K$, then by Theorem 4, $\underline{Q}(F|K) = 1$ implies $\underline{Q}^{\mathbf{c}}(F|K) = 1$. If $F \cap K \neq K$, the proof of condition *(i)* goes along the same line of the proof of condition *(i)* of Theorem 4 by replacing $\mathcal{P}^{\mathbf{j}}$ with $\mathcal{P}^{\mathbf{jc}}$. For condition *(ii)*, if $\underline{P}^{\mathbf{jc}}(K) = 0$ then $\underline{P}^{\mathbf{j}}(K) = 0$, moreover, $\underline{Q}^{\mathbf{c}}(\cdot|K) \geq \underline{Q}(\cdot|K)$. By coherence we have that $\underline{P}^{\mathbf{j}}_{|\mathcal{A}_{\mathcal{L}}} = \overline{P}^{\mathbf{j}}_{|\mathcal{A}_{\mathcal{L}}} = \pi$, moreover, for every $F \in \mathcal{A}$ and $H_i \in \mathcal{L}$, it holds $\underline{P}^{\mathbf{j}}(F \cap H_i) = \overline{P}^{\mathbf{j}}(F \cap H_i) = \sigma(F|H_i)\pi(H_i)$, thus every finitely additive probability $\tilde{P}$ on $\mathcal{A}$ such that $\underline{P}^{\mathbf{jc}} \leq \tilde{P} \leq \overline{P}^{\mathbf{jc}}$ belongs to $\mathcal{P}^{\mathbf{jc}}$. Since $\mathcal{L}$-conglomerability does not affect the conditional probability of conditional events $F|K$'s with null joint probability on K, the proof goes along the same line of the proof of condition *(ii)* of Theorem 4, taking $\underline{P}^{\mathbf{jc}}$ in place of $\underline{P}^{\mathbf{j}}$. This shows that $\underline{Q}^{\mathbf{c}}(\cdot|K) = \underline{Q}(\cdot|K)$. □

Theorem 5 shows that restricting to the subset $\mathcal{Q}^{\mathbf{c}}$ of $\mathcal{L}$-conglomerable full conditional probabilities extending $\{\pi, \sigma\}$, its lower envelope $\underline{Q}^{\mathbf{c}}$ inherits the $\mathcal{L}$-conglomerability constraint imposed on the elements of $\mathcal{Q}^{\mathbf{c}}$, only in correspondence of those conditional events $F|K$'s whose conditioning event K has positive lower $\mathcal{L}$-conglomerable joint probability. On the converse, the lower envelope $\underline{Q}^{\mathbf{c}}$ is determined just by coherence on the conditional events with conditioning event having zero lower $\mathcal{L}$-coglomerable joint probability, as follows by condition *(ii)* of Theorem 5.

If $\sigma(F|\cdot)$ is Stieltjes integrable with respect to π, for every $F \in \mathcal{A}_{\mathcal{L}}$, then condition *(i)* of Theorem 5, reduces to

$$\underline{Q}^{\mathbf{c}}(F|K) = \frac{P^{\mathbf{jd}}(F \cap K)}{P^{\mathbf{jd}}(K)}.$$

In particular, under the assumption of a σ such that $\sigma(F|\cdot)$ is Stieltjes integrable with respect to π, for every $F \in \mathcal{A}_{\mathcal{L}}$, the previous Theorem 5 generalizes Theorem 8 in [16], in which $\mathcal{E}$ is assumed to be finite and $\mathcal{A} = \langle \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}} \rangle$.

A simplification of condition *(i)* of Theorem 5 is obtained also in the case the functions on $\mathcal{L}$ defined as $X(\cdot) = \sigma(F \cap H|\cdot)$ and $(1 - Y(\cdot)) = (1 - \sigma(F^c \cap H|\cdot))$ are *comonotonic* (see, e.g., [27]), i.e., for every $H_h, H_k \in \mathcal{L}$,

$$[X(H_h) - X(H_k)] \cdot [(1 - Y(H_h)) - (1 - Y(H_k))] \geq 0, \tag{15}$$

as shown by the following proposition. In particular, this happens for all conditional events in $\mathcal{A}_{\mathcal{L}} \times \mathcal{A}^0_{\mathcal{E}}$ related to "posterior probabilities".

Proposition 2. *For every $F|K \in \mathcal{A} \times \mathcal{A}^0$ such that $F \cap K \neq K$ and $\underline{P}^{\mathbf{jc}}(K) > 0$, if $X(\cdot) = \sigma(F \cap H|\cdot)$ and $(1 - Y(\cdot)) = (1 - \sigma(F^c \cap H|\cdot))$ are comonotonic then*

$$\underline{Q}^{\mathbf{c}}(F|K) \quad = \quad \frac{\underline{P}^{\mathbf{jc}}(F \cap K)}{\underline{P}^{\mathbf{jc}}(F \cap K) + \overline{P}^{\mathbf{jc}}(F^c \cap K)}.$$

Proof. Consider the core $\mathcal{P}_{\pi_*}$ of the inner measure π_* induced by π on $\langle\mathcal{L}\rangle^*$, defined as in (2), and define $\pi^*(A) = 1 - \pi_*(A^c)$, for every $A \in \langle\mathcal{L}\rangle^*$. By Proposition 6.26 in [58] there exists $\tilde{\pi} \in \mathcal{P}_{\pi_*}$ such that $\int X(H_i)\tilde{\pi}(\mathrm{d}H_i) = \oint X(H_i)\pi_*(\mathrm{d}H_i)$ and $\int(1 - Y(H_i))\tilde{\pi}(\mathrm{d}H_i) = \oint(1 - Y(H_i))\pi_*(\mathrm{d}H_i)$. Thus, since $\int(1 - Y(H_i))\tilde{\pi}(\mathrm{d}H_i) = 1 - \int Y(H_i)\tilde{\pi}(\mathrm{d}H_i)$ and $\oint(1 - Y(H_i))\pi_*(\mathrm{d}H_i) = 1 - \oint Y(H_i)\pi^*(\mathrm{d}H_i)$, it follows $\oint Y(H_i)\pi^*(\mathrm{d}H_i) = \int Y(H_i)\tilde{\pi}(\mathrm{d}H_i)$. This implies

$$\begin{aligned}
\underline{P}^{\mathbf{jc}}(F \cap K) &= \oint X(H_i)\pi_*(\mathrm{d}H_i) = \int X(H_i)\tilde{\pi}(\mathrm{d}H_i) = L^{\mathbf{jc}}(F, K),\\
\overline{P}^{\mathbf{jc}}(F^c \cap K) &= \oint Y(H_i)\pi^*(\mathrm{d}H_i) = \int Y(H_i)\tilde{\pi}(\mathrm{d}H_i) = U^{\mathbf{jc}}(F^c, K),
\end{aligned}$$

and the conclusion follows. □

Example 6 (Example 5 continued)**.** *Since $\underline{P}^{\mathbf{jc}}(B) = \underline{P}^{\mathbf{jc}}(C) = 0$, then by condition (ii) of Theorem 5, $\underline{Q}(A|B) = \underline{Q}^{\mathbf{c}}(A|B)$, $\overline{Q}(A|B) = \overline{Q}^{\mathbf{c}}(A|B)$, $\underline{Q}(A|C) = \underline{Q}^{\mathbf{c}}(A|C)$, $\overline{Q}(A|C) = \overline{Q}^{\mathbf{c}}(A|C)$.*

Since $\mathcal{A}_{\mathcal{L}} = \langle\mathcal{L}\rangle^*$, we have that $\underline{P}^{\mathbf{jc}}(E) = P^{\mathbf{jd}}(E) = \int \sigma(E|H_\theta)\pi(\mathrm{d}H_\theta) = \frac{2}{n+1}$ and $\underline{P}^{\mathbf{jc}}(D \cap E) = P^{\mathbf{jd}}(D \cap E) = \int \sigma(D \cap E|H_\theta)\pi(\mathrm{d}H_\theta) = \frac{1}{n+1}$, thus $\underline{Q}^{\mathbf{c}}(D|E) = \overline{Q}^{\mathbf{c}}(D|E) = \frac{1}{2}$.

