Jointly continuous utility functions on submetrizable k_ω-spaces

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Abstract

A Hausdorff topological space X is a submetrizable k_ω-space if it is the *inductive limit of an increasing sequence of metric compact subspaces of* X. These spaces have nice properties and they seem to be very interesting in the study of the utility representation problem. Every closed preorder on a submetrizable k_ω-space has a continuous utility representation and some theorems on the existence of jointly continuous utility functions have been recently proved.

When the commodity space is locally compact second countable, Back proved the existence of a continuous map from the space of total preorders topologized by closed convergence (Fell topology) to the space of utility functions with different choice sets (partial maps) endowed with a generalization of the compact-open topology. In this paper we generalize Back's Theorem to submetrizable k_ω-spaces with a family of not necessarily total preorders.

The continuous utility representation theorems on submetrizable k_ω-spaces have some economic applications. In fact, an example of a submetrizable k_ω-space is the space of tempered distributions, which has been used to define a new state preference model in the infinite dimensional case.

Key words: Submetrizable k_ω-spaces, jointly continuous utility functions, space of tempered distributions, state preference model.

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1 Introduction

The submetrizable k_ω-spaces seem destined to play an important role in the study of the utility representation problem.

A Hausdorff topological space X is a submetrizable k_ω-space if it is the *inductive limit of an increasing sequence of metric compact subspaces of* X. It is well known that X is a k_ω-space if and only if X is a hemicompact k-space. An interesting survey on k_ω-spaces can be found in [14,17]. These spaces have been used in economic models in the infinite dimensional case (see [11]).

A first representation theorem for submetrizable k_ω-spaces was proved, using different techniques, in [2,13]: a closed (not necessarily total) preorder on a submetrizable k_ω-space has a continuous utility representation. This result generalizes a theorem proved by Levin in 1983 [20] in locally compact second countable spaces.

In [10] the authors gave some representation theorems on topological spaces that are very close to submetrizable k_ω-spaces. Their result and the one cited above are equivalent in the case of a total preorder without jumps, otherwise they are independent.

The k_ω-spaces have nice properties. In [21] the author proves that every k_ω-space, equipped with a closed preorder, is normally preordered. Hence, every continuous and isotone functions defined on a compact subset of a k_ω-space X has a continuous and isotone extension to all of X ([22]).

An interesting problem is to find topological conditions under which a space admits a countable family of continuous utility functions that represents every given closed preorder (countable multi-utility).

In [3] it is proved that each closed preorder defined on a normally preordered space with a countable netweight has a countable utility representation. Therefore, every closed preorder on a submetrizable k_ω-space has a countable utility (multi-utility) representation too.

The submetrizable k_ω-spaces are interesting commodity spaces also in the study of the existence of *jointly continuous utility functions*. Given a family S of preorders on X, the jointly continuous utility representation problem consists in finding suitable topologies on S and X ensuring the existence of a continuous function $u : \mathrm{S} \times X \to \mathbb{R}$ such that for every $\preceq \in \mathrm{S}$ $u(\preceq, \cdot)$ is a utility function for $\preceq$.

Previous results of Levin [20] and Back [1] were proved in locally compact second countable commodity spaces.

In [13] the existence of a jointly continuous utility function is proved under the assumptions that X is submetrizable k_ω-space and the space of preorders is metrizable or both spaces are hemicompact, submetrizable and their product is a k-space.

Levin's result has been extended to a space $\mathcal{P}$ of closed preorders defined on (closed) subsets $D \subset X$ (see Corollary 1, in [20]).

Using Levin's Theorems, Back in [1] proved the existence of a continuous map from the space of total preorders topologized by closed convergence (Fell topology) to the space of utility functions with different choice sets endowed with a new topology on graphs of partial maps which is a generalization of the compact-open topology. The classical topologies of function spaces are defined only on functions with the same domain while the utility functions considered by Back are functions with "moving" domain, i.e. partial maps. A natural approach is to identify partial maps with their graphs and consider topologies on graphs. Using this topology, Back found a jointly continuous utility representation theorem. The same topology has interesting applications in other fields of mathematics including differential equations [4–6,9,16] and dynamic programming models [19]. Its topological properties and relations with other known topologies have been studied in [7,8,18].

In [12] Back's Theorem has been generalized to non-metrizable commodity spaces with a family of not necessarily total preorders. In that paper, X is a regular space having a weaker locally compact second countable topology. The topology on the space of preorders, defined by means of a suitable convergence structure, coincides with the topology of closed convergence when the commodity space is locally compact second countable.

The submetrizable k_ω-spaces are regular spaces but they do not necessarily admit a weaker locally compact second countable topology. In fact the space of tempered distributions S' is an example of this.

