http://dx.doi.org/10.1016/j.jpaa.2015.05.023

INDUCTIVELY COMPUTABLE UNIONS OF FAT LINEAR SUBSPACES

GIULIANA FATABBI, BRIAN HARBOURNE AND ANNA LORENZINI

ABSTRACT. This paper introduces complexes of linear varieties, called inclics (for INductively Constructible LInear ComplexeS). By assigning an order of vanishing (i.e., a multiplicity) to each member of the complex, we obtain fat linear varieties (fat points if all of the linear varieties are points). The scheme theoretic union of these fat linear varieties gives an inclic scheme X. For such a scheme, we show there is an inductive procedure for computing the Hilbert function and a resolution of its defining ideal I_X, regardless of the choice of multiplicities. As an application, we show how our results allow the computation of the Hilbert functions and of a resolution of fat points with all but one point having support in a hyperplane. We also explicitly compute the Waldschmidt constants $\widehat{\alpha}(I_X)$ for galactic inclics X; these are special inclics constructed starting from a star configuration to which we add general points in a larger projective space.

1. INTRODUCTION

There is a long tradition of research on ideals of unions of linear varieties in projective spaces. Such an ideal is the intersection of ideals generated by linear forms. Examples include square free monomial ideals, ideals of star configurations [GHM] and ideals of finite sets of points. Research started with the radical case (see [D, DS, GO, HH, HS, L] for example) but there is also a lot of interest in ideals of schemes of linear varieties with assigned multiplicities, including but not limited to fat points (see [CHT, DHST, Fa, FHL, FaL, FrL, Fr, FMN, GHV1, V] for just a few examples). The ideals in the uniform multiplicity case are symbolic power ideals; ideals in this special case are also of interest and are receiving increasing attention (see [BH, BH2, GHM, GMS, GHV2, HaHu, M] for example), but there are few cases where the Hilbert functions of arbitrary symbolic powers can be determined.

In this paper we introduce *inclic schemes*. These are schemes whose components comprise a complex of linear varieties called an inclic (for INductively Constructible LInear Complex; see Remark 3.3 for examples motivating this terminology). An inclic scheme is obtained by arbitrarily assigning a multiplicity (i.e., an assigned order of vanishing) to each component. Our main foundational results provide a recursive procedure for computing Hilbert functions and free resolutions of ideals of inclic schemes. These results can be applied to the case of fat points with all but one point having support in a hyperplane. In certain cases this procedure can also be applied to compute Hilbert functions and free resolutions of arbitrary symbolic powers of radical ideals. This substantially extends the range of examples of ideals for which this is possible. As a further application, we define galactic schemes and explicitly compute the Waldschmidt constants of certain galactic schemes built up from star configurations. (A Waldschmidt constant is an asymptotic measure of the initial degrees of the symbolic powers of an ideal. These have arisen in a range of previous

Date: March 28, 2015.

2000 *Mathematics Subject Classification.* Primary: 13F20, 14C20; Secondary: 13A02, 14N05.

Key words and phrases. star configurations, galaxies, inclic schemes, symbolic powers, fat points, projective space, Hilbert functions, resolutions, Betti numbers.

The second author's work on this project was sponsored by the National Security Agency under Grant/Cooperative agreement "Advances on Fat Points and Symbolic Powers," Number H98230-11-1-0139. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notice. The second author also thanks the University of Perugia and his coauthors for their generous hospitality during his visit in 2010 when this work was initiated. And all of the authors thank the referee for his very helpful comments.

research, such as [BH, Ch, DHST, GHV1, HaHu, M, W], and are related to work on multiplier ideals; see [EV, HaHu] and [La, Proposition 10.1.1 and Example 10.1.3].) In section 2 we set up our technical notation. In sections 3 and 4 we prove our main foundational results about, respectively, Hilbert functions and free resolutions for ideals of inclics. In section 5 we apply the results of the previous sections to the case of fat points with all but one point having support in a hyperplane and in section 6 we compute galactic Waldschmidt constants.

2. Preliminaries and notation

To define inclics, let $n > 0$ be an integer. We work in the projective space $\mathbb{P}^n$ over an arbitrary field K (some results will require the characteristic to be 0). An *inclic*

$$\mathcal{C} = \mathcal{C}(n, r, s; L_0, L_1, \dots, L_r, H_0, H_1, \dots, H_s)$$

is a collection of linear subvarieties $L_0, L_1, \dots, L_r, H_0, H_1, \dots, H_s \subsetneq \mathbb{P}^n$ such that the following conditions hold:

(C1) $H_0, H_1, \dots, H_s$ are distinct hyperplanes;
(C2) $L_i \subseteq H_0$ for $i > 0$, but $L_0 \not\subseteq H_0$;
(C3) if $L_i \subseteq L_j$, then $i = j$; and
(C4) for all $i \geq 0$ and $j > 0$ we have $L_i \not\subseteq H_j$.

Given such an inclic, an *inclic scheme* is a scheme of the form $X = \sum_{i\geq 0} l_i L_i + \sum_{j>0} h_j H_j$, by which we mean the scheme defined by the ideal $I_X = (\cap_{i\geq 0} I_{L_i}^{l_i}) \bigcap (\cap_{j>0} I_{H_j}^{h_j})$, where l_i and h_j are non-negative integers and for any linear subvariety $V \subseteq \mathbb{P}^n$, the ideal $I_V \subseteq R = K[\mathbb{P}^n] = K[x_0, \dots, x_n]$ is the ideal generated by all forms vanishing on V. We note that ideals such as I_V are homogeneous and saturated. Moreover, if $I = \sqrt{I_X}$, then $I = (\cap_{i\geq 0} I_{L_i}) \bigcap (\cap_{j>0} I_{H_j})$, and for any $m \geq 1$, the symbolic power $I^{(m)}$ is $I^{(m)} = (\cap_{i\geq 0} I_{L_i}^m) \bigcap (\cap_{j>0} I_{H_j}^m)$, so $I_X = I^{(m)}$ in the case that $l_i = h_i = m$ for all i and j.

