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The controllability of an impulsive integro-differential process with nonlocal feedback controls

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Abstract: The controllability of an impulsive process governed by a parametric integro-differential equation involving a Volterra operator is shown. The model is studied via the existence of impulsive mild solutions for a semilinear integro-differential inclusion. A discussion on the impulse functions is presented.

Keywords: Integro-differential equations; feedback controls; impulses; population dynamics

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1 Introduction

It is known that the spreading disease in a territory is a worldwide problem and the mode of spread of the disease may take different forms. In 1927 Kermack and McKendrik [19] proposed a model driven by a nonlinear integral equation, later extensively studied. Kendall [18] generalized the Kermack-McKendrik model to a space-dependent integro-differential equation in which it was assumed that the infected individuals become immediately infectious, without considering the incubation period. Following this line, most of the research literature on epidemic models assumes that the disease incubation is negligible leading to the SIR models. Of course, the literature is wide; we just refer, as examples, to the papers [16] where the spatial dynamics of a class of integro-differential equations are studied, and [25] where the authors combine pest control and infectious disease investigating the control problem in the management of an epidemic to control a pest population.
Actually, in real phenomena often happens that incubation time assumes a relevance in the study of the disease, as well as the pregnancy time in the evolution of a population or the maturation delay of the individual (i.e the time between birth and the moment when the individual is involved in the reproductive process). A notable example is the Nicholson's blowflies equation

$$N'(t) = -\delta N(t) + pN(t - T_D)e^{-aN(t-T_D)} \tag{1}$$

(T_D is the fixed delay). It was first used by Gurney, Blythe and Nisbet [15] to explain Nicholson's experimental data [21] with the Australian sheep blowflie and later widly studied, also in the case of a distributed delay (see, e.g. [12], [13], [14]).

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In [15] the authors present a new class of equations, of which (1) is a special case. They observe that the insect population growth in an isolated laboratory culture can be better described by the equation

$$N'(t) = -\delta N(t) + R(N(t - T_D)), \tag{2}$$

than by the classical time-delayed logistic model $N'(t) = rN(t)\left[1 - \frac{N(t-T_D)}{K}\right]$.
Following them, in this paper we study a process driven by the parametric integro-differential equation

$$\frac{\partial u}{\partial t}(t,z) = -b(t,z)u(t,z) + g\left(t, u(t,z), \int_0^t \frac{e^{s/T}}{T} u(s,z)\,ds\right), \tag{3}$$

where $u : [0,T] \times [0,1] \to I\!R$, $b : [0,T] \times [0,1] \to I\!R$, and $g : [0,T] \times I\!R \times I\!R \to I\!R$ are given functions.
In a biological setting, $u(t,z)$ and $b(t,z)$ represent the density of a population and the removal coefficient (including death and migration) at time t and position z; g is the population development law, affected by a distributed delay which accounts for a memory-effect expressed by the Volterra integral operator; the parameter T measures the relevance of the delay. Note that if $b(t,z) = \delta$ and $g(t,p,q) = aqe^{-t/T}$, then equation (3) becomes

$$\frac{\partial u}{\partial t}(t,z) = -\delta u(t,z) + a\int_0^t \frac{e^{-(t-s)/T}}{T} u(s,z)\,ds,$$

i.e. a particular case of (2), but restated for a dispersal delay. Indeed, the importance of events decreases exponentially the further one looks into the past. Moreover, the kernel $k(\tau) = \frac{e^{-\tau/T}}{T}$ clearly satisfies $k(\tau) \geq 0$ and $\int_0^\infty k(\tau)d\tau = 1$, so $\int_{\tau_1}^{\tau_2} k(\tau)d\tau$ can be interpretated as the probability that the delay is between τ_1 and τ_2.
The problem we consider in this note is subject to feedback controls, which often appear in models from the life sciences. We refer for instance to the book [11]; there the authors illustrate many situations in systems biology in which feedback control theory is used. In our paper the control is expressed by a further term we add to the right hand-side of equation (3), leading to

$$\frac{\partial u}{\partial t}(t,z) = -b(t,z)u(t,z) + g\left(t, u(t,z), \int_0^t \frac{e^{s/T}}{T} u(s,z)\,ds\right) + \omega(t,z), \tag{4}$$

where ω is subject to the condition

$$\omega(t,\cdot) \in W(u(t,\cdot)). \tag{5}$$

By its formulation (see (9)), the set of functions $W(u(t,\cdot))$ depends on the weighted values of $u(t,\cdot)$ all over the habitat normalized to $[0,1]$; that is we deal with a control strategy with a nonlocal nature. For example we can consider a population in a controlled environment sensitive to external factors such as temperature (like a pest population in a greenhouse or a bacterial culture in an *in vitro* culture medium).
In our problem, we also consider the presence of impulse functions representing the action of instantaneous external forces on the system at fixed times

$$u\left(t_i^+, z\right) = u\left(t_i, z\right) + \mathcal{I}_i(u\left(t_i, z\right)), \; i = 1, \ldots, p. \tag{6}$$

Actually many real phenomena, in Biology, Medicine, Physics or other where abrupt perturbations occur or sudden behavioral changes happen, can be described by means of impulse functions. A wide production appeared on this subject in the last years (see, e.g. [8], [9], [10], [22]) even in the integro-differential case (see, e.g. [1]).

We study the model by using techniques from the theory of ordinary differential equations/inclusions and from the semigroup theory as well. This approach allows us to provide the existence of mild stratiegies solving the problem, even in the case when classical solutions do not exist.

The paper is organized as follows. In Section 2 we state the problem given by (4)-(6).

Section 3 is devoted to the study of the existence of strategies for the model. We reformulate the problem as a multivalued impulsive Cauchy problem (see (15)) driven by the *ordinary* semilinear integro-differential inclusion in the space $L^2([0,1])$

$$v'(t) \in A(t)v(t) + \tilde{g}\left(t, v(t), \int_0^t \frac{e^{s/T}}{T} v(s)ds\right) + W(v(t)). \tag{7}$$

In Sections 3.1 and 3.2 we list the set of assumptions on the functions appearing in the model and deduce the corresponding properties on the maps of the associated problem. Successively, in Section 3.3 we discuss the properties of the multimap $F = \tilde{g} + W$. Then, after the proof of the existence of an impulsive mild solution for the problem governed by the ordinary integro-differential inclusion (7), we deduce the existence of a strategy which solves the model. An example of nonlinearity including a dispersal kernel is provided.