In this paper we generalize Back's Theorem to submetrizable k_ω-spaces with a family of not necessarily total preorders. Using that every submetrizable k_ω-space (X, τ) is a quotient space of a locally compact second countable space $(\hat{X}, \eta)$, we have associated in a natural way to the space of partial preorders $\mathcal{P}$ on X a suitable space of partial preorders $\widetilde{\mathcal{P}}$ on $\hat{X}$ satisfying the hypotheses of Back's Theorem. So, we have proved the existence of a continuous map $\nu : \mathcal{P} \to \mathcal{U}$ such that $\nu(\preceq)$ is a utility function for every $\preceq \in \mathcal{P}$. The topology on $\mathcal{U}$ is just the topology used by Back, while the topology in $\mathcal{P}$ is a Fell-type topology defined using the subspace topologies $\tau_{|F_n}$, X being the inductive limit of $(F_n)_n$.

An important example of a submetrizable k_ω-space is the space S' of tempered distributions ([14], Example 3.3). In [11] the space of tempered distributions S' has been used to define a new state preference model. The author extends the classical finite dimensional state preference model of Financial Analysis to a model for a portfolio of goods where the space of states of the world is the space $\mathbb{R}^m$. Precisely, considered n goods and a continuously infinite set of world states, he defines an $\mathcal{S}$ -linear state preference model to be a system $(n, \mathbb{R}^m, A)$ where $A : \mathbb{R}^n \to S'_m$ is a linear operator. So, this new model is based on the space of tempered distributions in which the author has introduced suitable structures. In Decision Theory one of the most important objectives is the representation of preorders by continuous utility functions. Our previous results [13] and the theorems of the present paper strengthen the idea of using the space of tempered distributions S' in Decision Theory.

2 Notation and preliminaries

Let (X, τ) be a topological space and let $CL((X, \tau))$ be the set of all non-empty closed subsets of (X, τ).
We will denote by $F(\tau)$ the Fell topology on $CL((X, \tau))$, that is the topology having as a subbase all sets of the form

$$U^- = \{B \in CL((X, \tau)) : B \cap U \neq \emptyset\},\ U \in \tau \text{ and}$$

$$(K^c)^+ = \{B \in CL((X, \tau)) : B \cap K = \emptyset\}, K \text{ compact in } (X, \tau).$$

With the same meaning $F(\tau \times \tau)$ will denote the Fell topology on $CL((X, \tau) \times (X, \tau))$.

A *preorder* $\preceq$ on $D \subset X$ is a binary relation $\preceq \subset D \times D$ which is reflexive and transitive, we say D is the domain of $\preceq$ and we set $D = D(\preceq)$. An anti-symmetric preorder is said to be an *order*. A *total preorder* $\preceq$ is a preorder such that if $x, y \in D$ then $(x, y) \in \preceq$ or $(y, x) \in \preceq$.
A preorder $\preceq$ on $D \subset X$ is said to be *closed* if $\preceq$ is a closed subset of the topological product $D \times D$. Note that $\preceq$ is closed in $X \times X$ if and only if $\preceq$ is closed in $D \times D$ and D is a closed subset of X.
The preorder $\preceq$ is said to be *continuous* with respect to $\tau|_D$ if for every $a \in D$ the sets $[a, +\infty[= \{x \in D : (a, x) \in \preceq\}$ and $]-\infty, a] = \{x \in D : (x, a) \in \preceq\}$ are $\tau|_D$-closed. We recall that if the preorder $\preceq$ is total, then $\preceq$ is closed iff it is continuous iff $\tau_\preceq \subset \tau$ where $\tau_\preceq$ is the *order topology* generated by $\preceq$ on D.
A preorder $\preceq$ is called *locally non-satiated* if for each $x \in D(\preceq)$ and each neighbourhood U of x there is $y \in U$ such that $(x, y) \in \prec$.
In the following we deal with preorders defined on closed subsets of X and we put

$$\mathcal{P} = \{\preceq :\ \preceq \text{ is a preorder on } D(\preceq) \subset X \text{ and } \preceq \in CL((X, \tau) \times (X, \tau))\}.$$

We denote by $\mathcal{P}_{\text{lns}} \subset \mathcal{P}$ the subset consisting of the locally non-satiated preorders.

As in Back's paper [1] we denote by $\mathcal{U}_\tau$ the space of all continuous partial maps (D, u) where $D \subset X$ is non-empty and closed and $u : D \to \mathbb{R}$ is continuous. Let τ_c be the topology on $\mathcal{U}_\tau$ which has as a subbase all sets of the type

$$[G] = \{(D, u) \in \mathcal{U}_\tau :\ D \cap G \neq \emptyset\}$$

$$[K : I] = \{(D, u) \in \mathcal{U}_\tau :\ u(D \cap K) \subset I\}$$

where G is an open subset of X, $K \subset X$ is compact and $I \subset \mathbb{R}$ is open (possibly empty).
A partial map $(D, u) \in \mathcal{U}_\tau$ is said to be a *utility function* for a preorder $\preceq$

if $D = D(\preceq)$ and for every $x, y \in D$, $(x, y) \in \preceq$ implies $u(x) \leq u(y)$ and $(x, y) \in \prec$ implies $u(x) < u(y)$.