If $s = 0$ and each L_i is a point, then the inclic is just a choice of r points L_i, $0 < i \leq r$, of the hyperplane H_0 and one point L_0 that is not in H_0. The case of a finite set of points, all of which are in a hyperplane, is dealt with in [FHL]. Requiring that $L_0 \not\subseteq H_0$ thus takes us beyond [FHL].

Another special case of an inclic is related to what we call a *galaxy*. To define a galaxy, we start with a star configuration $S(n, e, u)$. We recall [GHM] that the star configuration $S(n, e, u)$ is defined by a set of $u \geq n$ distinct hyperplanes $A_1, \dots, A_u \cong \mathbb{P}^{n-1}$ in $\mathbb{P}^n$ such that, for each $1 \leq i \leq n$, the intersection of any i of the hyperplanes has dimension at most $n - i$. The star configuration of codimension $e \leq n$ is the set $S(n, e, u)$ of the $\binom{u}{e}$ linear varieties arising as intersections of e arbitrary distinct choices $A_{i_1}, \dots, A_{i_e}$ of the hyperplanes. Let $N \geq 1$ be an integer and regard $\mathbb{P}^n$ as a linear subvariety of $\mathbb{P}^{n+N}$. The galaxy $\mathcal{G} = \mathcal{G}(n, N, e, h) = \mathcal{G}(n, N, e, h; S(n, e, u), \mathcal{H})$ consists of $S(n, e, u)$ and a choice of h general points $\mathcal{H} = \{P_1, \dots, P_h\} \in \mathbb{P}^{n+N}$; in particular, for each i, P_{i+1} is not in the span of $\mathbb{P}^n$ and $P_1, \dots, P_i$. We refer to $S(n, e, u)$ as the galactic center, to $\mathbb{P}^n$ as the galactic (n-)plane, and to $\mathcal{H}$ as the galactic halo. (These astronomical references were prompted by the connection to star configurations and give useful intuition, this intuition only goes so far. For example, the restriction that the halo $\mathcal{H}$ consists of general points means that $h \leq N$; i.e., the galactic halo is relatively sparse.) When $h = N$, then we get an inclic in which the components of $S(n, e, u)$ and the points $P_1, \dots, P_{h-1}$ are the linear varieties L_i, and there is only one hyperplane, H_0 (so again $s = 0$), this hyperplane being the hyperplane containing the linear span of $\mathbb{P}^n$ and the points $P_1, \dots, P_{h-1}$, but not containing P_h.

For any homogeneous ideal $I \subseteq R$, the Hilbert function of I is the function $h(I, t)$ of t defined as $h(I, t) = \dim_K I_t$, where I_t is the K-vector space span of all forms in I of degree t. If $I_X \subsetneq R$ is the saturated ideal defining a subscheme $X \subseteq \mathbb{P}^n$, the Hilbert function of X is the function $h(X, t) = h(R, t) - h(I_X, t) = \binom{t+n}{n} - h(I_X, t)$. In all cases, we adopt the understanding that Hilbert functions are 0 when $t < 0$.

An important value associated to any homogeneous ideal $(0) \neq I \subseteq R$ is $\alpha(I)$, defined to be the least degree t such that $h(I,t) \neq 0$. In case I is of the form $I = \cap_j I_{V_j}^{m_j}$ for a finite set of linear varieties V_j, none of which contains the other, we define the mth symbolic power $I^{(m)}$ of I as $I^{(m)} = \cap_j I_{V_j}^{mm_j}$. We then define the *Waldschmidt constant* (introduced by Waldschmidt in [W] in case I is the ideal of a finite set of points) to be

$$\widehat{\alpha}(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}.$$

This limit exists by Fekete's Lemma (see Section 1.10 of [S]), but in general is hard to compute and not many specific values are known.

Another asymptotic measure related to $\widehat{\alpha}(I)$ is the *resurgence* [BH, BH2, GHV1], defined for any homogeneous ideal $(0) \neq I \subsetneq R = K[\mathbb{P}^n]$ as

$$\rho(I) = \sup\Big\{\frac{m}{r} : I^{(m)} \not\subseteq I^r\Big\}.$$

In general it is difficult to determine for which m and r we have $I^{(m)} \subseteq I^r$. The interest of $\rho(I)$ is that it is the largest real c such that we always have $I^{(m)} \subseteq I^r$ for $m/r > c$, but it is difficult to compute. It is not a priori even clear that it exists. It is known and easy to see that $1 \leq \rho(I)$. Much deeper is the fact that $I^{(m)} \subseteq I^r$ whenever $m/r \geq n$ [ELS, HoHu] from which it follows that $\rho(I) \leq n$ and hence $\rho(I)$ exists. This raises the issue of getting better bounds. One of the main results for bounding and sometimes computing $\rho(I)$ is that of [BH] which says that $\frac{\alpha(I)}{\widehat{\alpha}(I)} \leq \rho(I)$, and, if I defines a 0-dimensional subscheme of $\mathbb{P}^n$, that $\rho(I) \leq \frac{\mathrm{reg}(I)}{\widehat{\alpha}(I)}$, where $\mathrm{reg}(I)$ is the Castelnuovo-Mumford regularity of I, but these bounds depend on $\widehat{\alpha}(I)$ which has so far been computed in relatively few cases, so obtaining additional cases for which $\widehat{\alpha}(I)$ can be computed is of interest.