5. Conditionally $\mathcal{L}$-conglomerable extensions

The $\mathcal{L}$-conglomerability property expressed by formula (9) amounts to imposing a set of constraints on a joint probability $P(\cdot)$ on $\mathcal{A}$ involving a strategy σ. In turn, such property constrains also the conditional probability of those conditional events $F|K \in \mathcal{A} \times \mathcal{A}^0$ such that $P(K) > 0$. In other terms, if $Q(\cdot|\cdot)$ is a full conditional probability on $\mathcal{A}$ extending $\{P, \sigma\}$, then the value of $Q(F|K)$ is constrained by $\mathcal{L}$-conglomerability when $P(K) > 0$, but it is not when $P(K) = 0$, as shown in the following example.

Example 7. *Let $\Omega = \mathbb{N}$, $\mathcal{A} = \wp(\mathbb{N})$, $\mathcal{L} = \{H_i = \{2i-1, 2i\}\}_{i\in\mathbb{N}}$ and $\mathcal{A}_{\mathcal{L}} = \langle\mathcal{L}\rangle^*$. Consider the countably additive joint probability $P(\cdot)$ on $\mathcal{A}$ such that, for $i \in \mathbb{N}$,*

$$P(\{2i-1\}) = \frac{1}{2^i} \quad \textit{and} \quad P(\{2i\}) = 0,$$

and denote $\pi = P_{|\mathcal{A}_{\mathcal{L}}}$.

Since $P(H_i) > 0$ for every $H_i \in \mathcal{L}$, the only strategy σ on $\mathcal{A} \times \mathcal{L}$ coherent with $P(\cdot)$ is defined, for every $F|H_i \in \mathcal{A} \times \mathcal{L}$, as

$$\sigma(F|H_i) = \frac{P(F \cap H_i)}{P(H_i)} = \begin{cases} 1 & \textit{if } (2i-1) \in F, \\ 0 & \textit{otherwise}, \end{cases}$$

and, for every $F \in \mathcal{A}$, $\sigma(F|\cdot)$ is (trivially) an $\mathcal{A}_{\mathcal{L}}$-continuous function on $\mathcal{L}$. Simple computations show that $P(F) = \sum_{\{i\}\subseteq F} P(\{i\}) = \int \sigma(F|H_i)\pi(\mathrm{d}H_i)$, *which implies that $P(\cdot)$ is $\mathcal{L}$-disintegrable (or, equivalently, $\mathcal{L}$-conglomerable) with respect to σ.*

Since $\{P,\sigma\}$ is coherent, then it can be extended (not in a unique way) to a full conditional probability on $\mathcal{A}$. Denote $E = \{2i : i \in \mathbb{N}\}$, $D = \{4i : i \in \mathbb{N}\}$ and let $\mathcal{U}_D$ be a non-principal ultrafilter on $\wp(E)$ containing D, whose existence follows by Zorn's lemma. Notice that, for every $K \in \mathcal{A}^0$, $P(K) = 0$ if and only if $K \subseteq E$.

Let μ be the finitely additive measure defined on $\wp(E)$ such that, for every $A \in \wp(E)$, $\mu(A) = 1$ if $A \in \mathcal{U}_D$ and 0 otherwise. Let ν be the countably additive measure defined on $\wp(E)$ such that, for $i \in \mathbb{N}$, $\nu(\{2i\}) = \frac{1}{2^i}$. Finally, let γ be the finitely additive measure on $\wp(E)$ defined as $\gamma = \frac{1}{2}\mu + \frac{1}{2}\nu$, which is strictly positive on $\wp(E)^0$ and such that $\gamma(E) = 1$.

Let $Q(\cdot|\cdot)$ be the function defined, for every $F|K \in \mathcal{A} \times \mathcal{A}^0$, as

$$Q(F|K) = \begin{cases} \frac{P(F\cap K)}{P(K)} & \textit{if } P(K) > 0 \textit{ (i.e., if } K \in \wp(\mathbb{N}) \setminus \wp(E)\textit{)}, \\ \frac{\gamma(F\cap K)}{\gamma(K)} & \textit{otherwise.} \end{cases}$$

It is easy to verify that $Q(\cdot|\cdot)$ is a full conditional probability on $\mathcal{A}$ extending $\{P,\sigma\}$, and so $\{\pi,\sigma\}$. In particular, $Q(\cdot|\cdot)$ is a $\mathcal{L}$-disintegrable full conditional probability extending $\{\pi,\sigma\}$, according to Definition 6.

For every $F|K \in \mathcal{A} \times \mathcal{A}^0$ such that $P(K) > 0$, considering the bounded finitely additive measure π_K on $\mathcal{A}_\mathcal{L}$ such that $\pi_K = \frac{\pi}{P(K)}$ we have

$$Q(F|K) = \frac{P(F\cap K)}{P(K)} = \int \sigma(F \cap K|H_i)\pi_K(\mathrm{d}H_i),$$

i.e., also the conditional probability $Q(F|K)$ is constrained by $\mathcal{L}$-disintegrability.

Nevertheless, since $P(E) = 0$ and $D \subseteq E$, it holds that $Q(D|E) = \gamma(D) = \frac{2}{3}$, moreover, $\sigma(D|H_i) = 0$ for every $H_i \in \mathcal{L}$. This implies that the conditional probability $Q(D|E)$ is not constrained by $\mathcal{L}$-disintegrability, and even more, no form of $\mathcal{L}$-disintegrability can hold for $Q(D|E)$ since for every bounded finitely additive measure τ on $\mathcal{A}_\mathcal{L}$ it holds

$$Q(D|E) = \frac{2}{3} \neq 0 = \int \sigma(D|H_i)\tau(\mathrm{d}H_i).$$

The previous situation happens, in particular, for all those $K \in \mathcal{A}^0_\mathcal{L}$ such that $\pi(K) = 0$, thus, starting from the above considerations, the goal is to provide a reinforcement of the conglomerability condition asking it to hold for all $F|K \in \mathcal{A} \times \mathcal{A}^0_\mathcal{L}$ and to study its effect on the envelopes of the set of full conditional probabilities extending $\{\pi,\sigma\}$.

Definition 7. *A full conditional probability $\tilde{Q}(\cdot|\cdot)$ on $\mathcal{A}$ extending $\{\pi,\sigma\}$ is* **conditionally $\mathcal{L}$-conglomerable** *if, denoting with $\tilde{\pi}^{\mathbf{f}} = \tilde{Q}_{|\mathcal{A}_\mathcal{L}\times\mathcal{A}^0_\mathcal{L}}$, for every $F|K \in \mathcal{A} \times \mathcal{A}^0_\mathcal{L}$ and every $B \in \mathcal{A}_\mathcal{L}$ such that $B \subseteq K$ it holds*

$$\tilde{\pi}^{\mathbf{f}}(B|K) \inf_{H_i\subseteq B} \sigma(F|H_i) \leq \tilde{Q}(F \cap B|K) \leq \tilde{\pi}^{\mathbf{f}}(B|K) \sup_{H_i\subseteq B} \sigma(F|H_i). \quad (16)$$

Note that $\tilde{\pi}^{\mathbf{f}}$ is a full conditional probability on $\mathcal{A}_{\mathcal{L}}$ extending the prior probability π.

In the case $\sigma(F|\cdot)$ is Stieltjes integrable with respect to $\tilde{\pi}^{\mathbf{f}}(\cdot|K)$, for every $F \in \mathcal{A}$ and $K \in \mathcal{A}_{\mathcal{L}}^0$, then a conditionally $\mathcal{L}$-conglomerable $\tilde{Q}(\cdot|\cdot)$ can be expressed, for every $F|K \in \mathcal{A} \times \mathcal{A}_{\mathcal{L}}^0$, as

$$\tilde{Q}(F|K) = \int \sigma(F|H_i)\tilde{\pi}^{\mathbf{f}}(\mathrm{d}H_i|K), \tag{17}$$

and is said *conditionally $\mathcal{L}$-disintegrable.* In particular, this happens if $\sigma(F|\cdot)$ is $\mathcal{A}_{\mathcal{L}}$-continuous for every $F \in \mathcal{A}$.

Conditional $\mathcal{L}$-conglomerability ($\mathcal{L}$-disintegrability) is essentially determined by the set of full conditional prior probabilities

$$\mathcal{P}^{\mathbf{f}} = \{\tilde{\pi}^{\mathbf{f}} = \tilde{Q}_{|\mathcal{A}_{\mathcal{L}} \times \mathcal{A}_{\mathcal{L}}^0} : \tilde{Q} \in \mathcal{Q}\},$$

which is a non-empty compact subset of $[0,1]^{\mathcal{A}_{\mathcal{L}} \times \mathcal{A}_{\mathcal{L}}^0}$ endowed with the product topology, whose lower envelope $\underline{\pi}^{\mathbf{f}} = \min \mathcal{P}^{\mathbf{f}}$ is characterized in Corollary 2 that is an immediate consequence of Theorem 5.