In [1] Back proved the following

Theorem 2.1 *Let (X, τ) be a locally compact and second countable space. There exists a continuous map $\nu : (\mathcal{P}, F(\tau \times \tau)) \to (\mathcal{U}_\tau, \tau_c)$ such that $\nu(\preceq)$ is a utility function for $\preceq$, for every $\preceq \in \mathcal{P}$. Any such map ν is actually a homeomorphism of $\mathcal{P}_{lns}$ onto $\nu(\mathcal{P}_{lns})$.*

Back proved the theorem in the case of total preorders. In [12] it was point out that using a similar proof, it is possible to prove the existence of a continuous map $\nu : (\mathcal{P}, F(\tau \times \tau)) \to (\mathcal{U}_\tau, \tau_c)$, also in the case the preorders are not necessarily total.
Of course, the hypothesis of totalness cannot be dropped to prove that ν is a homeomorphism of $\mathcal{P}_{\text{lns}}$ onto $\nu(\mathcal{P}_{\text{lns}})$.

The aim of this paper is to generalize Back's Theorem for submetrizable k_ω-spaces. First, we argue on some properties and characterizations of the submetrizable k_ω-spaces.

Let (X, τ) be a Hausdorff space.
We recall that X is called a *k-space* if $A \subset X$ is open if and only if $A \cap K$ is open in K for every compact set K of X, X is *hemicompact* if there is a sequence $\{K_n\}$ of compact subsets of X which is cofinal in the set of all compact subsets of X, X is *submetrizable* if there exists a weaker metrizable topology on X.

Proposition 2.2 *Let (X, τ) be a Hausdorff space and let $A \subset X$. The following conditions are equivalent:*

i) for every compact $K \subset X$, $A \cap K$ is an open subset of K;

ii) for every locally compact $F \subset X$, $A \cap F$ is an open subset of F.

Proof. It is sufficient to prove $i) \Rightarrow ii)$. Let $A \subset X$ be such that $A \cap K$ is open in K, for every compact $K \subset X$. Let F be a locally compact subset of X. $A \cap F$ is open in F if and only if $(A \cap F) \cap K_F$ is open in K_F, for every K_F compact subset of F. But, K_F is compact in F if and only if K_F is compact in X, so $A \cap F$ is open in F.

Theorem 2.3 *Let (X, τ) be a Hausdorff space. The following conditions are equivalent:*

i) (X, τ) *is a k-space;*

ii) $A \subset X$ *is open if and only if, for every locally compact* $F \subset X$, $A \cap F$ *is an open subset of* F.

Definition 2.4 *Let* (X, τ) *be a Hausdorff space and* $\mathcal{S} = (S_n)_n$ *a countable increasing family of subsets of* X. (X, τ) *is said to be the inductive limit of the family* $\mathcal{S} = (S_n)_n$ *if* $X = \bigcup_{n \in I\!N} S_n$ *and any subset* A *of* X *is open if and only if, for every* $n \in I\!N$, $A \cap S_n$ *is an open subset of* S_n.
We will say that $(S_n)_n$ determines the topology of X.

Proposition 2.5 *Let* (X, τ) *be the inductive limit of a countable increasing family* $\mathcal{S} = (S_n)_n$. *Then every compact subset* K *of* X *is contained in some* S_n.

Proof. Let K be a compact subset of X which is not contained in any S_n. There exists a sequence $(x_n)_n$ of elements of K such that $x_n \notin S_n$ for every n. Let $L = \{x_n : n \in I\!N\}$. The set $L \cap S_n$ is finite and closed, for every $n \in I\!N$, so L is closed in X and hence compact because it is contained in K. The relative topology of L is discrete because L is the inductive limit of the family $(L \cap S_n)_n$ consisting of finite sets. This is a contradiction because every discrete compact space is finite.

Proposition 2.6 *Let* (X, τ) *be a Hausdorff space.* X *is a k-space if and only it is the inductive limit of an increasing sequence* $(S_n)_n$ *of k-subspaces of* X.

Proof. Let $(S_n)_n$ be a family of k-subspaces of X and let X be the inductive limit of $(S_n)_n$. Let $A \subset X$ and let $A \cap K$ be open in K for every compact $K \subset X$. For every $n \in I\!N$ and for every compact $K \subset S_n$, $(A \cap S_n) \cap K = A \cap K$ is open in K hence $A \cap S_n$ is open in S_n and A is open in X.