Hereafter we study fat inclic schemes for some fixed hyperplane $H_0 \subset \mathbb{P}^n$. Clearly, we may choose coordinates such that $I_{H_0} = (x_0)$, so $R' = K[\mathbb{P}^{n-1}] = K[H_0] = K[x_1,\dots,x_n]$. Since $L_0 \not\subseteq H_0$, we may also assume that $I_{L_0} = (x_{k+1},\dots,x_n) \subset K[\mathbb{P}^n] = R$ for some $0 \leq k < n$. We fix such a choice of coordinates for the rest of this article. We denote the linear forms defining H_j for $j > 0$ by η_j. We also take Y to be the fat subscheme $Y = l_1L_1 + \cdots + l_rL_r$ of $\mathbb{P}^n$. In addition we define $Y' = Y \cap H_0$ and $Y'_i = Y_i \cap H_0$ for $Y_i = l_1(i)L_1 + \cdots + l_r(i)L_r$, where $l_j(i) = \max(0, l_j - i)$. Thus $Y'_0 = Y'$ and $I_{Y'_i} = I_{Y_i} \cap K[x_1,\dots,x_n]$. Moreover, $I_{Y_i} = I_Y : (x_0^i)$. We set $Z = l_0L_0$, $W = Y \cup Z$, $X = W \bigcup \cup_{j>0} h_jH_j$, $L'_0 = L_0 \cap H_0$, $Z' = Z \cap H_0$, $W' = W \cap H_0$ and $X' = X \cap H_0$.

The following notation will be useful. Let $J' \subseteq R' = K[x_1,\dots,x_n]$ be a homogeneous ideal; keeping in mind that $I_{L_0} = (x_{k+1},\dots,x_n)$, we set $J'^{(k,t)} = J' \cap (I_{L'_0})^t$. Thus $(J'^{(k,t)})_i = (J')_i \cap ((I_{L'_0})^t)_i$. Note that in the special case that $k = 0$ (i.e., that L_0 is the point p defined in $\mathbb{P}^n$ by $(x_1,\dots,x_n)$), we have $(J'^{(0,t)})_i = J'_i$ for $i \geq t$ and $(J'^{(0,t)})_i = 0$ for $i < t$; in short, if we know J', then we immediately know $J'^{(0,t)}$ for all t.

Note that R has a bi-grading; i.e., the direct sum $R = \oplus_{ij} R_{ij}$ has the property that $R_{ij}R_{st} = R_{i+s,j+t}$, where R'_i is the K-vector space span of the forms in $R' = K[x_1,\dots,x_n]$ of total degree i, and R_{ij} is the K-vector subspace $x_0^jR'_i \subset R$. We say an element $F \in R$ is bi-homogeneous if $F \in R_{ij}$ for some i and j, and we say an ideal $I \subseteq R$ is bi-homogeneous if $I = \oplus_{ij} I_{ij}$, where $I_{ij} = I \cap R_{ij}$. As usual, I is bi-homogeneous if and only if I has bi-homogeneous generators, and intersections, sums and products of bi-homogeneous ideals are bi-homogeneous.

3. Hilbert Function

We can now state and prove our main theorem about Hilbert functions.

Theorem 3.1. *Let Y', Y_i', Z, W and X be as above, let $l' = \max(l_1, \dots, l_r)$. Then $I_X = \eta_1^{h_1} \cdots \eta_s^{h_s} I_W$ and I_W is bihomogeneous, decomposing as a direct sum of R'-modules as*

$$I_W = \oplus_j x_0^j (I_{Y_j'})^{(k,l_0)} = (\oplus_{0 \le j < l'} x_0^j (I_{Y_j'})^{(k,l_0)}) \bigoplus \oplus_{j \ge l'} x_0^j I_{Z'} = (\oplus_{0 \le j < l'} x_0^j (I_{Y_j'})^{(k,l_0)}) \bigoplus x_0^{l'} I_Z.$$

Moreover, $h(I_X, t) = h(I_W, t - \sum_{j>0} h_j)$ *and* $h(I_W, t) = \sum_{j=0}^{l'-1} h((I_{Y_j'})^{(k,l_0)}, t-j) + h(I_Z, t-l')$, *where* $h(I_Z, t-l') = 0$ *if* $t < l' + l_0$ *and* $h(I_Z, t-l') = \binom{t-l'+n}{n} - \sum_{0 \le i < l_0} \binom{t-l'-i+k}{k}\binom{i+n-k-1}{n-k-1}$ *for* $t \ge l' + l_0$.

Proof. It is obvious that $I_X = \eta_1^{h_1} \cdots \eta_s^{h_s} I_W$ and $h(I_X, t) = h(I_W, t - \sum_{j>0} h_j)$, so now we consider I_W and $h(I_W, t)$. To begin, note that the ideals $I_{L_i} \subset R$ are bi-homogeneous (having bi-homogeneous generators), so I_Y and $I_W = I_Y \cap I_Z$ are bi-homogeneous, hence $I_Y = \oplus_{ij} (I_Y)_{ij}$ and $I_W = \oplus_{ij} ((I_Y)_{ij} \cap (I_Z)_{ij})$. But $F \in (I_Y)_{ij}$ if and only if $F = x_0^j G$ where $G \in (I_{Y_j'})_i$; i.e., $(I_Y)_{ij} = x_0^j (I_{Y_j'})_i$. Thus $I_Y = \oplus_{ij} x_0^j (I_{Y_j'})_i$, and since $(I_Z)_{ij} = x_0^j (I_{Z'})_i$, we have

$$(*) \quad I_W = \oplus_{ij} ((x_0^j (I_{Y_j'})_i) \cap (I_Z)_{ij}) = \oplus_{ij} ((x_0^j (I_{Y_j'} \cap I_{Z'})_i) = \oplus_{ij} (x_0^j ((I_{Y_j'})^{(k,l_0)})_i) = \oplus_j x_0^j (I_{Y_j'})^{(k,l_0)}.$$