Eventually, in Section 4 we present a discussion on the impulse functions acting in the process. These functions in Biology are called "regulation functions" since their presence in the system leads to a regulation of the model. For example, in practical pest management the pesticide is not periodically used, but the human control acts on the pest population only if it overcomes prescribed thresholds at fixed times. We describe this kind of situation in Example 4.1. Then, we observe that in our results we can take completely arbitrary impulse functions and produce an example where they are not continuous.

To assist the reader, in the Appendix we list the symbols and the definitions used throughout the paper.

2 Position of the problem

We consider the following process with feedback controls described by a parametric integro-differential equation involving a Volterra operator and subject to impulses

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,z) = -b(t,z)u(t,z) + g\left(t, u(t,z), \displaystyle\int_0^t \frac{e^{s/T}}{T}\, u(s,z)\, ds\right) + \omega(t,z)\,, \\ \qquad\qquad\qquad\qquad t \in [0,T]\,,\ t \neq t_i\,,\ i = 1,\ldots,p\,,\ a.e.\, z \in [0,1], \\ \omega(t,\cdot) \in W(u(t,\cdot))\,,\ a.e.\, t \in [0,T], \\ u\left(t_i^+, z\right) = u\left(t_i, z\right) + \mathcal{I}_i(u\left(t_i, z\right))\ ,\ i = 1,\ldots,p\ ,\ a.e.\, z \in [0,1], \\ u(0,z) = \alpha_0(z)\,, a.e.\, z \in [0,1], \end{cases} \tag{8}$$

where: $T > 0$; $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$; $b : [0,T] \times [0,1] \to I\!R$; $g : [0,T] \times I\!R \times I\!R \to I\!R$; for every $i = 1,\ldots,p$, $\mathcal{I}_i : I\!R \to I\!R$ and $u(t_i^+,\cdot) = \lim_{s\to t_i^+} u(s,\cdot)$; $\alpha_0 \in L^2([0,1])$. The map $W : L^2([0,1]) \to \mathcal{P}(L^2([0,1]))$ is defined by

$$W(v) = \left\{ \begin{array}{l} \beta \in AC([0,1]) : \\ f_1(z, \int_0^1 \varphi(\zeta)v(\zeta)\, d\zeta) \leq \beta(z) \leq f_2(z, \int_0^1 \varphi(\zeta)v(\zeta)\, d\zeta) \\ \text{and } |\beta'(z)| \leq l(z), \text{ for a.a. } z \in [0,1] \end{array} \right\}, v \in L^2([0,1]), \tag{9}$$

where $f_1, f_2 : [0,1] \times I\!R \to I\!R$, $l \in L^1_+([0,1])$, and $\varphi \in L^2([0,1])$ are given functions.
The map W models the feedback control depending on the weighted values of $u(t,\cdot)$ all over the interval $[0,1]$; in other words, we have a strategy where the control at time t and position z depends on the value given by $\int_0^1 \varphi(\zeta)u(t,\zeta)\, d\zeta$, leading to a nonlocal structure of the control itself. An example of this kind of multimap can be found in [20].

For every $i = 1,\ldots,p$, the maps $\mathcal{I}_i : I\!R \to I\!R$ are the impulse functions and represent instantaneous external actions on the system.

Definition 2.1 A couple (u,ω) of functions $u,\omega : [0,T] \times [0,1] \to I\!R$ is said to be *an impulsive mild admissible pair for* (8) if: $u(t,\cdot) \in L^2([0,1])$ for every $t \in [0,T]$; $u(\cdot,z) \in PC([0,T], I\!R)$, for all $z \in [0,1]$; u satisfies the identity

$$\begin{aligned} u(t,z) \;&=\; e^{\int_0^t -b(\sigma,z)d\sigma}\alpha_0(z) + \sum_{0<t_i<t} e^{\int_{t_i}^t -b(\sigma,z)d\sigma}\mathcal{I}_i(u(t_i,z)) + \\ &+ \int_0^t e^{\int_s^t -b(\sigma,z)d\sigma}\left[g\left(s, u(s,z), \int_0^s \frac{e^{\sigma/T}}{T}\, u(\sigma,z)\, d\sigma\right) + \omega(s,z)\right] ds, \end{aligned}$$

for every $t \in [0,T]$, $z \in [0,1]$, where $\omega(s,\cdot) \in W(u(s,\cdot))$, a.e. $s \in [0,T]$.

3 Existence of strategies

Our approach to the problem provides for rewriting the feedback control process (8) as an impulsive problem driven by an ordinary integro-differential inclusion in the space $L^2([0,1])$.

To this aim, we put:

$v(t)(z) = u(t,z)$ and $w(t)(z) = \omega(t,z)$, $t \in [0,T], z \in [0,1]$; clearly, $v, w : [0,T] \to L^2([0,1])$;

$A(t) : L^2([0,1]) \to L^2([0,1]),\ t \in [0,T]$

$$A(t)v(z) = -b(t,z)v(z)\,,\ \ z \in [0,1]\,, v \in L^2([0,1]); \tag{10}$$

$\tilde{g} : [0,T] \times L^2([0,1]) \times L^2([0,1]) \to L^2([0,1])$

$$\tilde{g}(t,v,w)(z) = g(t,v(z),w(z)), t \in [0,T],\ v,w \in L^2([0,1]); \tag{11}$$

$I_i : L^2([0,1]) \to L^2([0,1]),\ i = 1,\dots,p,$

$$I_i(v)(z) = \mathcal{I}_i(v(z)),\ z \in [0,1], v \in L^2([0,1]). \tag{12}$$

So, we can pass from problem (8) to the next

$$\begin{cases} v'(t) = A(t)v(t) + \tilde{g}\left(t, v(t), \int_0^t \frac{e^{s/T}}{T} v(s)ds\right) + w(t),\ t \in [0,T]\,, \\ \hfill t \neq t_i\,,\ i = 1,\dots,p, \\ w(t) \in W(v(t)),\ a.e.\, t \in [0,T], \\ v(t_i^+) = v(t_i) + I_i(v(t_i))\,,\ i = 1,\dots,p, \\ v(0) = \alpha_0\ . \end{cases} \tag{13}$$