Definition 2.7 *A Hausdorff space* X *is said to be a* k_ω*-space if* X *is an inductive limit of a countable increasing family* $\mathcal{S} = (K_n)_n$ *where every* K_n *is compact or equivalently if* X *is a hemicompact k-space.*

Proposition 2.8 *Let* (X, τ) *be a Hausdorff space. TFAE:*

(i) X *is a submetrizable* k_ω*-space;*

(ii) X *is the inductive limit of an increasing sequence of metric compact subspaces;*

(iii) X *is the inductive limit of an increasing sequence of second countable and locally compact subspaces.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ Let X be a submetrizable k_ω-space. By definition X is the inductive limit of an increasing sequence $(K_n)_n$ where every K_n is compact. Since X is submetrizable, then every (Hausdorff) compact K_n is metrizable, hence second countable.
$(iii) \Rightarrow (i)$ Conversely, suppose that X is the inductive limit of a sequence $(S_n)_n$ of locally compact second countable subspaces. By Proposition 2.6, X is a k-space. Moreover, since every S_j is clearly hemicompact, let $(K_n^j)_n$ be a cofinal sequence in the family of all the compact subsets of S_j. Then, if $K \subset X$ is compact, by Proposition 2.5, K is contained in some S_j, hence in some K_n^j. Therefore, $(K_n^j)_{j,k\in N}$ is a countable family of compact subsets of X, which is cofinal with respect to the the family of all the compact subsets of K. Hence X is hemicompact.
Finally, we note that every compact subset of X is metrizable since, as already observed, it is contained in some S_j which is metrizable. Therefore X is submetrizable since it is a k_ω-space in which every compact subset is metrizable ([14], page 113).

Definition 2.9 *Let X, Y be topological spaces. Let $\{A_s\}_{s\in\mathcal{S}}$ be a cover of X and let $\{f_s\}_{s\in\mathcal{S}}$ be a family of continuous mappings, where $f_s : A_s \to Y$. We say that the mappings f_s are compatible if for every pair $s_1, s_2 \in \mathcal{S}$ we have $f_{s_1}|A_{s_1} \cap A_{s_2} = f_{s_2}|A_{s_1} \cap A_{s_2}$.*
The mapping $f : X \to Y$ defined by $f(x) = f_s(x)$ for $x \in A_s$ is called the combination of the mappings $\{f_s\}_{s\in\mathcal{S}}$ and is denoted by the symbol $\bigtriangledown_{s\in\mathcal{S}} f_s$.

We recall that if for every $s \in \mathcal{S}$, A_s is an open subset of X and f_s is continuous then $f = \bigtriangledown_{s\in\mathcal{S}} f_s$ is continuous, too.

Theorem 2.10 *Every submetrizable k_ω-space X is a quotient space of a locally compact second countable space.*

Proof. By Proposition 2.8 there is an increasing family $\mathcal{F} = \{F_n : n \in I\!N\}$ of locally compact second countable subsets of X that determines the topology of X.
For each $n \geq 1$, let $\widetilde{F}_n = \{n\} \times F_n$ and let $\hat{X} = \bigcup_{n\geq 1} \widetilde{F}_n$. The topological sum $\hat{X} = \bigoplus_{F_n\in\mathcal{F}} \widetilde{F}_n$ is locally compact and second countable and the mapping $\pi : \hat{X} \to X,\ (m, x) \mapsto x$ is continuous (π is the combination of the embeddings of F_n into $X, n \in I\!N$) and, for each $n \geq 1$, induces a homeomorphism of $\widetilde{F}_n$ onto F_n. Moreover π is a quotient map since X is a k-space.

3 Jointly continuous utility functions on submetrizable k_ω-spaces

In this section X will be a submetrizable k_ω-space and, as in Theorem 2.10, we will assume X is the quotient space of $\hat{X} = \bigoplus_{F_n\in\mathcal{F}} \widetilde{F}_n, \widetilde{F}_n = \{n\} \times F_n$,

where F_n are locally compact second countable subsets of X and the family $\{F_n : n \in I\!N\}$ has been chosen with the minimun cardinality.
$\pi : \hat{X} \to X$ will denote the corresponding quotient map.
The topology of $\hat{X} = \bigoplus_{F_n \in \mathcal{F}} \widetilde{F}_n$ will be denoted by η and hence the topological sum by $(\hat{X}, \eta)$.

Let $\mathcal{P} = \{\preceq \in CL((X,\tau) \times (X,\tau)) : \preceq$ is a preorder on $D(\preceq) \subset X\}$.
For every $\preceq \in \mathcal{P}$ we associate in a natural way a preorder $\tilde{\preceq}$ defined on a closed subset of $\hat{X}$. More precisely, $\tilde{\preceq}$ is so defined:
- $D(\tilde{\preceq}) = \pi^{-1}(D(\preceq))$
- for every $a, b \in D(\tilde{\preceq})$, $\; a \tilde{\preceq} b \;$ if and only if $\; \pi(a) \preceq \pi(b)$.