Remark 6. *The strategy σ has no role in the extension of π to $\mathcal{A}_{\mathcal{L}} \times \mathcal{A}_{\mathcal{L}}^0$. Indeed, for every $H_i \in \mathcal{L}$, every $\tilde{\pi}^{\mathbf{f}} \in \mathcal{P}^{\mathbf{f}}$ needs to satisfy $\tilde{\pi}^{\mathbf{f}}_{|\mathcal{A}_{\mathcal{L}} \times \{H_i\}} = \sigma_{|\mathcal{A}_{\mathcal{L}} \times \{H_i\}}$, which trivially holds by axiom* **(C3)**.

Corollary 2. *The lower envelope $\underline{\pi}^{\mathbf{f}}$ of the set $\mathcal{P}^{\mathbf{f}}$ of coherent extensions of $\{\pi, \sigma\}$ to $\mathcal{A}_{\mathcal{L}} \times \mathcal{A}_{\mathcal{L}}^0$ satisfies the following properties:*

(i) *$\underline{\pi}^{\mathbf{f}}(\cdot|K)$ is a totally monotone capacity on $\mathcal{A}_{\mathcal{L}}$, for every $K \in \mathcal{A}_{\mathcal{L}}^0$;*

(ii) *for every $F|K \in \mathcal{A}_{\mathcal{L}} \times \mathcal{A}_{\mathcal{L}}^0$ it holds $\underline{\pi}^{\mathbf{f}}(F|K) = 1$ when $F \cap K = K$, and if $F \cap K \neq K$, then*

$$\underline{\pi}^{\mathbf{f}}(F|K) = \begin{cases} \frac{\pi(F \cap K)}{\pi(K)} & \textit{if } \pi(K) > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

The previous result implies that, for $K \in \mathcal{A}_{\mathcal{L}}^0$, $\underline{\pi}^{\mathbf{f}}(\cdot|K)$ is a finitely additive probability if $\pi(K) > 0$, and otherwise it is a totally monotone capacity *vacuous at* K (i.e., for every $F \in \mathcal{A}_{\mathcal{L}}$ it holds $\underline{\pi}^{\mathbf{f}}(F|K) = 1$ if $K \subseteq F$ and 0 otherwise).

In what follows we restrict to the set

$$\mathcal{Q}^{\mathbf{cc}} = \{\tilde{Q} : \tilde{Q} \text{ is a conditionally } \mathcal{L}\text{-conglomerable f.c.p. on } \mathcal{A} \text{ extending } \{\pi, \sigma\}\},$$

which is such that $\mathcal{Q}^{\mathbf{cc}} \subseteq \mathcal{Q}^{\mathbf{c}} \subseteq \mathcal{Q}$ and whose topological structure is investigated in the following result.

Theorem 6. *The set $\mathcal{Q}^{\mathbf{cc}}$ is a non-empty compact subset of $[0,1]^{\mathcal{A} \times \mathcal{A}^0}$ endowed with the product topology.*

Proof. We prove first that $\mathcal{Q}^{\mathbf{cc}}$ is not empty. At this aim, consider the set $\mathcal{P}^{\mathbf{f}} = \{\tilde{\pi}^{\mathbf{f}}(\cdot|\cdot)\}$ of conditional prior probabilities full on $\mathcal{A}_{\mathcal{L}}$ extending $\{\pi, \sigma\}$. Remark 6 implies that every $\tilde{\pi}^{\mathbf{f}}$ in $\mathcal{P}^{\mathbf{f}}$ can be coherently extended to $\langle\mathcal{L}\rangle^* \times \mathcal{A}^0_{\mathcal{L}}$ without being affected from σ, obtaining a set $\mathcal{W}^{\mathbf{f}} = \{\tilde{\nu}^{\mathbf{f}}(\cdot|\cdot)\}$ of conditional probabilities on $\langle\mathcal{L}\rangle^* \times \mathcal{A}^0_{\mathcal{L}}$ with lower and upper envelopes $\underline{\nu}^{\mathbf{f}} = \min \mathcal{W}^{\mathbf{f}}$ and $\overline{\nu}^{\mathbf{f}} = \max \mathcal{W}^{\mathbf{f}}$. The following lemma shows that, for every $K \in \mathcal{A}^0_{\mathcal{L}}$, $\underline{\nu}^{\mathbf{f}}(\cdot|K)$ is a totally monotone capacity on $\langle\mathcal{L}\rangle^*$.

Lemma 1. *Let $\mathcal{A}$ be an algebra of subsets of Ω, P a full conditional probability on $\mathcal{A}$, and $\mathcal{P} = \{\tilde{P}(\cdot|\cdot)\}$ the set of all conditional probabilities extending P on $\langle\mathcal{A}\rangle^* \times \mathcal{A}^0$. The lower envelope $\underline{P} = \min \mathcal{P}$ is such that for every $K \in \mathcal{A}^0$, $\underline{P}(\cdot|K)$ coincides with the inner measure on $\langle\mathcal{A}\rangle^*$ generated by $P(\cdot|K)$, thus is a totally monotone capacity.*

Proof of Lemma 1. For every $F|K \in \langle\mathcal{A}\rangle^* \times \mathcal{A}^0$, Theorem 1 implies that we can restrict to finite subfamilies $\mathcal{F} \subseteq \mathcal{A} \times \mathcal{A}^0$ of the form $\mathcal{F} = \mathcal{B} \times \mathcal{B}^0$, with $\mathcal{B} \subseteq \mathcal{A}$ finite subalgebra containing K. Let us denote with B the maximal element of $\mathcal{B}$ with respect to inclusion such that $B \subseteq F$. In turn, this implies

$$\underline{P}(F|K) = \sup\{P(B|K) \,:\, B \subseteq F, B \in \mathcal{A}\},$$

so $\underline{P}(\cdot|K)$ coincides with the inner measure on $\langle\mathcal{A}\rangle^*$ generated by $P(\cdot|K)$ and is therefore a totally monotone capacity. $\square$

Thus, for a fixed $\tilde{\pi}^{\mathbf{f}}$ in $\mathcal{P}^{\mathbf{f}}$, for every $K \in \mathcal{A}^0_{\mathcal{L}}$ the proof goes along the same line of the proof of Thereom 2 using $\tilde{\pi}^{\mathbf{f}}(\cdot|K)$ in place of $\pi(\cdot)$, and the claim follows.

To prove $\mathcal{Q}^{\mathbf{cc}}$ is compact, it is sufficient to consider a net $(\tilde{Q}_\alpha)_\alpha$ in $\mathcal{Q}^{\mathbf{cc}}$ converging pointwise to $\tilde{Q}$. Denote with $(\tilde{\pi}^{\mathbf{f}}_\alpha)_\alpha$ and $\tilde{\pi}^{\mathbf{f}}$ the restrictions of $(\tilde{Q}_\alpha)_\alpha$ and $\tilde{Q}$ on $\mathcal{A}_{\mathcal{L}} \times \mathcal{A}^0_{\mathcal{L}}$, respectively. The compactness of the set $\mathcal{Q}$ of all full conditional probabilities on $\mathcal{A}$ extending $\{\pi, \sigma\}$ implies that $\tilde{Q}$ is a full conditional probability on $\mathcal{A}$ extending $\{\pi, \sigma\}$, moreover, since the pointwise limits of nets preserve non-strict inequalities and σ is fixed, for every $F|K \in \mathcal{A} \times \mathcal{A}^0_{\mathcal{L}}$ and every $B \in \mathcal{A}_{\mathcal{L}}$ such that $B \subseteq K$, it follows

$$\tilde{\pi}^{\mathbf{f}}(B|K) \inf_{H_i \subseteq B} \sigma(F|H_i) \leq \tilde{Q}(F \cap B|K) \leq \tilde{\pi}^{\mathbf{f}}(B|K) \sup_{H_i \subseteq B} \sigma(F|H_i),$$

that is, $\tilde{Q}$ is an element of $\mathcal{Q}^{\mathbf{cc}}$ and the claim follows. $\square$

Let $\underline{Q}^{\mathbf{cc}} = \min \mathcal{Q}^{\mathbf{cc}}$ and $\overline{Q}^{\mathbf{cc}} = \max \mathcal{Q}^{\mathbf{cc}}$ and since $\underline{P}^{\mathbf{jc}} = \underline{Q}^{\mathbf{cc}}_{|\mathcal{A} \times \{\Omega\}}$ and $\overline{P}^{\mathbf{jc}} = \overline{Q}^{\mathbf{cc}}_{|\mathcal{A} \times \{\Omega\}}$, Example 4 implies that $\underline{Q}^{\mathbf{cc}}(\cdot|K)$, for $K \in \mathcal{A}^0$, is generally not 2-monotone.