But for $j \ge l'$ we have $I_{Y_j'} = R'$ and hence $(I_{Y_j'})^{(k,l_0)} = I_{Z'}$, so

$$I_W = \oplus_j x_0^j (I_{Y_j'})^{(k,l_0)} = (\oplus_{0 \le j < l'} x_0^j (I_{Y_j'})^{(k,l_0)}) \bigoplus \oplus_{j \ge l'} x_0^j I_{Z'} = (\oplus_{0 \le j < l'} x_0^j (I_{Y_j'})^{(k,l_0)}) \bigoplus x_0^{l'} I_Z.$$

The fact that $h(I_W, t) = \sum_{j=0}^{l'-1} h((I_{Y_j'})^{(k,l_0)}, t-j) + h(I_Z, t-l')$ is now immediate, keeping in mind that the Hilbert function is computed with respect to the singly graded structure of R; i.e., $(I_W)_t = \oplus_{i+j=t} (I_W)_{ij}$. But the value of $h(I_Z, t-l')$ is known; the formula given in the statement of the theorem comes from [DHST, Lemma 2.1]. □

Recall that $I_{L_0} = (x_{k+1}, \dots, x_n)$. The case with $k = 0$ (i.e., that L_0 is the point p defined in $\mathbb{P}^n$ by $(x_1, \dots, x_n)$), is particularly simple; in this case, if we know the Hilbert functions of Y_j' for all j, then we know the Hilbert functions of W and hence X.

Corollary 3.2. *Under the hypotheses of Theorem 3.1, let $\lambda = \min(l'-1, t-l_0)$. If we also have $k = 0$, then*

$$h(I_W, t) = \sum_{j=0}^{\lambda} h(I_{Y_j'}, t-j) + \sum_{j=l'}^{t-l_0} \binom{t-j+n-1}{n-1},$$

which is $h(I_W, t) = \sum_{j=0}^{\lambda} h(I_{Y_j'}, t-j)$ *for* $t < l' + l_0$ *and*

$$h(I_W, t) = \sum_{j=0}^{\lambda} h(I_{Y_j'}, t-j) + \binom{t-l'+n}{n} - \binom{l_0+n-1}{n}$$

for $t \ge l' + l_0$.

Proof. This follows immediately from Theorem 3.1, since $(I_{Z'})_i = R_i'$ (so $h(I_{Z'}, t-j) = \binom{t-j+n-1}{n-1}$ and $((I_{Y_j'})^{(0,l_0)})_{t-j} = (I_{Y_j'})_{t-j}$ for $t - j \ge l_0$, that is, for $j \le t - l_0$). □

Remark 3.3. Examples for which we would know the Hilbert functions of $Y_j' \subset H_0$ for all j can be constructed inductively. For example, start with a flag of projective spaces $V_1 \subset V_2 \subset \cdots \subset V_n$, each contained in the next as a linear subvariety, with $V_i \simeq \mathbb{P}^i$. Let $U_1 \subset V_1$ be any finite set of points $u_{11}, \dots, u_{1s}$. Let U_2 consist of a point $u_{21} \in V_2 \setminus V_1$ together with any lines $u_{22}, \dots, u_{2s_2} \subset V_2$ not containing u_{21} or any component of U_1 (i.e., not containing u_{1i} for any i). Continue in this way, so U_i consists of a point $u_{i1} \in V_i \setminus V_{i-1}$ and a finite set of hyperplanes $u_{ij} \subset V_i$ not containing u_{i1} and not containing any of the components of U_j for $j < i$. Then $U_1 \cup \cdots \cup U_n$ defines an inclic and for any multiplicities m_{ij} we can inductively compute $h(I_X, t)$ for any t, for $X = \sum_{ij} m_{ij} u_{ij}$. Indeed, define $X_1 = \sum_j m_{1j}u_{1j}$, and then $X_2 = X_1 + \sum_j m_{2j}u_{2j}$, and in general $X_k = X_{k-1} + \sum_j m_{kj}u_{kj}$. Since we know $h((X_1)_i, t)$ for all i and t, Theorem 3.1 gives us $h((X_2)_i, t)$ for all i and t, and similarly $h((X_k)_i, t)$ for each k in turn for all i and t, and hence eventually $h(X_n, t)$ for all t.

Our result also handles other constructions. For example, instead of starting with points in $\mathbb{P}^1$, we could start with a star configuration of points in $\mathbb{P}^2$ (i.e., the points of pair-wise intersection of a finite set of lines, no three of which meet at any single point, see [GHM]). Let S be the scheme theoretic sum of the points of the star and consider the scheme iS. The Hilbert function of iS is known for all i ([CHT]), so we can proceed as above to construct an X_n, as long as in this case we assign the same multiplicity to each point of S (the Hilbert function is not always known if the multiplicities of the points of S are allowed to vary).

Recall that given a closed subscheme $X \subset \mathbb{P}^n$ with corresponding ideal I_X, we define $\alpha(X) = \alpha(I_X)$ to be the least degree t such that there is a non-trivial form $F \in (I_X)_t$.