Proposition 3.1 *$\tilde{\preceq}$ is a closed preorder defined on $D(\tilde{\preceq}) \subset \hat{X}$.*

Proof. Let $p = \pi \times \pi : \hat{X} \times \hat{X} \to X \times X$. p is a continuous, surjective map and induces a homeomorphism of $\widetilde{F}_n \times \widetilde{F}_n$ onto $F_n \times F_n$. Clearly we have

$$\tilde{\preceq} = \{(x,y) \in D(\tilde{\preceq}) \times D(\tilde{\preceq}) : (\pi(x), \pi(y)) \in \preceq\} =$$

$$= \{(x,y) \in \pi^{-1}(D(\preceq)) \times \pi^{-1}(D(\preceq)) : p(x,y) \in \preceq\} = p^{-1}(\preceq).$$

Now we put $\widetilde{\mathcal{P}} = \{\tilde{\preceq} : \preceq \in \mathcal{P}\} \subset CL(\hat{X} \times \hat{X})$ and consider the topological subspace $(\widetilde{\mathcal{P}}, F(\eta \times \eta))$.

Proposition 3.2 *Consider the following assertions:*

(a) *the net $\{\tilde{\preceq}_\sigma : \sigma \in \Sigma\} \subset \widetilde{\mathcal{P}}$ is $F(\eta \times \eta)$ - convergent to $\tilde{\preceq}$*
(b) *the net $\{D(\tilde{\preceq}_\sigma) : \sigma \in \Sigma\} \subset CL((\hat{X}, \eta))$ is $F(\eta) -$ convergent to $D(\tilde{\preceq})$*
(c) *the net $\{D(\preceq_\sigma) : \sigma \in \Sigma\} \subset CL((X, \tau))$ is $F(\tau) -$ convergent to $D(\preceq)$.*

Then $(a) \Rightarrow (b) \Rightarrow (c)$.

Proof. $(a) \Rightarrow (b)$
Let U be open in $(\hat{X}, \eta)$ such that $D(\tilde{\preceq}) \in U^-$, that is $D(\tilde{\preceq}) \cap U \neq \emptyset$. Then $\tilde{\preceq} \cap (U \times U) \neq \emptyset$. Since $\{\tilde{\preceq}_\sigma : \sigma \in \Sigma\}$ $F(\eta \times \eta))$ - converges to $\tilde{\preceq}$ there is $\sigma_0 \in \Sigma$ such that $\tilde{\preceq}_\sigma \cap (U \times U) \neq \emptyset$ for every $\sigma \geq \sigma_0$ and so $D(\tilde{\preceq}_\sigma) \cap U \neq \emptyset$ for every $\sigma \geq \sigma_0$.
Let now K be a compact set in $(\hat{X}, \eta)$ such that $D(\tilde{\preceq}) \in (K^c)^+$, that is $D(\tilde{\preceq}) \cap K = \emptyset$. Then $\tilde{\preceq} \cap (K \times K) = \emptyset$. Since $\{\tilde{\preceq}_\sigma : \sigma \in \Sigma\}$ $F(\eta \times \eta))$ - converges to $\tilde{\preceq}$ there is $\sigma_0 \in \Sigma$ such that $\tilde{\preceq}_\sigma \cap (K \times K) = \emptyset$ for every $\sigma \geq \sigma_0$ and so $D(\tilde{\preceq}_\sigma) \cap K = \emptyset$ for every $\sigma \geq \sigma_0$.
$(b) \Rightarrow (c)$
Let U be open in (X, τ) such that $D(\preceq) \in U^-$, that is $D(\preceq) \cap U \neq \emptyset$. Then

$\pi^{-1}(D(\preceq)\cap U)=\pi^{-1}(D(\preceq)\cap\pi^{-1}(U)=D(\widetilde{\preceq})\cap\pi^{-1}(U)\neq\emptyset$.
Since $\{D(\widetilde{\preceq}_\sigma):\sigma\in\Sigma\}$ $F(\eta)$- converges to $D(\widetilde{\preceq})$, there is $\sigma_0\in\Sigma$ such that $D(\widetilde{\preceq}_\sigma)\cap\pi^{-1}(U)\neq\emptyset$ for every $\sigma\geq\sigma_0$ and hence
$\pi(D(\widetilde{\preceq}_\sigma)\cap\pi^{-1}(U))\subset D(\preceq_\sigma)\cap U\neq\emptyset$ for every $\sigma\geq\sigma_0$.
Let now K be a compact set in (X,τ) such that $D(\preceq)\in(K^c)^+$, that is $D(\preceq)\cap K=\emptyset$. Let F_n be a locally compact second countable subset of X such that $K\subset F_n$. Then $(D(\widetilde{\preceq}))\cap(\pi^{-1}(K)\cap\widetilde{F}_n)=\emptyset$ and hence there is $\sigma_0\in\Sigma$ such that $D(\widetilde{\preceq}_\sigma)\cap(\pi^{-1}(K)\cap\widetilde{F}_n)=\emptyset$ for every $\sigma\geq\sigma_0$ that is $D(\preceq_\sigma)\cap K=\emptyset$ for every $\sigma\geq\sigma_0$.