Let us also consider the set

$$\mathcal{Q}^{\mathbf{cd}} = \{\tilde{Q} \,:\, \tilde{Q} \text{ is a conditionally } \mathcal{L}\text{-disintegrable f.c.p. on } \mathcal{A} \text{ extending } \{\pi, \sigma\}\},$$

which, depending on the integrability of σ, can be empty and coincides with $\mathcal{Q}^{\mathbf{cc}}$ if, for every $F \in \mathcal{A}$, $\sigma(F|\cdot)$ is Stieltjes integrable with respect to $\tilde{\pi}^{\mathbf{f}}(\cdot|K)$

for every $K \in \mathcal{A}^0_{\mathcal{L}}$ and every $\tilde{\pi}^{\mathbf{f}} \in \mathcal{P}^{\mathbf{f}}$. In particular, $\mathcal{Q}^{\mathbf{cd}} = \mathcal{Q}^{\mathbf{cc}}$ if, for every $F \in \mathcal{A}$, $\sigma(F|\cdot)$ is $\mathcal{A}_{\mathcal{L}}$-continuous.

Theorem 7. *For every $F|K \in \mathcal{A} \times \mathcal{A}^0$ it holds*

$$\underline{Q}^{\mathbf{cc}}(F|K) = \min\left\{\underline{Q}^{\mathbf{cd}}_{\{\tilde{\pi},\rho\}}(F|K) \,:\, \tilde{\pi} \in \mathcal{P}_{\pi_*}\right\},$$

where $\mathcal{B} = \langle \mathcal{A} \cup \langle \mathcal{L}\rangle^*\rangle$, ρ is any strategy on $\mathcal{B} \times \mathcal{L}$ extending σ and $\underline{Q}^{\mathbf{cd}}_{\{\tilde{\pi},\rho\}}$ is the lower envelope of the class $\mathcal{Q}^{\mathbf{cd}}_{\{\tilde{\pi},\rho\}}$ of conditionally $\mathcal{L}$-disintegrable full conditional probabilities on $\mathcal{B}$ extending $\{\tilde{\pi},\rho\}$.

Proof. Consider the core $\mathcal{P}_{\pi_*}$ of the inner measure π_* induced by π on $\langle \mathcal{L}\rangle^*$, defined as in (2). For every $\tilde{\pi} \in \mathcal{P}_{\pi_*}$, the assessment $\{\tilde{\pi},\rho\}$ is coherent and $\rho(F|\cdot)$ is trivially $\langle \mathcal{L}\rangle^*$-continuous, for every $F \in \mathcal{B}$, so, every conditionally $\mathcal{L}$-disintegrable full conditional probability on $\mathcal{B}$ extending $\{\tilde{\pi},\rho\}$ is conditionally $\mathcal{L}$-conglomerable, as well as its restriction on $\mathcal{A}\times\mathcal{A}^0$, which is an element of $\mathcal{Q}^{\mathbf{cc}}$. This implies that, for every $F|K \in \mathcal{A} \times \mathcal{A}^0$, $\underline{Q}^{\mathbf{cc}}(F|K)$ is attained minimizing $\underline{Q}^{\mathbf{cd}}_{\{\tilde{\pi},\rho\}}(F|K)$, varying $\tilde{\pi} \in \mathcal{P}_{\pi_*}$. □

The previous theorem shows that the lower bound of conditionally $\mathcal{L}$-conglomerable extensions can be expressed in terms of lower bounds of conditionally $\mathcal{L}$-disintegrable extensions computed with respect to an extension of the prior π on $\langle \mathcal{L}\rangle^*$. Hence, in what follows we focus on conditional $\mathcal{L}$-disintegrability.

To avoid cumbersome integrability requirements, in the rest of this section the following assumption is made:

(A) $\sigma(F|\cdot)$ is $\mathcal{A}_{\mathcal{L}}$-continuous, for every $F \in \mathcal{A}$.

Under **(A)**, $\mathcal{Q}^{\mathbf{cd}}$ is not empty and it holds $\mathcal{Q}^{\mathbf{cd}} = \mathcal{Q}^{\mathbf{cc}}$, thus we can consider the envelopes $\underline{Q}^{\mathbf{cd}} = \min \mathcal{Q}^{\mathbf{cd}}$ and $\overline{Q}^{\mathbf{cd}} = \max \mathcal{Q}^{\mathbf{cd}}$.

The following result characterizes the lower envelope $\underline{Q}^{\mathbf{cd}}$, relying on the following functions, defined for $F \in \mathcal{A}$, $K \in \mathcal{A}^0$ and $A \in \mathcal{A}^0_{\mathcal{L}}$ with $K \subseteq A$ as

$$\begin{aligned} L^{\mathbf{cd}}(F,K;A) &= \min\left\{\tilde{Q}(F \cap K|A) \,:\, \tilde{Q} \in \mathcal{Q}^{\mathbf{cd}}, \tilde{Q}(F^c \cap K|A) = \overline{Q}^{\mathbf{cd}}(F^c \cap K|A)\right\},\\ U^{\mathbf{cd}}(F,K;A) &= \max\left\{\tilde{Q}(F \cap K|A) \,:\, \tilde{Q} \in \mathcal{Q}^{\mathbf{cd}}, \tilde{Q}(F^c \cap K|A) = \underline{Q}^{\mathbf{cd}}(F^c \cap K|A)\right\}, \end{aligned}$$

for which it holds

$$\underline{Q}^{\mathbf{cd}}(F \cap K|A) \leq L^{\mathbf{cd}}(F,K;A) \leq U^{\mathbf{cd}}(F,K;A) \leq \overline{Q}^{\mathbf{cd}}(F \cap K|A).$$

Theorem 8. *The lower envelope $\underline{Q}^{\mathbf{cd}}(\cdot|\cdot)$ is such that for every $F|K \in \mathcal{A}\times\mathcal{A}^0$, $\underline{Q}^{\mathbf{cd}}(F|K) = 1$ when $F \cap K = K$, and if $F \cap K \neq K$, then:*

(i) if $K \in \mathcal{A}^0_{\mathcal{L}}$, then

$$\underline{Q}^{\mathbf{cd}}(F|K) = \oint \sigma(F|H_i)\underline{\pi}^{\mathbf{f}}(\mathrm{d}H_i|K) = \begin{cases} \dfrac{\int \sigma(F \cap K|H_i)\pi(\mathrm{d}H_i)}{\pi(K)} & \text{if } \pi(K) > 0,\\[2ex] \inf\limits_{H_i \subseteq K} \sigma(F|H_i) & \text{otherwise;} \end{cases}$$

(ii) if $K \in \mathcal{A}^0 \setminus \mathcal{A}^0_{\mathcal{L}}$, then if there exists $A \in \mathcal{A}^0_{\mathcal{L}}$ such that $K \subseteq A$ and $\underline{Q}^{\mathbf{cd}}(K|A) > 0$ we have that

$$\underline{Q}^{\mathbf{cd}}(F|K) = \min\left\{\frac{\underline{Q}^{\mathbf{cd}}(F\cap K|A)}{\underline{Q}^{\mathbf{cd}}(F\cap K|A) + U^{\mathbf{cd}}(F^c, K; A)}, \frac{L^{\mathbf{cd}}(F,K;A)}{L^{\mathbf{cd}}(F,K;A) + \overline{Q}^{\mathbf{cd}}(F^c\cap K|A)}\right\},$$

otherwise $\underline{Q}^{\mathbf{cd}}(F|K) = 0$.