Now, we define the map $F : [0,T] \times L^2([0,1]) \times L^2([0,1]) \to \mathcal{P}(L^2([0,1]))$ as

$$F(t,v,w) = \tilde{g}(t,v,w) + W(v)\,,\ (t,v,w) \in [0,T] \times L^2([0,1]) \times L^2([0,1]), \tag{14}$$

therefore, problem (13) becomes the multivalued impulsive Cauchy problem

$$\begin{cases} v'(t) \in A(t)v(t) + F\left(t, v(t), \int_0^t \frac{e^{s/T}}{T} v(s)ds\right)\ ,\ t \in [0,T]\,,\ t \neq t_i\,,\ i = 1,\dots,p, \\ v(t_i^+) = v(t_i) + I_i(v(t_i))\,,\ i = 1,\dots,p, \\ v(0) = \alpha_0. \end{cases} \tag{15}$$

3.1 Assumptions on the *linear* part

We assume that the map $b : [0,T] \times [0,1] \to \mathbb{R}$ of problem (8) satisfies the conditions

(b.1) b is measurable;

(b.2) there exists $s \in L^1_+([0,T])$ such that

$$0 < b(t,z) \leq s(t)\,,\ \text{for every } t \in [0,T],\ \text{a.e. } z \in [0,1];$$

(b.3) for every $z \in [0,1]$, the function $b(\cdot,z) : [0,T] \to \mathbb{R}$ is continuous.

Consequently, on the family $\{A(t)\}_{t\in[0,T]}$ of problems (13) and (15) we have the following results.

Proposition 3.1 *(cf. [20, Section 3.1]) Under conditions (b.1), (b.2), for every $t \in [0,T]$ the map $A(t)$ defined in* (10) *is a well-posed linear operator.*

Proposition 3.2 *Under conditions (b.1)-(b.3), the family $\{A(t)\}_{t\in[0,T]}$ defined by* (10) *generates the evolution system $\{T(t,s)\}_{0\le s\le t\le T}$ of bounded linear operators $T(t,s) : L^2([0,1]) \to L^2([0,1])$, $0 \le s \le t \le T$, defined by*

$$[T(t,s)v](z) = e^{\int_s^t -b(\sigma,z)d\sigma} v(z)\,,\ \ z \in [0,1],\, v \in L^2([0,1]). \tag{16}$$

Proof. First of all, each operator $T(t,s)$ is well-defined. Indeed, let us fix $v \in L^2([0,1])$ and consider the map $T(t,s)v : [0,1] \to I\!R$ defined by (16). By (b.1) and the Tonelli's Theorem, the map $z \mapsto e^{\int_s^t -b(\sigma,z)d\sigma}$ is measurable, so $T(t,s)v$ is measurable too. Moreover, since $e^{\int_s^t -b(\sigma,z)d\sigma} < 1$ (see (b.2)), by (16) we have $\int_0^1 [T(t,s)v(z)]^2 dz < \|v\|_2^2$, and so $T(t,s)v \in L^2([0,1])$.

Then, it is easily seen that each $T(t,s)$ is a bounded and linear operator.

Moreover, it is immediate to check that the family $\{T(t,s)\}_{0\le s\le t\le T}$ has property (T1) (see Section 4). Now we show that also (T2) is satisfied. Fix $v \in L^2([0,1])$, $(\bar{t},\bar{s})$ with $0 \le \bar{s} \le \bar{t} \le T$ and $\{(t_n,s_n)\}_n$ with $0 \le s_n \le t_n \le T$ such that $(t_n,s_n) \to (\bar{t},\bar{s})$; then, let us define

$$h_n(z) = \left[e^{\int_{s_n}^{t_n} -b(\sigma,z)d\sigma} - e^{\int_{\bar{s}}^{\bar{t}} -b(\sigma,z)d\sigma}\right]^2 v^2(z),\ \ z \in [0,1],\ \ n \in \mathbb{N}.$$

Note that $(h_n)_n$ are L^1-functions point-wise converging in $[0,1]$ to zero; further, by (b.2) we have that $|h_n(z)| \le 4v^2(z)$ for every $z \in [0,1]$. Therefore, by using the dominated convergence theorem we obtain

$$\lim_{n\to\infty} \|T(t_n,s_n)v - T(\bar{t},\bar{s})v\|_2^2 = 0,$$

and so $T(t_n,s_n)v \to T(\bar{t},\bar{s})v$ in $L^2([0,1])$.

Then we can conclude that the family $\{T(t,s)\}_{0\le s\le t\le T}$ is an evolution system.

Finally, $\{T(t,s)\}_{0\le s\le t\le T}$ is generated by the family $\{A(t)\}_{t\in[0,T]}$ defined in (10), i.e. (see [7]) on the region $D(A) = L^2([0,1])$, each operator $T(t,s)$ is strongly differentiable relative to t and s, while

$$\frac{\partial T(t,s)}{\partial t} = A(t)T(t,s) \qquad \text{and} \qquad \frac{\partial T(t,s)}{\partial s} = -T(t,s)A(s).$$

To prove these, we fix $v \in L^2([0,1])$ and $z \in [0,1]$. By (16), (b.3) and (10) we have the identities

$$\begin{aligned} \frac{\partial T(t,s)}{\partial t} v(z) &= \frac{\partial}{\partial t} e^{\int_s^t -b(\sigma,z)d\sigma} v(z) = -b(t,z) e^{\int_s^t -b(\sigma,z)d\sigma} v(z) = A(t)T(t,s)v(z), \\ \frac{\partial T(t,s)}{\partial s} v(z) &= \frac{\partial}{\partial s} e^{\int_s^t -b(\sigma,z)d\sigma} v(z) = e^{\int_s^t -b(\sigma,z)d\sigma} b(s,z) v(z) = -T(t,s)A(s)v(z). \end{aligned}$$

□

Remark 3.1 It is worthy to note that the evolution system defined by (16) is not compact.

3.2 Assumptions on the *nonlinear* part

On the map $g : [0,T] \times I\!R \times I\!R \to I\!R$ of problem (8) we suppose that:

(g.1) for every $\epsilon > 0$ there exists a compact $C_\epsilon \subset [0,T]$ such that $\lambda([0,T] \setminus C_\epsilon) < \epsilon$ and $g_{| C_\epsilon \times I\!R \times I\!R}$ is continuous; here λ is the Lebesgue measure on $I\!R$;

(g.2) there exists a function $\psi \in L^2([0,1])$ such that

$$|g(t, v(z), w(z))| \leq \psi(z), \text{ for a.e. } z \in [0,1], \text{ every } t \in [0,T], \text{ every } v, w \in L^2([0,1]);$$

(g.3) there exists a function $h \in L^1_+([0,T])$ such that for every bounded $\Omega_1, \Omega_2 \subset L^2([0,1])$,

$$\chi_2(g(t, \Omega_1(\cdot), \Omega_2(\cdot))) \leq h(t) \left[\chi_2(\Omega_1) + \chi_2(\Omega_2)\right], \text{ for a.e. } t \in [0,T],$$

where χ_2 is the Hausdorff measure of noncompactness in $L^2([0,1])$.