We will introduce on $\mathcal{P}$ a Fell-type topology defined using the topologies $\tau_{|F_n}$, $n\in I\!N$, that is the topology generated by all sets of the form

$$U^-=\{\preceq\in\mathcal{P}:\preceq\cap U\neq\emptyset\},\ U \text{ open in } (F_n,\tau_{|F_n})\times(F_n,\tau_{|F_n}) \quad\text{for some}\quad n\in I\!N$$

$$(K^c)^+=\{\preceq\in\mathcal{P}:\preceq\cap K=\emptyset\},\ K \text{ compact in } (X,\tau)\times(X,\tau).$$

This topology will be denoted by $\bigcup_n F(\tau_{|F_n}\times\tau_{|F_n})$.

Theorem 3.3 *The map* $\Gamma:(\mathcal{P},\bigcup_n F(\tau_{|F_n}\times\tau_{|F_n})\to(\widetilde{\mathcal{P}},F(\eta\times\eta))$ *defined by* $\Gamma(\preceq)=p^{-1}(\preceq)$ *is a homeomorphism.*

Proof. $\Gamma:\mathcal{P}\to\widetilde{\mathcal{P}}$ is a bijective map, in fact $\Gamma(\preceq)=p^{-1}(\preceq)=p^{-1}(\bigcup_n(\preceq\cap F_n\times F_n))=\bigcup_n p^{-1}(\preceq\cap F_n\times F_n)$ and p^{-1} is a homeomorphism of $F_n\times F_n$ onto $\widetilde{F}_n\times\widetilde{F}_n$, $n\in I\!N$.
The map Γ is continuous.
Let U be open in $(\hat{X},\eta)\times(\hat{X},\eta)$ and $U^-=\{\widetilde{\preceq}\in\widetilde{\mathcal{P}}:\ \widetilde{\preceq}\cap U\neq\emptyset\}\in F(\eta\times\eta)$.
$U^-=\{\widetilde{\preceq}\in\widetilde{\mathcal{P}}:\ \bigcup_n(\widetilde{\preceq}\cap U\cap\widetilde{F}_n\times\widetilde{F}_n)\neq\emptyset\}=\{\widetilde{\preceq}\in\widetilde{\mathcal{P}}:\ \bigcup_n p^{-1}(\preceq\cap U_n)\neq\emptyset\}$, where $U_n=p(U\cap\widetilde{F}_n\times\widetilde{F}_n)$ is open in $\tau_{|F_n}\times\tau_{|F_n}$.
So $\Gamma^{-1}(U^-)=\{\preceq\in\mathcal{P}:\ \bigcup_n(\preceq\cap U_n)\neq\emptyset\}=\bigcup_n\{\preceq\in\mathcal{P}:\preceq\cap U_n\neq\emptyset\}=\bigcup_n(U_n)^-\in\bigcup_n F(\tau_{|F_n}\times\tau_{|F_n})$.
Let $\widetilde{K}$ be compact in $(\hat{X},\eta)\times(\hat{X},\eta)$ and $(\widetilde{K^c})^+=\{\widetilde{\preceq}\in\widetilde{\mathcal{P}}:\ \widetilde{\preceq}\cap\widetilde{K}=\emptyset\}$.
Let $K=p(\widetilde{K})$ compact in $(X,\tau)\times(X,\tau)$. $\Gamma^{-1}(\widetilde{K^c})^+=\{\preceq\in\mathcal{P}:\ p^{-1}(\preceq)\cap p^{-1}(K)=\emptyset\}=\{\preceq\in\mathcal{P}:\ p^{-1}(\preceq\cap K)=\emptyset\}=\{\preceq\in\mathcal{P}:\preceq\cap K=\emptyset\}=(K^c)^+$.
Γ is an open map.
Let U be open in $(F_n,\tau)\times(F_n,\tau)$ for some n and let $U^-=\{\preceq\in\mathcal{P}:\preceq\cap U\neq\emptyset\}\in\bigcup_n F(\tau_{|F_n}\times\tau_{|F_n})$. Let $\widetilde{U}=p^{-1}(U)$, open in $(\hat{X},\eta)\times(\hat{X},\eta)$.
It is immediate to prove that $\Gamma(U^-)=\{\widetilde{\preceq}\in\widetilde{\mathcal{P}}:\ \widetilde{\preceq}\cap\widetilde{U}\neq\emptyset\}=\widetilde{U}^-$. Similarly it is shown that $\Gamma(K^c)^+=(\widetilde{K^c})^+$ where K is compact in $(X,\tau)\times(X,\tau)$ and $\widetilde{K}=p^{-1}(K)$ is compact in $(\hat{X},\eta)\times(\hat{X},\eta)$.