Proof. Condition *(i)*. Given $\tilde{\pi}^{\mathbf{f}}$ in $\mathcal{P}^{\mathbf{f}}$, for every $F|K \in \mathcal{A} \times \mathcal{A}^0_{\mathcal{L}}$ define the function

$$\tilde{P}(F|K) = \int \sigma(F|H_i)\tilde{\pi}^{\mathbf{f}}(\mathrm{d}H_i|K),$$

which is a conditional probability on $\mathcal{A} \times \mathcal{A}^0_{\mathcal{L}}$ extending $\{\tilde{\pi}^{\mathbf{f}}, \sigma\}$. Varying $\tilde{\pi}^{\mathbf{f}}$ in $\mathcal{P}^{\mathbf{f}}$, we obtain a set $\mathcal{P}^{\mathbf{cd}}$, whose elements are restrictions on $\mathcal{A} \times \mathcal{A}^0_{\mathcal{L}}$ of elements in $\mathcal{Q}^{\mathbf{cd}}$, thus $\mathcal{P}^{\mathbf{cd}}$ is a non-empty compact subset of $[0,1]^{\mathcal{A}\times\mathcal{A}^0_{\mathcal{L}}}$ endowed with the product topology, whose lower envelope is $\underline{P}^{\mathbf{cd}} = \min \mathcal{P}^{\mathbf{cd}}$.

Corollary 2 implies that, for every $K \in \mathcal{A}^0_{\mathcal{L}}$, $\underline{\pi}^{\mathbf{f}}(\cdot|K)$ is a totally monotone capacity, so, for every $F|K \in \mathcal{A} \times \mathcal{A}^0_{\mathcal{L}}$ it holds $\underline{P}^{\mathbf{cd}}(F|K) = 1$ when $F \cap K = K$, and if $F \cap K \neq K$, then

$$\underline{P}^{\mathbf{cd}}(F|K) = \oint \sigma(F|H_i)\underline{\pi}^{\mathbf{f}}(\mathrm{d}H_i|K) = \begin{cases} \frac{\int \sigma(F\cap K|H_i)\pi(\mathrm{d}H_i)}{\pi(K)} & \text{if } \pi(K) > 0, \\ \inf\limits_{H_i \subseteq K} \sigma(F|H_i) & \text{otherwise,} \end{cases}$$

where the last equality follows by the properties of the Choquet integral [27].

Condition *(ii)*. Each $\tilde{P}$ in $\mathcal{P}^{\mathbf{cd}}$ can be further extended through coherence to a full conditional probability $\tilde{Q}$ on $\mathcal{A}$ which is conditionally $\mathcal{L}$-disintegrable. The extension $\tilde{Q}$ is generally not unique so we have a set

$$\mathcal{Q}_{\tilde{P}} = \left\{\tilde{Q} \,:\, \tilde{Q} \text{ is a f.c.p on } \mathcal{A} \text{ extending } \tilde{P}\right\},$$

whose lower envelope is $\underline{Q}_{\tilde{P}} = \min \mathcal{Q}_{\tilde{P}}$.

Lemma 2. *The lower envelope $\underline{Q}_{\tilde{P}}(\cdot|\cdot)$ is such that for every $F|K \in \mathcal{A} \times \mathcal{A}^0$ it holds $\underline{Q}_{\tilde{P}}(F|K) = 1$ when $F \cap K = K$, and if $F \cap K \neq K$, then:*

$$\underline{Q}_{\tilde{P}}(F|K) = \begin{cases} \frac{\tilde{P}(F\cap K|A)}{\tilde{P}(K|A)} & \textit{if } \exists\, A \in \mathcal{A}^0_{\mathcal{L}} \textit{ s.t. } K \subseteq A \textit{ and } \tilde{P}(K|A) > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof of Lemma 2. The proof is trivial in case $F \cap K = K$ or there exists $A \in \mathcal{A}^0_{\mathcal{L}}$ such that $K \subseteq A$ and $\tilde{P}(K|A) > 0$, thus suppose $F \cap K \neq K$ and $\tilde{P}(K|A) = 0$ for every $A \in \mathcal{A}^0_{\mathcal{L}}$ with $K \subseteq A$.

Under this hypothesis, let $\mathcal{G} = \mathcal{A} \times \mathcal{A}^0_{\mathcal{L}}$. By Theorem 1, for every $F|K \in \mathcal{A} \times \mathcal{A}^0$, the interval of coherent extensions $\mathbb{I}_{F|K} = \left[\underline{Q}_{\tilde{P}}(F|K), \overline{Q}_{\tilde{P}}(F|K)\right]$ can be computed in terms of finite subfamilies of $\mathcal{G}$.

Since for every $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{G}$ and $\operatorname{card} \mathcal{F}_2 < \aleph_0$ one has $\underline{Q}^{\mathcal{F}_1}_{\tilde{P}}(F|K) \leq \underline{Q}^{\mathcal{F}_2}_{\tilde{P}}(F|K)$, we can restrict to finite subfamilies of $\mathcal{G}$ of the form $\mathcal{F} = \mathcal{B} \times \mathcal{B}^0_{\mathcal{L}}$ where $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{B}_{\mathcal{L}} \subseteq \mathcal{A}_{\mathcal{L}} \cap \mathcal{B}$ are finite algebras with $\{F, K\} \subseteq \mathcal{B}$. Indeed, every finite subfamily can be suitably enlarged in order to meet this form. Now, Corollary 2 in [18] implies that $\underline{Q}^{\mathcal{F}}_{\tilde{P}}(F|K) = 0$ and since this holds for every finite subfamily $\mathcal{F}$ the proof follows. □

The set $\mathcal{Q}^{\mathbf{cd}}$ of conditionally $\mathcal{L}$-disintegrable full conditional probabilities on $\mathcal{A}$ extending $\{\pi, \sigma\}$ can be expressed as

$$\mathcal{Q}^{\mathbf{cd}} = \bigcup \{\mathcal{Q}_{\tilde{P}} \,:\, \tilde{P} \in \mathcal{P}^{\mathbf{cd}}\}.$$

Lemma 3. *The lower envelope $\underline{Q}^{\mathbf{cd}}(\cdot|\cdot)$ is such that for every $F|K \in \mathcal{A} \times (\mathcal{A}^0 \setminus \mathcal{A}^0_{\mathcal{L}})$ it holds $\underline{Q}^{\mathbf{cd}}(F|K) = 1$ when $F \cap K = K$, and if $F \cap K \neq K$, then if there exists $A \in \mathcal{A}^0_{\mathcal{L}}$ such that $K \subseteq A$ and $\underline{P}^{\mathbf{cd}}(K|A) > 0$ we have that*

$$\begin{aligned}\underline{Q}^{\mathbf{cd}}(F|K) \quad &= \quad \min\left\{\frac{\underline{P}^{\mathbf{cd}}(F \cap K|A)}{\underline{P}^{\mathbf{cd}}(F \cap K|A) + U^{\mathbf{cd}}(F^c, K; A)}, \right. \\ &\qquad\qquad \left. \frac{L^{\mathbf{cd}}(F, K; A)}{L^{\mathbf{cd}}(F, K; A) + \overline{P}^{\mathbf{cd}}(F^c \cap K|A)}\right\},\end{aligned}$$

otherwise $\underline{Q}^{\mathbf{cd}}(F|K) = 0$.

Proof of Lemma 3. The statement is trivial if $F \cap K = K$, thus suppose $F \cap K \neq K$.

If there exists $A \in \mathcal{A}^0_{\mathcal{L}}$ with $K \subseteq A$ such that $\underline{P}^{\mathbf{cd}}(K|A) > 0$, then it holds $\tilde{P}(K|A) > 0$ for every $\tilde{P} \in \mathcal{P}^{\mathbf{cd}}$, and so we have

$$\underline{Q}^{\mathbf{cd}}(F|K) = \min\left\{\frac{\tilde{P}(F \cap K|A)}{\tilde{P}(F \cap K|A) + \tilde{P}(F^c \cap K|A)} \,:\, \tilde{P} \in \mathcal{P}^{\mathbf{cd}}\right\}.$$

The conclusion follows since the real function $\frac{x}{x+y}$ is increasing in x and decreasing in y, so the minimum is attained in correspondence of $\frac{\underline{P}^{\mathbf{cd}}(F\cap K|A)}{\underline{P}^{\mathbf{cd}}(F\cap K|A)+U^{\mathbf{cd}}(F^c,K;A)}$ or $\frac{L^{\mathbf{cd}}(F,K;A)}{L^{\mathbf{cd}}(F,K;A)+\overline{P}^{\mathbf{cd}}(F^c\cap K|A)}$.