The next proposition provide the properties of the map $\tilde{g}$ of problem (13).

Proposition 3.3 *If (g.1)-(g.2) hold, then the map* $\tilde{g} : [0,T] \times L^2([0,1]) \times L^2([0,1]) \to L^2([0,1])$ *defined in* (11) *satisfies the Scorza-Dragoni property.*

Proof. Fix $\epsilon > 0$ and consider the set C_ϵ from (g.1). Consider any $(t_0, v_0, w_0) \in C_\epsilon \times L^2([0,1]) \times L^2([0,1])$ and let $\{(t_n, v_n, w_n)\}_n \subset C_\epsilon \times L^2([0,1]) \times L^2([0,1])$ be a sequence such that $(t_n, v_n, w_n) \to (t_0, v_0, w_0)$ in $C_\epsilon \times L^2([0,1]) \times L^2([0,1])$.
By (11) and the continuity of g on the set $C_\epsilon \times I\!R \times I\!R$, from the above convergence we have

$$\tilde{g}(t_n, v_n, w_n)(z) = g(t_n, v_n(z), w_n(z)) \to g(t_0, v_0(z), w_0(z)) = \tilde{g}(t_0, v_0, w_0)(z), \ a.e.\, z \in [0,1].$$

This convergence and hypothesis (g.2) yield that we can apply the dominated convergence theorem, so that $\tilde{g}(t_n, v_n, w_n) \to \tilde{g}(t_0, v_0, w_0)$ in $L^2([0,1])$. □

On the feedback control map $W : L^2([0,1]) \to \mathcal{P}(L^2([0,1]))$, we assume that the functions $f_1, f_2 : [0,1] \times I\!R \to I\!R$ satisfy the following conditions:

(f.1) $f_i(\cdot, r) \in AC([0,1])$, for every $r \in I\!R$, $i = 1,2$;

(f.2) $\left|\frac{\partial f_i}{\partial z}(z,r)\right| \leq l(z)$, for a.e. $z \in [0,1]$ and every $r \in I\!R$, $i = 1,2$;

(f.3) $f_1(z,r) \leq f_2(z,r)$, for every $z \in [0,1]$ and every $r \in I\!R$;

(f.4) there exists $c > 0$ such that

$$\left| f_i\left(z, \int_0^1 \varphi(\zeta) v(\zeta)\, d\zeta\right)\right| \leq c\, \|v\|_2, \text{ a.e. } z \in [0,1], \text{ every } v \in L^2([0,1]), \ i = 1,2;$$

(f.5) $f_1(z, r_0) \geq \limsup_{r \to r_0} f_1(z,r)$, for every $z \in [0,1]$, $r_0 \in I\!R$;
$f_2(z, r_0) \leq \liminf_{r \to r_0} f_2(z,r)$, for every $z \in [0,1]$, $r_0 \in I\!R$.

Proposition 3.4 *Suppose that (f.1)-(f.5) hold. Then, the multimap* $W : L^2([0,1]) \to \mathcal{P}(L^2([0,1]))$ *defined in* (9) *has the following properties:*

(W.1) W is nonempty closed convex valued;

(W.2) W is compact, i.e. it maps bounded sets into relatively compact sets;

(W.3) W is lower semicontinuous.

Proof. Throughout the proof we use the notation

$$r_v = \int_0^1 \varphi(\zeta)v(\zeta)d\zeta, \ v \in L^2([0,1]).$$

(W.1) Let us fix $v \in L^2([0,1])$. By (f.1) and (f.2) it follows that the maps $\beta_1 := f_1(\cdot, r_v)$ and $\beta_2 := f_2(\cdot, r_v)$ belong to $W(v)$, so this set is nonempty. Moreover, by using [3, Theorem 0.3.4] and the Mazur's Theorem, we can proceed as in the proof of [20, Proposition 3.2] and obtain that the set $W(v)$ is closed and convex.

(W.2) Let Ω be a bounded subset of $L^2([0,1])$ and consider a sequence $(\beta_n)_n$ in $W(\Omega)$. Hence, there exists $\{v_n\}_n \subset \Omega$ such that $\beta_n \in W(v_n)$ for every $n \in \mathbb{N}$ and

$$f_1(z, r_{v_n}) \leq \beta_n(z) \leq f_1(z, r_{v_n}), \text{ for every } z \in [0,1]. \tag{17}$$

We observe that the sequence $(v_n)_n$ is bounded in the reflexive Banach space $L^2([0,1])$, so there exists a subsequence, that we denote as the sequence, which weakly converges to a function v in $L^2([0,1])$. Now, let $(\beta_n)_n$ be the subsequence corresponding to the obtained $(v_n)_n$; since $|\beta_n'(z)| \leq l(z)$ for a.e. $z \in [0,1]$ and every $n \in \mathbb{N}$, and (17) holds, we can say that there exist $\gamma \in AC([0,1])$ and a further subsequence $(\beta_n)_n$ uniformly convergent to γ in $[0,1]$ such that $(\beta_n')_n$ weakly converges to γ' in $L^1([0,1])$ (see [3, Theorem 0.3.4]). Hence by the Mazur's Theorem we can say that $|\gamma'(z)| \leq l(z)$ for a.e. $z \in [0,1]$ (see the proof of [20, Proposition 3.2]). Further, by (f.4) we have the following estimate:

$$|\beta_n(z)| \ \ \leq \ \ |f_1(z, r_{v_n})| + |f_2(z, r_{v_n})| \leq 2c\|v_n\|_2, \text{ for a.e. } z \in [0,1] \text{ and every } n \in \mathbb{N};$$

recalling that the set $\{v_n\}_n$ is bounded in $L^2([0,1])$, we have that the sequence $(\beta_n)_n$ is almost everywhere bounded on $[0,1]$ by a constant function. Therefore, we can apply [4, Theorem 7.2] and obtain that $\beta_n \to \gamma$ in $L^2([0,1])$. So, the set $W(\Omega)$ is relatively compact in $L^2([0,1])$.