Let $\mathcal{V}=\{(D(\widetilde{\preceq}),\widetilde{u})\in\mathcal{U}_\eta:\widetilde{u}$ is a utility function for some $\widetilde{\preceq}\in\widetilde{\mathcal{P}}\}\subset\mathcal{U}_\eta$, where $\mathcal{U}_\eta$ is the space of all continuous partial maps defined on $D(\widetilde{\preceq})\subset\hat{X}$ and η_c is the topology which has as a subbase all sets of the type $[G],[K:I]$

defined in $(\hat{X}, \eta)$. Let

$$L : \mathcal{V} \to \mathcal{U}_\tau$$

be defined as follows

$$L((D(\widetilde{\preceq}), \widetilde{u})) = (D(\preceq), u) \quad \text{where} \quad u(x) = \widetilde{u}(\pi^{-1}(x)).$$

Proposition 3.4 *The map L is well defined and $L((D(\widetilde{\preceq}), \widetilde{u})) = (D(\preceq), u)$ is a utility function for $\preceq$.*

Proof. Since all the elements of $\pi^{-1}(x)$ are equivalent with respect to $\preceq$, the map L is well defined. Because $D(\preceq)$ is a k_ω-space then $\pi_{|D(\widetilde{\preceq})} : D(\widetilde{\preceq}) \to D(\preceq)$ is a quotient map and $u : D(\preceq) \to I\!R$ is continuous:

$$\begin{array}{ll} D(\widetilde{\preceq}) & \xrightarrow{\widetilde{u}} I\!R \\ \downarrow \pi_{|D(\widetilde{\preceq})} & \nearrow u \\ D(\preceq) & \end{array}$$

Now it is easy to prove that $(D(\preceq), u) = L((D(\widetilde{\preceq}), \widetilde{u}))$ is a utility function for $\preceq$.

Proposition 3.5 *The map $L : \mathcal{V} \to \mathcal{U}_\tau$ is continuous.*

Proof. Let $\{(D(\widetilde{\preceq_\sigma}), \widetilde{u_\sigma}) : \sigma \in \Sigma\}$ η_c - converges to $(D(\widetilde{\preceq}), \widetilde{u})$.
We want to prove that $\{(D(\preceq_\sigma), u_\sigma) = L((D(\widetilde{\preceq_\sigma}), \widetilde{u_\sigma})) : \sigma \in \Sigma\}$ τ_c - converges to $(D(\preceq), u) = L((D(\widetilde{\preceq}), \widetilde{u}))$.
Note that $(D, u) \in [U]$ if and only if $D \in U^-$ and $(D, u) \in [K : \emptyset]$ if and only if $D \in (K^c)^+$. Thus if U is an open in (X, τ) such that $(D(\preceq), u) \in [U]$, or K is a compact set in (X, τ) and $(D(\preceq), u) \in [K : \emptyset]$, by Proposition 3.2 there is $\sigma_0 \in \Sigma$ such that $(D(\preceq_\sigma), u_\sigma) \in [U]$ or $(D(\preceq_\sigma), u_\sigma) \in [K : \emptyset]$ for every $\sigma \geq \sigma_0$.
Let now K be a compact set in (X, τ) and I be a nonempty open set in $I\!R$ such that $u(K \cap D(\preceq)) \subset I$. If $D(\preceq) \cap K = \emptyset$ we are done by above.
Suppose that $D(\preceq) \cap K \neq \emptyset$. We would like to prove that there is $\sigma_0 \in \Sigma$ such that $u_\sigma(K \cap D(\preceq_\sigma)) \subset I$ for every $\sigma \geq \sigma_0$.
Let F_n be a locally compact second countable subset of X such that $K \subset F_n$. Because $\pi^{-1}(K) \cap \widetilde{F}_n$ is compact in $(\hat{X}, \eta)$ and π induces a homeomorphism of $\widetilde{F}_n$ onto F_n, we have $\widetilde{u}(\pi^{-1}(K) \cap \widetilde{F}_n \cap D(\widetilde{\preceq})) \subset I$ that is $(D(\widetilde{\preceq}), \widetilde{u}) \in [\pi^{-1}(K) \cap \widetilde{F}_n : I]$.
There is $\sigma_0 \in \Sigma$ such that $\widetilde{u}_\sigma(\pi^{-1}(K) \cap \widetilde{F}_n \cap D(\widetilde{\preceq}_\sigma)) \subset I$ for every $\sigma \geq \sigma_0$ and so, by the homeomorphism from $\widetilde{F}_n$ to F_n, $u_\sigma(K \cap D(\preceq_\sigma)) \subset I$ for every $\sigma \geq \sigma_0$.

Theorem 3.6 *Let (X, τ) be a submetrizable k_ω-space. There exists a con-*

tinuous map

$$\nu : (\mathcal{P}, \bigcup_n F(\tau_{|F_n} \times \tau_{|F_n})) \to (\mathcal{U}_\tau, \tau_c)$$

such that $\nu(\preceq)$ is a utility function for $\preceq$, for every $\preceq \in \mathcal{P}$.