Otherwise, for all $A \in \mathcal{A}^0_{\mathcal{L}}$ with $K \subseteq A$ it holds $\underline{P}^{\mathbf{cd}}(K|A) = 0$, which implies for every such A the existence of $\tilde{P}_A \in \mathcal{P}^{\mathbf{cd}}$ such that $\tilde{P}_A(K|A) = 0$ and so $\tilde{P}_A(K|B) = 0$ for every $B \in \mathcal{A}^0_{\mathcal{L}}$ with $A \subseteq B$.

We show the existence of $\tilde{P}_0 \in \mathcal{P}^{\mathbf{cd}}$ such that $\tilde{P}_0(K|A) = 0$ for all $A \in \mathcal{A}^0_{\mathcal{L}}$ with $K \subseteq A$. At this aim, recall that the compactness of $\mathcal{P}^{\mathbf{cd}}$ is equivalent to the fact that every family of non-empty closed subsets of $\mathcal{P}^{\mathbf{cd}}$ with the finite intersection property has non-empty intersection.

For an arbitrary finite subalgebra $\mathcal{B}_{\mathcal{L}} \subseteq \mathcal{A}_{\mathcal{L}}$ define $K^*_{\mathcal{B}_{\mathcal{L}}} = \bigcap\{B \in \mathcal{B}^0_{\mathcal{L}} : K \subseteq B\}$, which belongs to $\mathcal{B}^0_{\mathcal{L}}$ since $\mathcal{B}_{\mathcal{L}}$ is finite. Introduce the collection

$$\mathbf{D}_0 = \left\{\mathcal{D}_0^{\mathcal{B}_{\mathcal{L}}} = \left\{\tilde{P} \in \mathcal{P}^{\mathbf{cd}} : \tilde{P}(K|K^*_{\mathcal{B}_{\mathcal{L}}}) = 0\right\} : \mathcal{B}_{\mathcal{L}} \subseteq \mathcal{A}_{\mathcal{L}}, \operatorname{card}\mathcal{B}_{\mathcal{L}} < \aleph_0\right\},$$

which is easily seen to be a family of non-empty closed subsets of $\mathcal{P}^{\mathbf{cd}}$.

We show that $\mathbf{D}_0$ has the finite intersection property. For any $\mathcal{B}_{\mathcal{L}1}, \ldots, \mathcal{B}_{\mathcal{L}n}$ finite subalgebras of $\mathcal{A}_{\mathcal{L}}$, the corresponding generated algebra $\mathcal{B}'_{\mathcal{L}} = \langle\bigcup_{i=1}^n \mathcal{B}_{\mathcal{L}i}\rangle$ is still a finite subalgebra of $\mathcal{A}_{\mathcal{L}}$, moreover, $K^*_{\mathcal{B}'_{\mathcal{L}}} \subseteq K^*_{\mathcal{B}_{\mathcal{L}i}}$ for $i = 1, \ldots, n$. It is easily seen that, for $i = 1, \ldots, n$, it holds $K \cap K^*_{\mathcal{B}_{\mathcal{L}i}} \subseteq K \cap K^*_{\mathcal{B}'_{\mathcal{L}}}$ and $K^c \cap K_{\mathcal{B}_{\mathcal{L}i}} \supseteq K^c \cap K^*_{\mathcal{B}'_{\mathcal{L}}}$, that is $K|K^*_{\mathcal{B}_{\mathcal{L}i}} \subseteq_{GN} K|K^*_{\mathcal{B}'_{\mathcal{L}}}$, according to the definition of inclusion relation for conditional events $\subseteq_{GN}$ given in [35]. Hence, for every $\tilde{P} \in \mathcal{D}_0^{\mathcal{B}'_{\mathcal{L}}}$ we have $\tilde{P}(K|K^*_{\mathcal{B}'_{\mathcal{L}}}) = 0$ and by the monotonicity of $\tilde{P}$ with respect to $\subseteq_{GN}$ relation [20], it follows $\tilde{P}(K|K^*_{\mathcal{B}_{\mathcal{L}i}}) = 0$ for $i = 1, \ldots, n$, and so $\tilde{P} \in \mathcal{D}_0^{\mathcal{B}_{\mathcal{L}i}}$ for $i = 1, \ldots, n$. This implies $\bigcap_{i=1}^n \mathcal{D}_0^{\mathcal{B}_{\mathcal{L}i}} \neq \emptyset$ and so $\mathbf{D}_0$ satisfies the finite intersection property which, in turn, implies $\bigcap \mathbf{D}_0 \neq \emptyset$, i.e., there exists $\tilde{P}_0 \in \bigcap \mathbf{D}_0$ such that $\tilde{P}_0(K|A) = 0$ for every $A \in \mathcal{A}^0_{\mathcal{L}}$ with $K \subseteq A$.

Finally, Lemma 2 implies $\underline{Q}^{\mathbf{cd}}(F|K) = \underline{Q}_{\tilde{P}_0}(F|K) = 0$. □

Hence, condition *(ii)* follows by Lemma 3. □

Also in this case, a simplification of condition *(ii)* of Theorem 8 is obtained in the case the functions on $\mathcal{L}$ defined as $X(\cdot) = \sigma(F \cap H|\cdot)$ and $(1 - Y(\cdot)) = (1 - \sigma(F^c \cap H|\cdot))$ are comonotonic (see (15)).

Proposition 3. *For every $F|K \in \mathcal{A} \times \mathcal{A}^0$ such that $F \cap K \neq K$, $K \in \mathcal{A}^0 \setminus \mathcal{A}^0_{\mathcal{L}}$ and there exists $A \in \mathcal{A}^0_{\mathcal{L}}$ such that $K \subseteq A$ and $\underline{Q}^{\mathbf{cd}}(K|A) > 0$, if $X(\cdot) = \sigma(F \cap H|\cdot)$ and $(1 - Y(\cdot)) = (1 - \sigma(F^c \cap H|\cdot))$ are comonotonic then*

$$\underline{Q}^{\mathbf{cd}}(F|K) = \frac{\underline{Q}^{\mathbf{cd}}(F \cap K|A)}{\underline{Q}^{\mathbf{cd}}(F \cap K|A) + \overline{Q}^{\mathbf{cd}}(F^c \cap K|A)}.$$

Proof. The proof goes along the same line of the proof of Proposition 2. By Corollary 2 we have that $\underline{\pi}^{\mathbf{f}}(\cdot|A)$ is a totally monotone capacity on $\mathcal{A}_{\mathcal{L}}$ inducing a core $\mathcal{P}_{\underline{\pi}^{\mathbf{f}}(\cdot|A)} = \{\tilde{\pi}^{\mathbf{f}}(\cdot|A)\}$ of finitely additive probability measures on $\mathcal{A}_{\mathcal{L}}$, defined as in (2), moreover, the functions $X(\cdot)$ and $(1 - Y(\cdot))$ are comonotonic and $\mathcal{A}_{\mathcal{L}}$-continuous.

By Proposition 6.26 in [58] there exists $\tilde{\pi}^{\mathbf{f}}(\cdot|A) \in \mathcal{P}_{\underline{\pi}^{\mathbf{f}}(\cdot|A)}$ such that

$$\begin{aligned}
\underline{Q}^{\mathbf{cd}}(F \cap K|A) &= \oint X(H_i)\underline{\pi}^{\mathbf{f}}(\mathrm{d}H_i|A) = \int X(H_i)\tilde{\pi}^{\mathbf{f}}(\mathrm{d}H_i|A) = L^{\mathbf{cd}}(F, K; A),\\
\overline{Q}^{\mathbf{cd}}(F^c \cap K|A) &= \oint Y(H_i)\overline{\pi}^{\mathbf{f}}(\mathrm{d}H_i|A) = \int Y(H_i)\tilde{\pi}^{\mathbf{f}}(\mathrm{d}H_i|A) = U^{\mathbf{cd}}(F^c, K; A),
\end{aligned}$$

and the conclusion follows. □

Example 8 (Example 5 continued)**.** *Since $\mathcal{A}_{\mathcal{L}} = \langle \mathcal{L} \rangle^*$ it immediately follows that for every $F \in \mathcal{A}$, $\sigma(F|\cdot)$ is an $\mathcal{A}_{\mathcal{L}}$-continuous function on $\mathcal{L}$.*

For $A|B$, since $B \in \mathcal{A}^0_{\mathcal{L}}$ and $\pi(B) = 0$, it follows $\underline{Q}^{\mathbf{cd}}(A|B) = \left(\frac{1}{10}\right)^n$, $\underline{Q}^{\mathbf{cd}}(A^c|B) = 1 - \left(\frac{9}{10}\right)^n$, and so $\overline{Q}^{\mathbf{cd}}(A|B) = \left(\frac{9}{10}\right)^n$ that coincide with the probability bounds determined by the whole set of coherent extensions.