Lemma 3.4. *With the previous notation, there is a least $j \geq 0$ such that $\alpha((I_{Y'_j})^{(k,l_0)}) = l_0$. Let $l' = \max(l_1, \dots, l_r)$ and let d be this least j. If, moreover, $\mathrm{char}(K) = 0$, then*

$$0 = \alpha(I_{Y'_{l'}}) < \alpha(I_{Y'_{l'-1}}) < \alpha(I_{Y'_{l'-2}}) < \dots < \alpha(I_{Y'_0})$$

and

$$l_0 = \alpha((I_{Y'_d})^{(k,l_0)}) < \alpha((I_{Y'_{d-1}})^{(k,l_0)}) < \alpha((I_{Y'_{d-2}})^{(k,l_0)}) < \dots < \alpha((I_{Y'_0})^{(k,l_0)}).$$

Proof. By definition, $(I_{Y'_j})^{(k,l_0)} \subset (x_{k+1}, \dots, x_n)^{l_0}$ so $\alpha((I_{Y'_j})^{(k,l_0)}) \geq l_0$ for all j. But for $j \geq l'$ we have $((I_{Y'_j})^{(k,l_0)}) = (x_{k+1}, \dots, x_n)^{l_0}$ so $\alpha((I_{Y'_j})^{(k,l_0)}) = l_0$. Thus there is a least j such that $\alpha((I_{Y'_j})^{(k,l_0)}) = l_0$ so d is defined.

Since $Y'_{l'} = \varnothing$, we have $I_{Y'_{l'}} = (1)$, so $\alpha(I_{Y'_{l'}}) = 0$. Now assume $\mathrm{char}(K) = 0$. Consider any non-zero homogeneous element $F \in I_{Y'_j}$ for $j < l'$. By Euler's identity, not all of the partials of F are 0. However, they all belong to $I_{Y'_{j+1}}$ and the non-zero ones have degree $\deg(F) - 1$. Therefore $\alpha(I_{Y'_{j+1}}) \leq \alpha(I_{Y'_j}) - 1$, so we have

$$0 = \alpha(I_{Y'_{l'}}) < \alpha(I_{Y'_{l'-1}}) < \alpha(I_{Y'_{l'-2}}) < \dots < \alpha(I_{Y'_0})$$

as claimed.

The argument for the second claim is similar. Let $j < d$ and consider any non-zero homogeneous element $F \in (I_{Y'_j})^{(k,l_0)} \subseteq I_{Y'_j}$, so $\deg(F) > l_0$, since also $F \in (I_{L'_0})^{l_0}$. Again not all of the partials of F are 0 but they all belong to $I_{Y'_{j+1}}$. Since $\deg(F) > l_0$, they all also belong to $(x_{k+1}, \dots, x_n)^{l_0}$ and hence to $(I_{Y'_{j+1}})^{(k,l_0)}$. Therefore $\alpha((I_{Y'_{j+1}})^{(k,l_0)}) \leq \alpha((I_{Y'_j})^{(k,l_0)}) - 1$, so we have

$$l_0 = \alpha((I_{Y'_d})^{(k,l_0)}) < \alpha((I_{Y'_{d-1}})^{(k,l_0)}) < \alpha((I_{Y'_{d-2}})^{(k,l_0)}) < \dots < \alpha((I_{Y'_0})^{(k,l_0)}).$$

□

Corollary 3.5. *Let $L_0, L_1, \dots, L_r, H_0, H_1, \dots, H_s \subsetneq \mathbb{P}^n$ be an inclic, and let $W = \sum_{i\geq 0} l_i L_i$ and $X = \sum_{i\geq 0} l_i L_i + \sum_{j>0} h_j H_j$ for non-negative integers l_i and h_i. Let Y'_i be as above, and let $l' = \max(l_1, \dots, l_r)$ and $h = h_1 + \dots + h_s$. Then $\alpha(X) = h + \alpha(W)$ and $\max(l', l_0) \leq \alpha(W) \leq l' + l_0$. Moreover, there is a least $j \geq 0$ such that $\alpha((I_{Y'_j})^{(k,l_0)}) = l_0$. Taking this least j to be d, we have*

$$\alpha(W) \leq l_0 + d,$$

with $\alpha(W) = l_0 + d$ if $\mathrm{char}(K) = 0$.

Proof. Since $I_X = \eta_1^{h_1} \cdots \eta_s^{h_s} I_W$, where η_i is the linear form defining H_i, we see that $\alpha(X) = h + \alpha(W)$. Since $l_1, \ldots, l_r, l_0 \leq \alpha(W)$, the lower bound $\max(l', l_0) \leq \alpha(W)$ holds. Since $x_0^{l'} x_n^{l_0} \in I_W$, the upper bound $\alpha(W) \leq l' + l_0$ also holds.

To get more precise information, note by (*) in the proof of Theorem 3.1 that

$$\alpha(W) = \alpha(I_W) = \min_j \alpha(x_0^j (I_{Y'_j})^{(k,l_0)}) = \min_j \{j + \alpha((I_{Y'_j})^{(k,l_0)})\}.$$

By definition, $(I_{Y'_j})^{(k,l_0)} \subset (x_{k+1}, \ldots, x_n)^{l_0}$ so $\alpha((I_{Y'_j})^{(k,l_0)}) \geq l_0$ for all j. But for $j \geq l'$ we have $((I_{Y'_j})^{(k,l_0)}) = (x_{k+1}, \ldots, x_n)^{l_0}$ so $\alpha((I_{Y'_j})^{(k,l_0)}) = l_0$ and hence $\alpha(I_W) \leq l' + l_0$. Thus there is a least j such that $\alpha((I_{Y'_j})^{(k,l_0)}) = l_0$ so d is defined. Thus we have $\alpha(I_W) \leq d + l_0$, and in addition we have $d + l_0 \leq j + \alpha((I_{Y'_j})^{(k,l_0)})$ for all $j \geq d$.