(W.3) Let us fix $v \in L^2([0,1])$, $\beta \in W(v)$, $\{v_n\}_n \subset L^2([0,1])$ with $v_n \to v$ in $L^2([0,1])$. We can say that W is lower semicontinuous in v if there exists $(\beta_n)_n$, $\beta_n \in W(v_n)$, such that $\beta_n \to \beta$ in $L^2([0,1])$ (see [17, Theorem 1.1.2]). Clearly, since $(v_n)_n$ weakly converges to v, we have $r_{v_n} \to r_v$ in $I\!R$. Define

$$\begin{aligned} p_n(z) &= \max\{\beta(z), f_1(z, r_{v_n})\}, \text{ for every } z \in [0,1], \\ \beta_n(z) &= \min\{p_n(z), f_2(z, r_{v_n})\}, \text{ for every } z \in [0,1]. \end{aligned}$$

It can be shown (see the proof of [20, Proposition 3.4]) that $\beta_n \in W(v_n)$, $n \in \mathbb{N}$ and that, thanks to (f.5), $\beta_n(z) \to \beta(z)$ for every $z \in [0,1]$. Moreover, by (f.4) we have that

$$|\beta_n(z)| \leq 2c\|v_n\|_2, \text{ for a.e. } z \in [0,1] \text{ and every } n \in I\!N,$$

hence the set $\{\beta_n\}_n$ is bounded in $L^2([0,1])$. Therefore, as in the previous item we have $\beta_n \to \beta$ in $L^2([0,1])$. So the lower semicontinuity of W is proven. □

3.3 Existence of admissible pairs

We provide the existence of impulsive mild admissible pairs for our problem. To this aim, we advance a result on the properties of the map F of problem (15).

Lemma 3.1 *Assume that maps $g : [0,T] \times I\!R \times I\!R \to I\!R$, $f_1, f_2 : [0,1] \times I\!R \to I\!R$ satisfy respectively conditions (g.1)-(g.3), (f.1)-(f.5). Then the map $F : [0,T] \times L^2([0,1]) \times L^2([0,1]) \to \mathcal{P}(L^2([0,1]))$ defined in* (14) *takes nonempty compact convex values and satisfies conditions*

(F1) for every $\epsilon > 0$ there exists a compact $C_\epsilon \subset [0,T]$ such that $\lambda([0,T] \setminus C_\epsilon) < \epsilon$ and $F_{|C_\epsilon \times L^2([0,1]) \times L^2([0,1])}$ is lower semicontinuous;

(F2) there exists $\alpha \in L^1_+([0,T])$ such that

$$\|F(t,v,w)\|_2 \leq \alpha(t)(1 + \|v\|_2 + \|w\|_2) \ , \ \textit{for a.e. } t \in [0,T] \textit{ and all } v,w \in L^2([0,1]);$$

(F3) there exists $h \in L^1_+([0,T])$ such that

$$\chi_2(F(t,\Omega_1,\Omega_2)) \leq h(t)\left[\chi_2(\Omega_1) + \chi_2(\Omega_2)\right] \ , \ \textit{for a.e. } t \in [0,T],$$

for every Ω_1, Ω_2 bounded subsets of $L^2([0,1])$.

Proof. Under our assumptions, we can apply Proposition 3.4. Therefore, by (W.1) and (W.2) we deduce that the multimap F defined in (14) takes nonempty compact convex values; while by hypothesis (g.1), (W.3) and by [17, Theorem 1.2.13] we can say that F satisfies condition (F1).
Further, by (g.2), (f.4) the following estimate holds

$$\|F(t,v,w)\|_2 \leq \|\psi\|_2 + 2c\|v\|_2 \leq (\|\psi\|_2 + 2c)(1 + \|v\|_2 + \|w\|_2),$$

for a.e. $t \in [0,T]$ and every $v, w \in L^2([0,1])$; therefore F satisfies (F2).
Finally, by (W.2) and (g.3) the multimap F satisfies (F3). □

Theorem 3.1 *Assume the maps $b : [0,T] \times [0,1] \to I\!R$, $g : [0,T] \times I\!R \times I\!R \to I\!R$, $f_1, f_2 : [0,1] \times I\!R \to I\!R$ to satisfy respectively conditions (b.1)-(b.3), (g.1)-(g.3), (f.1)-(f.5). Then there exists an impulsive mild admissible pair for* (8).

Proof. Our first goal is to prove the existence of an *impulsive mild solution for* (15), i.e. of a function $v \in PC([0,T], L^2([0,1]))$ such that

$$v(t) = T(t,0)\alpha_0 + \sum_{0<t_i<t} T(t,t_i)I_i(v(t_i)) + \int_0^t T(t,s)\tilde{f}(s)ds \, , \ t \in [0,T],$$

for $\tilde{f} \in L^1([0,T], L^2([0,1]))$, $\tilde{f}(s) \in F\left(s, v(s), \int_0^s \frac{e^{\sigma/T}}{T} v(\sigma)d\sigma\right)$ a.e. $s \in [0,T]$.
Here $\{T(t,s)\}_{0\leq s\leq t\leq T}$, $T(t,s) : L^2([0,1]) \to L^2([0,1])$, $0 \leq s \leq t \leq T$, is the evolution system generated by the family $\{A(t)\}_{t\in[0,T]}$ (see Proposition 3.2).
Further, by Lemma 3.1 we can apply [6, Theorem 3.1], so that there exists a Carathéodory function f which is a selection of F, i.e. $f(t,\cdot,\cdot)$ is continuous for every $t \in [0,T]$, $f(\cdot,v,w)$

is Borel-measurable (and then strongly measurable since $L^2([0,1])$ is separable) for every $v, w \in L^2([0,1])$, $f(t,v,w) \in F(t,v,w)$ for a.e. $t \in [0,T]$ and for every $v, w \in L^2([0,1])$.
From now on, we proceed by steps.