Proof. The space $(\hat{X}, \eta)$ is locally compact and second countable. By Back's Theorem [1] there is a continuous map

$$\widetilde{\nu} : (\widetilde{\mathcal{P}}, F(\eta \times \eta)) \to (\widetilde{\mathcal{V}}, \eta_c)$$

such that $\widetilde{\nu}(\widetilde{\preceq})$ is a utility function for $\widetilde{\preceq}$, for every $\widetilde{\preceq} \in \widetilde{\mathcal{P}}$.
Consider the following diagram

$$\begin{array}{ccc} (\widetilde{\mathcal{P}}, F(\eta \times \eta)) & \stackrel{\widetilde{\nu}}{\longrightarrow} & (\mathcal{V}, \eta_c) \\ \uparrow\Gamma & & \downarrow L \\ (\mathcal{P}, \bigcup_n F(\tau_{|F_n} \times \tau_{|F_n})) & \stackrel{\nu = L\circ\widetilde{\nu}\circ\Gamma}{\longrightarrow} & (\mathcal{U}_\tau, \tau_c) \end{array}$$

The map $\nu = L \circ \widetilde{\nu} \circ \Gamma : (\mathcal{P}, \bigcup_n F(\tau_{|F_n} \times \tau_{|F_n})) \to (\mathcal{U}_\tau, \tau_c)$ is continuous and $\nu(\preceq)$ is a utility function for $\preceq$, for every $\preceq \in \mathcal{P}$.

4 Applications

In [11] the author considers a market "M" with n goods and extends the classical finite dimensional state preference model to a model where the space of states of the world is the space $I\!R^m$. The market in question will be observed only two times, the initial time and the final time. The set of the goods is $\underline{n} = \{k \in \mathbb{N} : k \leq n\}$, the set of the states of the world is $I\!R^m$. An $\mathcal{S}$-linear state preference model is a system $(\underline{n}, I\!R^m, A)$ where $A : I\!R^n \to S'_m$ is a linear operator. S'_m is the space of tempered distributions defined as the continuous dual of the Schwartz space $S_m = S(I\!R^m)$. An n-vector x is a portfolio of the model and the temperate distribution $A(x)$ is the A-representation of x. The author justifies and motivates from an economic point of view the choice of the temperate distribution space S'_m to extend to the infinite dimensional case the classical finite-dimensional state preference model. As in the finite dimensional case, a preference relation on $I\!R^n$ generated by A is defined by:

$$x \geq_A x' \Leftrightarrow A(x - x') \geq 0_{S'_m}$$

that is

$$x \geq_A x' \Leftrightarrow A(x - x')(\phi) \geq 0, \text{ for every non-negative } \phi \in S_m.$$

In Decision Theory one of the most important objectives is the representation of preorders by means of continuous utility functions. The main representation theorems of partial preorders are proved for closed preorders, so it is interesting to study if the preorder generated by A is closed.
An example of partial closed preordered generated by A can be obtained for $m = 1$ by defined $A(x - x')(\phi) = \sum_{i=1}^{n}(x_i - x'_i)\phi(0)$.
We prove the set $\mathcal{B} = \{A(x) : A(x)\phi \geq 0, \quad \text{for every} \quad \phi \geq 0\}$ is closed in $A(I\!\!R^n) \subset S'$.
Of course $A(I\!\!R^n)$ is isomorphic to a subspace $I\!\!R^s$ of $I\!\!R^n$ with $s = n - dim(Ker(A))$. Hence $A(I\!\!R^n) = < A(v_1), A(v_2), ..., A(v_s) >$ with suitable $v_1, v_2, ..., v_s \in I\!\!R^n$. Therefore, each operator $A(x)$ can be identified with an s-tuple of $I\!\!R^s$ by the homeomorphism $(a_1^x, a_2^x, ..., a_s^x) \mapsto \sum_{i=1}^{s} a_i^x A(v_i)$ and $A(I\!\!R^n)$ is a closed subspace of S'_m. To prove that $\mathcal{B}$ is closed, we can proceed by sequences. Let $(A(x_k))_k$ be a sequence in $\mathcal{B}$ converging to an operator $L \in S'_m$. Hence $L \in A(I\!\!R^n)$ and $L = A(x) = \sum_{i=1}^{s} a_i^x A(v_i)$. Now, for every $\phi \geq 0, A(x_k)\phi \geq 0$ implies $A(x)\phi \geq 0$, that is $A(x) \in \mathcal{B}$. Then the preorder generated by A has a continuous representation (see Theorem 2 in [20]).
Now, we can consider a family $(M_\alpha)_{\alpha \in J}$ of markets, where M_α is a market with $n_\alpha \in I\!\!N$ goods. For every $\alpha \in J$, let $(n_\alpha, I\!\!R^m, A_{n_\alpha})$ be an $\mathcal{S}$-linear state preference model, where $A_{n_\alpha} : I\!\!R^{n_\alpha} \to S'_m$ is a linear operator. So, the sequence $(A_{n_\alpha})_\alpha$ generates a family $\mathcal{P}$ of closed preorders defined on closed subspaces of S'_m. Theorem 3.6 proves the existence of a continuous map ν from $\mathcal{P}$ to $\mathcal{U}_\tau$ such that for every $\preceq \in \mathcal{P}$ $\nu(\preceq)$ is a continuous utility function.

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