Now assume $\mathrm{char}(K) = 0$. By Lemma 3.4 we have

$$l_0 + d \leq \alpha((I_{Y'_{d-1}})^{(k,l_0)}) + (d-1) \leq \alpha((I_{Y'_{d-2}})^{(k,l_0)}) + (d-2) \leq \ldots \leq \alpha((I_{Y'_0})^{(k,l_0)}) + (d-d),$$

and hence $\alpha(I_{Y'_j{}^{(k,l_0)}}) + j \geq l_0 + d$ for all $j < d$, and therefore $\alpha(W) = l_0 + d$, as claimed. $\square$

4. Sets of generators and resolutions

Now we work out a free resolution for I_W (and hence I_X, which is the same as for I_W except for a shift in the grading). The idea is to mimic what is done in [FHL], using the structure of I_W as an R'-module as given in Theorem 3.1. This immediately gives graded generators over R', from which a set B of graded generators over R can be obtained, which we can reduce to a smaller set B^*. When $\mathrm{char}(K) = 0$ and L_0 is a single point, B^* is a minimal set of homogeneous generators over R. Using the approach of [FHL], the nonminimal generators B extend to a free resolution, as we will show, but it is not clear how in general to obtain a minimal set of generators for I_W much less a minimal free resolution.

We begin by finding a set of generators for I_W. By Theorem 3.1, I_W is a direct sum of the graded R'-modules $x_0^i (I_{Y'_j})^{(k,l_0)}$, so graded generators over R can be obtained by taking a minimal set B_j of graded R'-generators for each $x_0^j (I_{Y'_j})^{(k,l_0)}$. Thus $B_j = x_0^j B'_j$, where B'_j is a minimal set of graded generators for the R'-ideal $(I_{Y'_j})^{(k,l_0)}$. The union $\cup_j B_{j \geq 0}$ is a set of graded R-generators for I_W. This is an infinite set since there is no bound on j. This is because I_W is not a finitely generated R'-module, but, as Theorem 3.1 shows, $\oplus_{j \geq l'} x_0^j (I_{Y'_j})^{(k,l_0)} = x_0^{l'} I_Z$, and since $B_{l'}$ generates $x_0^{l'} (I_{Y'_{l'}})^{(k,l_0)} = x_0^{l'} I_{Z'}$ over R', which in turn generates $x_0^{l'} I_Z$ over R, we see that $B = \cup_{0 \leq j \leq l'} B$ is a finite set of graded R-generators for I_W.

These are typically redundant, however, since for all $j \geq d$ the initial degree of $(I_{Y'_j})^{(k,l_0)}$ is l_0, so we have

$$((I_{Y'_d})^{(k,l_0)})_{l_0} \subseteq ((I_{Y'_{d+1}})^{(k,l_0)})_{l_0} \subseteq \cdots .$$

Thus for example, B'_d contains a basis for $((I_{Y'_d})^{(k,l_0)})_{l_0}$ and B'_{d+1} contains a basis for $((I_{Y'_{d+1}})^{(k,l_0)})_{l_0}$, but the vector space span $x_0^{d+1}((I_{Y'_{d+1}})^{(k,l_0)})_{l_0}$ of B_{d+1} contains the vector space span $x_0^{d+1}((I_{Y'_d})^{(k,l_0)})_{l_0}$ of $x_0 B_d$. There may be other redundancies, but to avoid redundancies of this kind at least, for each $j \geq 0$ starting with $j = 0$ we pick a basis A'_j of $((I_{Y'_j})^{(k,l_0)})_{l_0}$, extend to a basis A'_{j+1} of $((I_{Y'_{j+1}})^{(k,l_0)})_{l_0}$, which we extend to a basis A'_{j+2} of $((I_{Y'_{j+2}})^{(k,l_0)})_{l_0}$, etc. (Note that $A'_j = \varnothing$ for $j < d$.) Now define $A_j = x_0^j A'_j$ for $j \geq d$ (and $A_j = \varnothing$ for $j < d$). For each $j \geq 0$, we then extend A'_j to a minimal set B'_j of graded generators for $((I_{Y'_j})^{(k,l_0)})_{l_0}$, and set $B^*_j = x_0^j (B'_j \setminus A'_{j-1})$ (so $B^*_j = x_0^j B'_j$ for $j \leq d$). Then $B^* = \cup_{0 \leq j \leq l'} B^*_j$ is a set of R-generators for I_W.

Theorem 4.1. *If* $\mathrm{char}(K) = 0$ *and* L_0 *is a single point, then the set* B^* *is in fact a minimal set of generators for* I_W*.*

Proof. It is clear by construction that B^* generates. We need to check that it is minimal. By construction, each element of B^* is bigraded; i.e., it is a power of x_0 times a homogeneous form in R', hence is in R_{ij}. So take an element $g \in B^*$. Thus $g = x_0^j f \in R_{ij}$ for some i and j, where $f \in B'_j \setminus A'_{j-1}$. Suppose the elements of B^* without g still generate I_W; i.e., suppose that g is an R-linear combination of the other elements of B^*. Note that $(I_W)_{rs} = (0)$ for $r < l_0$. Thus $i \geq l_0$.

If $i = l_0$, then $f \in A'_j$ and g being an R-linear combination of the other elements of B^* means g is a K-linear combination of the other elements of $x_0^j B_0^* \cup x_0^{j-1} B_1^* \cup \cdots \cup x_0^{j-j} B_j^*$ of R'-degree l_0, i.e., g is a K-linear combination of the other elements of $\cup_{d \leq s \leq j} x_0^j (A'_s \setminus A'_{s-1}) = x_0^j A'_j$, hence f is in the K-vector space span of the other elements of A'_j, but A'_j is a basis and $f \in A'_j$, so this is impossible.