Step 1. Consider the interval $[0, t_1]$ and the Cauchy problem

$$\begin{cases} v'(t) = A(t)v(t) + f\left(t, v(t), \int_0^t \frac{e^{s/T}}{T} v(s)ds\right), \ t \in [0, t_1], \\ \\ v(0) = \alpha_0. \end{cases} \tag{18}$$

By (F2) and (F3) we have that the following estimates hold for a.e. $t \in [0, t_1]$:

$$\|f(t,v,w)\|_2 \ \leq \ \alpha(t)(1 + \|v\|_2 + \|w\|_2), \text{ for all } v, w \in L^2([0,1]); \tag{19}$$

$$\chi_2(f(t, \Omega_1, \Omega_2)) \ \leq \ h(t)\left[\chi_2(\Omega_1) + \chi_2(\Omega_2)\right], \text{ for every bounded } \Omega_1, \Omega_2 \subset L^2([0,1]). \tag{20}$$

Therefore, recalling that f is a Carathéodory function, we can apply [5, Theorem 5.1] and claim that problem (18) has at least one mild solution, namely a continuous function $v_0 : [0, t_1] \to L^2([0,1])$ such that

$$v_0(t) = T(t,0)\alpha_0 + \int_0^t T(t,s) f\left(s, v_0(s), \int_0^s \frac{e^{\sigma/T}}{T} v_0(\sigma)d\sigma\right) ds, \text{ for all } t \in [0, t_1]. \tag{21}$$

Step 2. Consider the interval $[t_1, t_2]$ and the Cauchy problem associated to v_0

$$\begin{cases} v'(t) = A(t)v(t) + f_1\left(t, v(t), \int_{t_1}^t \frac{e^{s/T}}{T} v(s)ds\right), \ t \in [t_1, t_2], \\ v(t_1) = v_0(t_1) + I_1(v_0(t_1)), \end{cases} \tag{22}$$

where

$$f_1(t,v,w) = f\left(t, v, \int_0^{t_1} \frac{e^{s/T}}{T} v_0(s)ds + w\right), \ t \in [0,T], \ v, w \in L^2([0,1]). \tag{23}$$

Function f_1 satisfies for a.e. $t \in [t_1, t_2]$ the estimate analogous to (19)

$$\begin{aligned} \|f_1(t,v,w)\|_2 &= \left\| f\left(t, v, \int_0^{t_1} \frac{e^{s/T}}{T} v_0(s)ds + w\right)\right\|_2 \\ &\leq \alpha(t)\left(1 + \|v\|_2 + \left\|\int_0^{t_1} \frac{e^{s/T}}{T} v_0(s)ds + w\right\|_2\right) \\ &\leq \alpha(t)\left(1 + \left|\int_0^{t_1} \frac{e^{s/T}}{T} v_0(s)ds\right|\right)(1 + \|v\|_2 + \|w\|_2), \text{ for all } v, w \in L^2([0,1]) \end{aligned}$$

and the analogous of (20) as well. Moreover, f_1 is also a Carathéodory function, so we can apply [5, Theorem 5.1] again. Let $v_1 : [t_1, t_2] \to L^2([0,1])$ be a mild solution of (22),

$$\begin{aligned} v_1(t) &= T(t,t_1)[v_0(t_1) + I_1(v_0(t_1))] + \\ &+ \int_{t_1}^t T(t,s) f_1\left(s, v_1(s), \int_{t_1}^s \frac{e^{\sigma/T}}{T} v_1(\sigma)d\sigma\right) ds, \text{ for all } t \in [t_1, t_2]. \end{aligned} \tag{24}$$

Step 3. By using an iterative process, we can claim that for every $i = 1, \dots, p$ each Cauchy problem

$$\begin{cases} v'(t) = A(t)v(t) + f_i\left(t, v(t), \int_{t_i}^t \frac{e^{s/T}}{T} v(s)ds\right), \, t \in [t_i, t_{i+1}], \\ v(t_i) = v_{i-1}(t_i) + I_i(v_{i-1}(t_i)), \end{cases}$$

where $f_i(t, v, w) = f_{i-1}\left(t, v, \int_{t_{i-1}}^{t_i} \frac{e^{s/T}}{T} v_{i-1}(s)ds + w\right)$, $t \in [0, T]$, $v, w \in L^2([0, 1])$ (with $f_0 = f$), admits a mild solution

$$\begin{aligned} v_i(t) &= T(t, t_i)[v_{i-1}(t_i) + I_i(v_{i-1}(t_i))] + \\ &\quad + \int_{t_i}^t T(t, s) f_i\left(s, v_i(s), \int_{t_i}^s \frac{e^{\sigma/T}}{T} v_i(\sigma) d\sigma\right) ds \, , \text{ for all } t \in [t_i, t_{i+1}]. \end{aligned}$$

Step 4. We show that the piecewise continuous function

$$\bar{v}(t) = \begin{cases} v_0(t) \, , \; t \in [0, t_1], \\ v_1(t) \, , \; t \in]t_1, t_2], \\ \dots \\ v_p(t) \, , \; t \in]t_p, T] \end{cases} \tag{25}$$

is an impulsive mild solution for problem (15).
If $t \in [0, t_1]$, then by (21) and (25) we have

$$\begin{aligned} \bar{v}(t) = v_0(t) &= T(t, 0)\alpha_0 + \int_0^t T(t, s) f\left(s, v_0(s), \int_0^s \frac{e^{\sigma/T}}{T} v_0(\sigma) d\sigma\right) ds \\ &= T(t, 0)\alpha_0 + \int_0^t T(t, s) f\left(s, \bar{v}(s), \int_0^s \frac{e^{\sigma/T}}{T} \bar{v}(\sigma) d\sigma\right) ds \, ; \end{aligned}$$

if $t \in]t_1, t_2]$, then by (24), (23), (21), (25) and the properties of evolution systems (see Section 4), we have

$$\begin{aligned} \bar{v}(t) = v_1(t) &= T(t, t_1)[v_0(t_1) + I_1(v_0(t_1))] + \int_{t_1}^t T(t, s) f_1\left(s, v_1(s), \int_{t_1}^s \frac{e^{\sigma/T}}{T} v_1(\sigma) d\sigma\right) ds \\ &= T(t, t_1)\left[T(t_1, 0)\alpha_0 + \int_0^{t_1} T(t_1, s) f\left(s, v_0(s), \int_0^s \frac{e^{\sigma/T}}{T} v_0(\sigma) d\sigma\right) ds\right] + \\ &\quad + T(t, t_1) I_1(v_0(t_1)) + \\ &\quad + \int_{t_1}^t T(t, s) f\left(s, v_1(s), \int_0^{t_1} \frac{e^{\sigma/T}}{T} v_0(\sigma) d\sigma + \int_{t_1}^s \frac{e^{\sigma/T}}{T} v_1(\sigma) d\sigma\right) ds \\ &= T(t, 0)\alpha_0 + T(t, t_1) I_1(\bar{v}(t_1)) + \int_0^t T(t, s) f\left(s, \bar{v}(s), \int_0^s \frac{e^{\sigma/T}}{T} \bar{v}(\sigma) d\sigma\right) ds \, . \end{aligned}$$