For $A|C$, since $C \in \mathcal{A}^0_{\mathcal{L}}$ and $\pi(C) = 0$, it follows $\underline{Q}^{\mathbf{cd}}(A|C) = \left(\frac{1}{3}\right)^n$, $\underline{Q}^{\mathbf{cd}}(A^c|C) = 1 - \left(\frac{5}{6}\right)^n$, and so $\overline{Q}^{\mathbf{cd}}(A|C) = \left(\frac{5}{6}\right)^n$.

For $D|E$, we have that $\underline{Q}^{\mathbf{cd}}(E|\Omega) = P^{\mathbf{jd}}(E) = \int \sigma(E|H_\theta)\pi(\mathrm{d}H_\theta) = \frac{2}{n+1}$ and $\underline{Q}^{\mathbf{cd}}(D \cap E|\Omega) = P^{\mathbf{jd}}(D \cap E) = \int \sigma(D \cap E|H_\theta)\pi(\mathrm{d}H_\theta) = \frac{1}{n+1}$, thus $\underline{Q}^{\mathbf{cd}}(D|E) = \overline{Q}^{\mathbf{cd}}(D|E) = \underline{Q}^{\mathbf{c}}(D|E) = \overline{Q}^{\mathbf{c}}(D|E) = \frac{1}{2}$.

It is well-known (see, e.g., [37]) that the pointwise convex combination of two coherent conditional probabilities is generally not a coherent conditional probability. Now, we show that the same holds for full conditional probabilities extending $\{\pi, \sigma\}$. In case of a finite partition $\mathcal{L}$, the three characterized sets of extensions $\mathcal{Q}$, $\mathcal{Q}^{\mathbf{c}}$ and $\mathcal{Q}^{\mathbf{cc}}$ trivially coincide, moreover, the following example shows that none of them is generally closed under pointwise convex combinations.

Example 9. *Let $\Omega = \{\omega_1, \omega_2, \omega_2, \omega_4\}$ and take the finite partitions $\mathcal{L} = \{H_1 = \{\omega_1, \omega_3\}, H_2 = \{\omega_2, \omega_4\}\}$ and $\mathcal{E} = \{E_1 = \{\omega_1, \omega_2\}, E_2 = \{\omega_3, \omega_4\}\}$, for which $H_i \cap E_j \neq \emptyset$ for all i, j. Consider the algebras $\mathcal{A}_{\mathcal{E}} = \langle \mathcal{E} \rangle$, $\mathcal{A}_{\mathcal{L}} = \langle \mathcal{L} \rangle$ and $\mathcal{A} = \langle \mathcal{A}_{\mathcal{E}} \cup \mathcal{A}_{\mathcal{E}} \rangle = \wp(\Omega)$.*

Take the prior probability π on $\mathcal{A}_{\mathcal{L}}$ such that $\pi(H_1) = 0$ and $\pi(H_2) = 1$, together with the statistical model λ on $\mathcal{A}_{\mathcal{E}} \times \mathcal{L}$ such that

$$\lambda(E_1|H_1) = 0, \quad \lambda(E_2|H_1) = 1 \quad \text{and} \quad \lambda(E_1|H_2) = \lambda(E_2|H_2) = \frac{1}{2},$$

that uniquely extends to a strategy σ on $\mathcal{A} \times \mathcal{L}$ by Proposition 1. Since $\mathcal{L}$ is finite we have that $\mathcal{Q} = \mathcal{Q}^{\mathbf{c}} = \mathcal{Q}^{\mathbf{cc}}$.

In order to build $Q^1 \in \mathcal{Q}$, let ν_0 be the additive measure on $\wp(\Omega)$ such that $\nu_0(\{\omega_4\}) = 1$ and $\nu_0(\{\omega_1\}) = \nu_0(\{\omega_2\}) = \nu_0(\{\omega_3\}) = 0$, ν_1 be the additive measure on $\wp(\{\omega_1, \omega_2, \omega_3\})$ such that $\nu_1(\{\omega_2\}) = 1$ and $\nu_1(\{\omega_1\}) = \nu_1(\{\omega_3\}) = 0$, and ν_2 be the additive measure on $\wp(\{\omega_1, \omega_3\})$ such that $\nu_2(\{\omega_1\}) = \nu_2(\{\omega_3\}) = \frac{1}{2}$. For every $F|K \in \mathcal{A} \times \mathcal{A}^0$, let $\alpha_K \in \{0, 1, 2\}$ be the minimum index such that $\nu_{\alpha_K}(K) > 0$ and define

$$Q^1(F|K) = \frac{\nu_{\alpha_K}(F \cap K)}{\nu_{\alpha_K}(K)}.$$

Analogously, to build $Q^2 \in \mathcal{Q}$, let μ_0 be the additive measure on $\wp(\Omega)$ such that $\mu_0(\{\omega_4\}) = 1$ and $\mu_0(\{\omega_1\}) = \mu_0(\{\omega_2\}) = \mu_0(\{\omega_3\}) = 0$, μ_1 be the additive measure on $\wp(\{\omega_1, \omega_2, \omega_3\})$ such that $\mu_1(\{\omega_1\}) = \mu_1(\{\omega_3\}) = \frac{1}{2}$ and $\mu_1(\{\omega_2\}) = 0$, and μ_2 be the additive measure on $\wp(\{\omega_2\})$ such that $\mu_2(\{\omega_2\}) = 1$. For every

$F|K \in \mathcal{A} \times \mathcal{A}^0$, let $\beta_K \in \{0,1,2\}$ be the minimum index such that $\mu_{\beta_K}(K) > 0$ and define

$$Q^2(F|K) = \frac{\mu_{\beta_K}(F \cap K)}{\mu_{\beta_K}(K)}.$$

Simple computations show that $Q^1, Q^2 \in \mathcal{Q}$, and, in particular, taking $A = \{\omega_1\}$, $B = \{\omega_1, \omega_2\}$ and $C = \{\omega_1, \omega_2, \omega_3\}$ we get $Q^1(A|C) = Q^1(B|C) = \frac{1}{2}$, $Q^1(A|B) = Q^2(B|C) = 1$ and $Q^2(A|C) = Q^2(A|B) = 0$. Denoting $Q^* = \frac{1}{2}Q^1 + \frac{1}{2}Q^2$, where the convex combination is intended pointwise on the elements of $\mathcal{A} \times \mathcal{A}^0$, we have

$$Q^*(A|C) = \frac{1}{4} \neq \frac{3}{8} = Q^*(A|B) \cdot Q^*(B|C),$$

that is Q^* fails axiom **(C3)** *and so $Q^* \notin \mathcal{Q}$.*

Remark 7. *All the three sets $\mathcal{Q}$, $\mathcal{Q}^{\mathbf{c}}$ and $\mathcal{Q}^{\mathbf{cc}}$ are compact and, especially concerning $\mathcal{Q}^{\mathbf{c}}$ and $\mathcal{Q}^{\mathbf{cc}}$, their closure with respect to limits of nets is in contrast with Walley's approach, where conglomerability is not preserved by limits of nets, in general. Also in this case, the main difference rests upon the fact that we are imposing conglomerability on "precise" models while in Walley's theory, a suitable form of conglomerability is asked directly on the lower envelope.*

We now restrict to a situation in which the usual Bayes rule for densities is applicable: let us consider two partitions $\mathcal{L} = \{H_i\}_{i \in I}$ and $\mathcal{E} = \{E_j\}_{j \in J}$ of Ω such that $H_i \cap E_j \neq \emptyset$ for every i, j, $\mathcal{A}_{\mathcal{L}}$ and $\mathcal{A}_{\mathcal{E}}$ are σ-algebras, and $\mathcal{A} = \langle \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}} \rangle^{\sigma}$.