Say $i > l_0$. Then g is an R'-linear combination of other elements of the union of $x_0^{j-s} B_s^* = x_0^j (B'_s \setminus A'_{s-1})$ for $s \leq j$ of R'-degrees $r \leq i$. Thus f is in the ith homogeneous component of the R'-module generated by the union $\cup_{s \leq j} (B'_s \setminus A'_{s-1}) = \cup_{s \leq j} B'_s$. Since B'_s generates $(I_{Y'_s})^{(0,l_0)}$ and $(I_{Y'_s})^{(0,l_0)} \subseteq (I_{Y'_{s+1}})^{(0,l_0)}$, f must be in the ith homogeneous component of the R'-module generated by $B'_j \setminus \{f\}$, modulo $((I_{Y'_{j-1}})^{(0,l_0)})_i$. But $((I_{Y'_{j-1}})^{(0,l_0)})_i \subset R'_1((I_{Y'_j})^{(0,l_0)})$ (as will be seen in a moment). Thus the image of $B'_j \setminus \{f\}$ in the quotient $((I_{Y'_j})^{(0,l_0)})/(R'_1(I_{Y'_j})^{(0,l_0)})$ must generate what the image of B'_j generates, which is the whole quotient. However, homogeneous elements of $(I_{Y'_j})^{(0,l_0)}$ whose images generate this particular quotient also generate $(I_{Y'_j})^{(0,l_0)}$; i.e., $B'_j \setminus \{f\}$ generates $(I_{Y'_j})^{(0,l_0)}$. This is a contradiction, since B'_j is a minimal set of homogeneous generators for $(I_{Y'_j})^{(0,l_0)}$.

Thus B^* is in fact a minimal set of bihomogeneous generators for I_W; we just need to justify $((I_{Y'_{j-1}})^{(0,l_0)})_i \subset R'_1((I_{Y'_j})^{(0,l_0)})$ for $i > l_0$. Let $u \in ((I_{Y'_{j-1}})^{(0,l_0)})_i$. Then $u \in I_{Y'_{j-1}}$, so u vanishes on each component of Y'_{j-1} to order at least 1 more than is needed to be in $I_{Y'_j}$, but taking partials drops the order of vanishing at most 1, so (for each $1 \leq t \leq n$) $\partial u / \partial x_t$ has sufficient order of vanishing to be in $I_{Y'_j}$. Moreover, $u \in ((I_{Y'_{j-1}})^{(0,l_0)})_i \subseteq (I_{Z'})_i = (I_{L_0}^{l_0})_i = ((x_1, \ldots, x_n)^{l_0})_i$ for $i > l_0$, so $\partial u / \partial x_t$ vanishes to order at least l_0 on L_0. Thus $\partial u / \partial x_t \in (I_{Y'_j})^{(0,l_0)}$ for $1 \leq t \leq n$, hence by Euler's identity we have $u \in R'_1(I_{Y'_j})^{(0,l_0)}$. □

We now consider the problem of constructing a free resolution of I_W. We start with some preliminary results.

Lemma 4.2. *Let* M*,* N*,* F *and* G *be graded* R'*-modules with* F *a free* R'*-module, and let* $\alpha : F \to M$*,* $\beta : G \to N$ *and* $h : M \to N$ *be graded* R'*-homomorphisms with* $\mathrm{Im}(h \circ \alpha) \subseteq \mathrm{Im}(\beta)$*. Then there exists a graded homomorphism* $h_0 : F \to G$ *such that* $\beta \circ h_0 = h \circ \alpha$*, and hence* $h_0(\ker(\alpha)) \subseteq \ker(\beta)$ *(and so* $h_0((\ker \alpha)_i) \subseteq (\ker \beta)_i$ *for all* $i \geq 0$*).*

Proof. Let S be a minimal set of homogeneous generators for F. For each $f \in S$, $h(\alpha(f))$ is homogeneous and β is a graded map such that $\mathrm{Im}(h \circ \alpha) \subseteq \mathrm{Im}(\beta)$, so there is an element $g_f \in G_i$ where $i = \deg(f)$ such that $\beta(g_f) = h(\alpha(f))$. Setting $h_0(f) = g_f$ for each $f \in S$ gives a graded map $h_0 : F \to G$ such that $\beta \circ h_0 = h \circ \alpha$, and hence $h_0((\ker \alpha)_i) \subseteq (\ker \beta)_i$ for all $i \geq 0$. □

Corollary 4.3. *Let* M *and* N *be graded* R'*-modules and let* $F_\bullet$ *and* $G_\bullet$ *be minimal graded free resolutions of* M *and* N *respectively. Let* $\phi_j : F_j \to F_{j-1}$*,* $j \geq 0$*, be the differentials for* $F_\bullet$ *(where* ϕ_0 *is the augmentation map, so* F_{-1} *signifies* M*), and likewise let* $\gamma_j : G_j \to G_{j-1}$*,* $j \geq 0$*, be the differentials for* $G_\bullet$*, and let* $h_{-1} = h : M \to N$ *be a graded homomorphism. Then there exist graded homomorphisms* $h_j : F_j \to G_j$*,* $j \geq 0$*, compatible with the differentials of the resolutions.*

$$\begin{tikzcd}
G_{i+1,j+1}(-i-1) \arrow[r, "\phi_{i+1,j+1}"] & G_{i+1,j}(-i-1) \arrow[r, "\phi_{i+1,j}"] & G_{i+1,j-1}(-i-1) \\
G_{i,j+1}(-i-1) \arrow[r, "\phi_{i,j+1}"] \arrow[ur, "h_{i,j+1}"] \arrow[dr, "\mu_{i,j+1}"'] & G_{ij}(-i-1) \arrow[r, "\phi_{ij}"] \arrow[ur, "h_{ij}"] \arrow[dr, "\mu_{ij}"'] & G_{i,j-1}(-i-1) \\
G_{i,j+1}(-i) \arrow[r, "\phi_{i,j+1}"] & G_{ij}(-i) \arrow[r, "\phi_{ij}"] & G_{i,j-1}(-i)
\end{tikzcd}$$

FIGURE 1.