By proceeding iteratively, we have

$$\begin{aligned}
\bar{v}(t) &= T(t,0)\alpha_0 + \sum_{0<t_i<t} T(t,t_i)I_i(\bar{v}(t_i)) + \int_0^t T(t,s) f\left(s, \bar{v}(s), \int_0^s \frac{e^{\sigma/T}}{T}\bar{v}(\sigma)d\sigma\right) ds \\
&= T(t,0)\alpha_0 + \sum_{0<t_i<t} T(t,t_i)I_i(\bar{v}(t_i)) + \int_0^t T(t,s)\tilde{f}(s)\, ds,\ t \in [0,T], \qquad (26)
\end{aligned}$$

where we put

$$\tilde{f}(s) = f\left(s, \bar{v}(s), \int_0^s \frac{e^{\sigma/T}}{T}\bar{v}(\sigma)d\sigma\right),\ \ s \in [0,T]. \qquad (27)$$

Clearly $\tilde{f}(s) \in F\left(s, \bar{v}(s), \int_0^s \frac{e^{\sigma/T}}{T}\bar{v}(\sigma)d\sigma\right)$ for a.e. $s \in [0,T]$.
Moreover, since $\tilde{f}$ is the superposition of the Carathéodory function f with the piecewise continuous function $s \mapsto \left(\bar{v}(s), \int_0^s \frac{e^{\sigma/T}}{T}\bar{v}(\sigma)d\sigma\right)$, we have that $\tilde{f}$ is Borel-measurable; so, by the growth condition on f (by (19) and the analogous ones on the other intervals), we achieve that $\tilde{f} \in L^1([0,T], L^2([0,1]))$.
Therefore the function $\bar{v} \in PC([0,T], L^2([0,1]))$ is an impulsive mild solution for (15).
Conclusion. Recalling (14), we can write

$$\tilde{f}(s) \in \tilde{g}\left(s, \bar{v}(s), \int_0^s \frac{e^{\sigma/T}}{T}\bar{v}(\sigma)d\sigma\right) + W(\bar{v}(s)),\ \text{a.e. } s \in [0,T]. \qquad (28)$$

Let us define the functions $\bar{u}, \bar{\omega} : [0,T] \times [0,1] \to I\!R$ by

$$\begin{aligned}
\bar{u}(t,z) &= \bar{v}(t)(z); \\
\bar{\omega}(t,z) &= \tilde{f}(t)(z) - \tilde{g}\left(t, \bar{v}(t), \int_0^t \frac{e^{s/T}}{T}\bar{v}(s)ds\right)(z) \\
&= f\left(t, \bar{u}(t,z), \int_0^t \frac{e^{s/T}}{T}\bar{u}(s,z)ds\right) - g\left(t, \bar{u}(t,z), \int_0^t \frac{e^{s/T}}{T}\bar{u}(s,z)ds\right),
\end{aligned}$$

for all $t \in [0,T], z \in [0,1]$ (see (27) and (11)).
By the above definitions and by (28), we get that $\bar{\omega}(s,\cdot) \in W(\bar{u}(s,\cdot))$ for a.e. $s \in [0,T]$.
Finally, by (26), (16), (12) and the fact that

$$\tilde{f}(t)(z) = g\left(t, \bar{u}(t,z), \int_0^t \frac{e^{s/T}}{T}\bar{u}(s,z)ds\right) + \bar{\omega}(t,z),$$

the identity of Definition 2.1 is satisfied by the couple $(\bar{u}, \bar{\omega})$, which is then an impulsive mild admissible pair for (8). □

We provide an example of nonlinearity satisfying the properties needed in our existence result and involving a dispersal kernel.

Example 3.1 *Let us consider the function* $g : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$g(t,p,q) = \frac{e^{t/T}}{1+|q|},\ (t,p,q) \in [0,T] \times \mathbb{R} \times \mathbb{R}.$$

Then the integro-partial differential equation driving (8) *reads as*

$$\frac{\partial u}{\partial t}(t,z) = -b(t,z)u(t,z) + \frac{1}{e^{-t/T} + \left|\int_0^t \frac{e^{-(t-s)/T}}{T} u(s,z)\,ds\right|} + \omega(t,z), t \in [0,T],\ a.e.\ z \in [0,1].$$

We notice that the map g here defined trivially satisfies (g.1) and (g.2).
Further, we show that also (g.3) is satisfied. Let us recall that in the space $L^2([0,1])$ *the Hausdorff measure of noncompactness* χ_2 *is equivalent to the measure of noncompactness*

$$\chi^*(\Omega) = \lim_{h\to 0} \sup_{\theta\in\Omega} \left\{ \int_0^1 [\theta(z+h) - \theta(z)]^2 dz \right\}^{1/2},$$

in the sense that

$$\chi_2(\Omega) \le \chi^*(\Omega) \le 2\chi_2(\Omega), \tag{29}$$

for all bounded sets $\Omega \subset L^2([0,1])$ *(see [2, Theorem 2.2]).*
Fix $\Omega_1, \Omega_2 \subset L^2([0,1])$ *and* $t \in [0,T]$*; we have*

$$\begin{aligned}
\chi^*(g(t,\Omega_1(\cdot),\Omega_2(\cdot))) &= \lim_{h\to 0} \sup_{\theta\in g(t,\Omega_1(\cdot),\Omega_2(\cdot))} \left\{ \int_0^1 [\theta(z+h) - \theta(z)]^2 dz \right\}^{1/2} \\
&= \lim_{h\to 0} \sup_{q(\cdot)\in\Omega_2} \left\{ \int_0^1 \left[\frac{e^{t/T}}{1+|q(z+h)|} - \frac{e^{t/T}}{1+|q(z)|} \right]^2 dz \right\}^{1/2} \\
&= e^{t/T} \lim_{h\to 0} \sup_{q(\cdot)\in\Omega_2} \left\{ \int_0^1 \left[\frac{|q(z+h)| - |q(z)|}{(1+|q(z+h)|)(1+|q(z)|)} \right]^2 dz \right\}^{1/2} \\
&\le e^{t/T} \lim_{h\to 0} \sup_{q(\cdot)\in\Omega_2} \left\{ \int_0^1 [|q(z+h)| - |q(z)|]^2\, dz \right\}^{1/2} = e^{t/T}\chi^*(\Omega_2).
\end{aligned}$$

Hence, by using (29) *we have*

$$\begin{aligned}
\chi_2(g(t,\Omega_1(\cdot),\Omega_2(\cdot))) &\le \chi^*(g(t,\Omega_1(\cdot),\Omega_2(\cdot))) \le e^{t/T}\chi^*(\Omega_2) \\
&\le 2e^{t/T}\chi_2(\Omega_2) \le 2e^{t/T}\left[\chi_2(\Omega_1) + \chi_2(\Omega_2)\right],
\end{aligned}$$

so (g.3) is satisfied for $h(t) = 2e^{t/T}$, $t \in [0,T]$.