Remark 8. *For instance, we fall in the above situation by considering two measurable spaces $(\mathcal{X}, \mathcal{A}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{A}_{\mathcal{Y}})$ where $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{Y}}$ are σ-algebras containing, respectively, $\{\{x\} : x \in \mathcal{X}\}$ and $\{\{y\} : y \in \mathcal{Y}\}$, and taking the product space $(\mathcal{X} \times \mathcal{Y}, \mathcal{A}_{\mathcal{X}} \otimes \mathcal{A}_{\mathcal{Y}})$. The σ-algebras $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{Y}}$ are identified with the corresponding sub-σ-algebras of $\mathcal{A}_{\mathcal{X}} \otimes \mathcal{A}_{\mathcal{Y}}$ whose atoms are, respectively, $\{\{x\} \times \mathcal{Y} : x \in \mathcal{X}\}$ and $\{\mathcal{X} \times \{y\} : y \in \mathcal{Y}\}$ which correspond to the searched partitions of $\mathcal{X} \times \mathcal{Y}$.*

If π is countably additive on $\mathcal{A}_{\mathcal{L}}$ and λ is a statistical model on $\mathcal{A}_{\mathcal{E}} \times \mathcal{L}$ such that $\lambda(\cdot|H_i)$ is countably additive and absolutely continuous with respect to the same σ-finite measure μ on $\mathcal{A}_{\mathcal{E}}$, for every $H_i \in \mathcal{L}$, then $l(\cdot; H_i) = \frac{\mathrm{d}\lambda(\cdot|H_i)}{\mathrm{d}\mu}$ is the Radon-Nikodym derivative of $\lambda(\cdot|H_i)$ with respect to μ. Under the previous hypotheses the function λ is also called a *transition kernel* and $l(E_j; \cdot)$ is assumed to be $\mathcal{A}_{\mathcal{L}}$-measurable for every $E_j \in \mathcal{E}$.

For general algebras $\mathcal{A}_{\mathcal{L}}$ and $\mathcal{A}_{\mathcal{E}}$, coherence only implies that λ uniquely extends to a strategy on $\langle \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}} \rangle \times \mathcal{L}$ by Proposition 1. Nevertheless, the condition $H_i \cap E_j \neq \emptyset$ for every i, j allows to identify $\mathcal{L}$ and $\mathcal{E}$ with the points of two measurable spaces, and $\mathcal{A}_{\mathcal{L}}$ and $\mathcal{A}_{\mathcal{E}}$ with the corresponding σ-algebras, thus $\mathcal{A}$ is easily seen to be isomorphic to the product σ-algebra generated by $\mathcal{A}_{\mathcal{L}}$ and $\mathcal{A}_{\mathcal{E}}$. This implies that, fixing $H_i \in \mathcal{L}$, for every $F \in \mathcal{A}$ there exists

$F_{H_i} \in \mathcal{A}_{\mathcal{E}}$ such that $F \cap H_i = F_{H_i} \cap H_i$. Thus, for every strategy σ on $\mathcal{A} \times \mathcal{L}$ extending λ it must hold

$$\sigma(F|H_i) = \sigma(F \cap H_i|H_i) = \sigma(F_{H_i} \cap H_i|H_i) = \sigma(F_{H_i}|H_i) = \lambda(F_{H_i}|H_i),$$

so σ is uniquely determined by λ, $\sigma(\cdot|H_i)$ is countably additive on $\mathcal{A}$ for every $H_i \in \mathcal{L}$, and $\sigma(F|\cdot)$ is bounded and $\mathcal{A}_{\mathcal{L}}$-measurable for every $F \in \mathcal{A}$. In turn, this implies $P^{\mathbf{jd}}(\cdot) = \underline{Q}^{\mathbf{cd}}(\cdot|\Omega)$ is countably additive on $\mathcal{A}$, moreover, for every $B|E_j \in \mathcal{A}_{\mathcal{L}} \times \mathcal{E}$ such that $0 < \int l(E_j; H_i)\pi(\mathrm{d}H_i) < +\infty$ it holds

$$\underline{Q}^{\mathbf{cd}}(B|E_j) \leq \frac{\int l(E_j; H_i)\mathbf{1}_B(H_i)\pi(\mathrm{d}H_i)}{\int l(E_j; H_i)\pi(\mathrm{d}H_i)} \leq \overline{Q}^{\mathbf{cd}}(B|E_j),$$

where the involved integrals are in the Lebesgue sense. If further π is absolutely continuous with respect to a σ-finite measure ν on $\mathcal{A}_{\mathcal{L}}$, then $p = \frac{\mathrm{d}\pi}{\mathrm{d}\nu}$ is the Radon-Nikodym derivative of π with respect to ν and we obtain that the usual statement of Bayes theorem for densities produces a coherent value, as proved in [3], even if the inequalities above may be strict, as shown in the following example.

Example 10. *Consider two random quantities Θ and X ranging, respectively, on $\mathbf{\Theta} = \mathbf{X} = [0,1]$, and let $\mathcal{L} = \{H_\theta = (\Theta = \theta) \ : \ \theta \in \mathbf{\Theta}\}$ and $\mathcal{E} = \{E_x = (X = x) \ : \ x \in \mathbf{X}\}$ with $H_\theta \cap E_x \neq \emptyset$ for every θ, x. Let $\mathcal{A}_{\mathcal{L}}$ and $\mathcal{A}_{\mathcal{E}}$ be both isomorphic to the Borel σ-algebra on $[0,1]$, and $\mathcal{A} = \langle \mathcal{A}_{\mathcal{L}} \cup \mathcal{A}_{\mathcal{E}} \rangle^\sigma$.*

Let π and $\lambda(\cdot|H_\theta)$, for every $\theta \in \mathbf{\Theta}$, coincide with the Lebesgue measure on $[0,1]$. The statistical model λ extends uniquely to a strategy σ on $\mathcal{A} \times \mathcal{L}$.

Let $B = (\Theta \in [0,0.5])$ and $E_{0.5} = (X = 0.5)$, in order to be $\underline{Q}^{\mathbf{cd}}(B|E_{0.5}) \neq 0$ there must exist $A \in \mathcal{A}^0_{\mathcal{L}}$ such that $\underline{Q}^{\mathbf{cd}}(E_{0.5}|A) > 0$ where

$$\underline{Q}^{\mathbf{cd}}(E_{0.5}|A) = \begin{cases} \frac{\int \sigma(E_{0.5} \cap A|H_\theta)\pi(\mathrm{d}H_\theta)}{\pi(A)} = \frac{\int \lambda(E_{0.5}|H_\theta)\mathbf{1}_A(H_\theta)\pi(\mathrm{d}H_\theta)}{\pi(A)} & \text{if } \pi(A) > 0, \\ \inf\limits_{H_\theta \subseteq A} \lambda(E_{0.5}|H_\theta) & \text{otherwise.} \end{cases}$$

Notice that $l(E_x; H_\theta) = 1$ for every x, θ. Since $\lambda(E_{0.5}|H_\theta) = \int_{0.5}^{0.5} \mathrm{d}x = 0$, for every $\theta \in \mathbf{\Theta}$, it trivially holds that $\underline{Q}^{\mathbf{cd}}(E_{0.5}|A) = 0$ for every $A \in \mathcal{A}^0_{\mathcal{L}}$, so it must be $\underline{Q}^{\mathbf{cd}}(B|E_{0.5}) = 0$. Similarly, it is possible to show that $\overline{Q}^{\mathbf{cd}}(B|E_{0.5}) = 1 - \underline{Q}^{\mathbf{cd}}(B^c|E_{0.5}) = 1$, that is the conditional probability of $B|E_{0.5}$ ranges in $[0,1]$.

6. Conclusions

The problem of characterizing the set of full conditional probabilities extending a prior probability and a strategy plays a fundamental role in probability, statistics and mathematical economics, as highlighted in Section 1. Furthermore, it reveals to be crucial also in fuzzy set theory and, in particular, in probabilistic fuzzy reasoning [14, 17, 21, 33, 52, 54]. In this paper the envelopes of the following classes of extensions have been characterized: *(i)* the class of all full conditional probability extensions; *(ii)* the class of conglomerable full conditional probability extensions; *(iii)* the class of conditionally conglomerable full conditional probability extensions.

Analytical properties such as conglomerability and disintegrability gather particular attention also in other non-additive settings [15, 42, 43, 44, 62] both because of their mathematical convenience and their behavioural interpretation.

An open problem is to characterize the set of full conditional probabilities singled out by a lower prior probability $\underline{\pi}$ and a lower strategy $\underline{\sigma}$, meeting the further condition of (conditional) conglomerability. This could lead to develop a theory of *coherent lower conglomerable conditional probabilities* based on (conditionally) conglomerable extensions determined by the class of prior probabilities dominating $\underline{\pi}$ and the class of strategies dominating $\underline{\sigma}$. This study could help to deepen the connection with Walley's theory.

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