4 On the impulse functions

In the next example we present a case in which at a priori fixed times the solution trajectory is forced, due to the impulses' action, to come back in a fixed range.

Example 4.1 *For the sake of simplicity, let us suppose* $p = 1$.
Let $R > 0$ *be a given threshold value and* $\mathcal{I}_1 : I\!\!R \to I\!\!R$ *be the* regulation impulse function *defined by*

$$\mathcal{I}_1(\xi) = \begin{cases} 0 & , \ \xi \le R, \\ -\frac{n}{n+1}\xi & , \ \xi \in]nR, (n+1)R] \, , \, n \ge 1 \ . \end{cases}$$

We suppose that the initial condition $\alpha_0 \in L^2([0,1])$ *is such that* $\alpha_0(z) \le R$*, for every* $z \in [0,1]$.
By the definition of regulation impulse function, after the jump time the solution trajectory is forced to come back in the interval $]-\infty, R]$. *Indeed, if* $u : [0,T] \times [0,1] \to I\!\!R$ *is a solution for the problem* (8) *when* $p = 1$*, then it is easy to see that*

$$u(t_1^+, z) = u(t_1, z) + \mathcal{I}_1(u(t_1, z)) \le R \ , \ \textit{for every } z \in [0,1] \ .$$

We wish to underline that in our results we do not require any assumption on the impulse functions $\mathcal{I}_i$ and, as a consequence, on I_i; so in problem (8) we can consider any type of impulse function. This means that one can act on the system with any instantaneous external force. Hence, differently with respect to most of the existing literature, impulse functions which are not continuous can be admitted.

Example 4.2 *The impulse function* $\mathcal{I}_1$ *presented in Example 4.1 is such that its associated map* $I_1 : L^2([0,1]) \to L^2([0,1])$ *(see* (12)*) is not continuous in* $\hat{v} \in L^2([0,1])$*, where* $\hat{v}$ *is defined by*

$$\hat{v}(z) = R \, , \, z \in [0,1] \ .$$

In fact, considered the sequence $(v_m)_m$ *in* $L^2([0,1])$ *defined as*

$$v_m(z) = \frac{m+1}{m} R \ , \ \textit{for every } z \in [0,1] \, , \, m \in I\!\!N,$$

we have

$$\|v_m - \hat{v}\|_2^2 \to 0 \ ;$$

but, since $v_m(z) \in]R, 2R]$ *for every* $z \in [0,1]$ *and* $m \in I\!\!N$*, we obtain*

$$I_1(v_m)(z) = -\frac{m+1}{2m} R \ \textit{and} \ I_1(\hat{v})(z) = 0 \ , \ \textit{for every } z \in [0,1] \ ;$$

therefore we get

$$\|I_1(v_m) - I_1(\hat{v})\|_2^2 \to \frac{R^2}{4} \neq 0 \ .$$

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Declarations of interest: none.

Appendix: List of symbols and definitions

E real Banach space endowed with the norm $\|\cdot\|$;

$\mathcal{P}(E)$ the family of all nonempty subsets of E;

$C(J,E)$ the space of E-valued continuous functions on a closed bounded interval $J \subset \mathbb{R}$;

$AC(J,E)$ the space of E-valued absolutely continuous functions on J (shortly, $AC(J)$ if $E = \mathbb{R}$);

$L^p(J,E)$ the space of all E-valued functions on J such that their p-power is Bochner integrable with norm $\|v\|_p = \left[\int_J \|v(z)\|^p \, dz\right]^{\frac{1}{p}}$ (shortly, $L^p(J)$ if $E = \mathbb{R}$), $p = 1, 2$;

$L^1_+(J) = \{f \in L^1(J) \,:\, f(t) \geq 0, \text{ for a.a. } t \in J\}$;

$$PC([0,T],E) = \left\{ x : [0,T] \to E \;:\; \begin{array}{l} x_{|\,J_i} \text{ is continuous}, \, i = 0,\ldots,p, \text{ and} \\ \text{there exists } x(t_i^+) \in E\,,\, i = 1,\ldots,p \end{array} \right\},$$

where: $\{t_0,\ldots,t_{p+1}\} \subset [0,T]$ such that $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = b$; $J_0 = [0,t_1]$, $J_i =]t_i, t_{i+1}]$, $i = 1,\ldots,p$; $x(t_i^+) = \lim_{s \to t_i^+} x(s)$, $i = 1,\ldots,p$;

$(PC([0,T],E), \|\cdot\|_{PC})$ Banach space endowed with the norm $\|x\|_{PC} = \sup_{0 \leq t \leq T} \|x(t)\|$, $x \in PC([0,T],E)$;

χ the *Hausdorff measure of noncompactness* in E

$$\chi(\Omega) = \inf\{\varepsilon > 0 \;:\; \Omega \text{ has a finite } \varepsilon\text{-net}\}\ , \text{ for all bounded } \Omega \subset E;$$

$\{T(t,s)\}_{(t,s)\in\Delta}$ is an *evolution system* (see, e.g., [23]) if $T(t,s) : E \to E$ is a bounded linear operator, $(t,s) \in \Delta = \{(t,s) \in [0,T] \times [0,T] \;:\; s \leq t\}$, and the following properties hold

(T1) $T(s,s) = I, \quad T(t,r)T(r,s) = T(t,s)$ for $0 \leq s \leq r \leq t \leq T$;

(T2) $(t,s) \mapsto T(t,s)$ is strongly continuous on Δ (i.e. the map $\xi_x : (t,s) \mapsto T(t,s)x$ is continuous on Δ, for every $x \in E$).

$F : X \to \mathcal{P}(E)$, X a topological space, is *lower semicontinuous at* $x_0 \in X$ if, for all $V \subseteq E$ open such that $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U$;

the multimap F is *lower semicontinuous* if it is lower semicontinuous at every $x_0 \in X$.

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