Proof. We prove the statement by induction on j. Since $F_\bullet$ and $G_\bullet$ are resolutions, ϕ_0 and γ_0 are surjective, so we have $\mathrm{Im}(h_{-1} \circ \phi_0) \subseteq \mathrm{Im}(\gamma_0)$, hence the case $j = 0$ is immediate from Lemma 4.2, which also gives $h_0(\ker(\phi_0)) \subseteq \ker(\gamma_0)$. By induction we may assume $h_{t-1}(\ker(\phi_{t-1})) \subseteq \ker(\gamma_{t-1})$ for $1 \le t \le j$. Because $F_\bullet$ and $G_\bullet$ are resolutions, we thus have $\mathrm{Im}(h_{j-1} \circ \phi_j) = h_{j-1}(\mathrm{Im}(\phi_j)) = h_{j-1}(\ker(\phi_{j-1})) \subseteq \ker(\gamma_{j-1}) = \mathrm{Im}(\gamma_j)$. By Lemma 4.2 and induction we now have h_j for all $j \ge -1$ as desired, with $h_j(\ker(\phi_j)) \subseteq \ker(\gamma_j)$. $\square$

Now, consider Y', Y'_i, Z, W and X as in section 2 and a minimal free resolution of each $I^{(k,l_0)}_{Y'_i}$ over $R' = K[x_1, \dots, x_n]$:

$$\cdots \to G'_{i,j} \to G'_{i,j-1} \to \cdots \to G'_{i,0} \to (I_{Y'_i})^{(k,l_0)} \to 0,$$

where $G'_{i,j}$ is isomorphic as a graded R'-free module to $\oplus_\ell R'(-\ell)^{\beta_{i,j,\ell}}$ (for an appropriate graded Betti number $\beta_{i,j,\ell}$). We denote the differentials by $\phi'_{i,j}$ and graded generators of the component of $G'_{i,j}$ corresponding to $R'(-\ell)^{\beta_{i,j,\ell}}$ by $\{s_{i,j,l,k}\}$ where k runs over 1 to $\beta_{i,j,\ell}$. From this data we will construct a free graded R-resolution $F_\bullet$ of I_W.

To do so we will need maps between the resolutions of $I^{(k,l_0)}_{Y'_i}$ and $I^{(k,l_0)}_{Y'_{i+1}}$. Let $h'_i : I^{(k,l_0)}_{Y'_i} \hookrightarrow I^{(k,l_0)}_{Y'_{i+1}}$ be the inclusion induced by the inclusion $Y'_{i+1} \subseteq Y'_i$ of closed subschemes, which we also denote by $h'_{i,-1} = h'_i$, where we regard $I^{(k,l_0)}_{Y'_i}$ and $I^{(k,l_0)}_{Y'_{i+1}}$ as $G'_{i,-1}$ and $G'_{i+1,-1}$. Corollary 4.3 applied to the resolution of each $(I_{Y'_i})^{(k,l_0)}$ now gives maps $h'_{ij} : G'_{i,j} \to G'_{i+1,j}$ giving a morphism $h'_{i\bullet} : (G'_i)_\bullet \to (G'_{i+1})_\bullet$ of resolutions. We get free R-modules $G_i = G'_i \otimes_{R'} R$ by tensoring by R and we denote the induced map $h'_i \otimes \mathrm{id}_R : G_i \to G_{i+1}$ by h_i. With the maps $\phi_{i,j} = \phi'_{i,j} \otimes \mathrm{id}_R$ as differentials, $(G_i)_\bullet$ is a minimal graded free R-resolution of $I^{(k,l_0)}_{Y'_i} \otimes_{R'} R$. Finally we define $\mu_{ij} : G_{ij}(-1) \to G_{ij}$, given by multiplication by x_0. Putting these all together we get for all i and j the commutative diagrams shown in Figure 1.

We now can define the graded free R-modules F_j for our resolution $F_\bullet$ of I_W. Recalling that $l' = \max\{l_1, \dots, l_r\}$, define

$$F_0 = \bigoplus_{i=0}^{l'} G_{i,0}(-i)$$

and, for $j > 0$,

$$F_j = G_{l',j}(-l') \oplus \left(\bigoplus_{i=0}^{l'-1} \Big(G_{i,j}(-i) \oplus G_{i,j-1}(-i-1) \Big) \right).$$

The structure of F_0 follows from the fact that I_W is generated by the R'-submodules $x_0^i I^{(k,l_0)}_{Y'_i}$, $0 \le i \le l'$, so $G_{i,0}(-i)$ corresponds to $x_0^i I^{(k,l_0)}_{Y'_i}$. The jth syzygy module F_j being a sum of modules of the form $G_{i,j}(-i) \oplus G_{i,j-1}(-i-1)$ comes from the fact that there are two types of syzygies. The component $G_{i,j}(-i)$ corresponds to jth R'-syzygies of $I^{(k,l_0)}_{Y'_i}$, which naturally carry over to jth

R-syzygies of $x_0^i I_{Y_i'}^{(k,l_0)}$. The component $G_{i,j-1}(-i-1)$ corresponds to jth syzygies between $x_0^i I_{Y_i'}^{(k,l_0)}$ and $x_0^{i+1} I_{Y_{i+1}'}^{(k,l_0)}$. Recall that $I_{Y_i'}^{(k,l_0)} \subseteq I_{Y_{i+1}'}^{(k,l_0)}$, so an element $f \in I_{Y_i'}^{(k,l_0)}$ is also an element of $I_{Y_{i+1}'}^{(k,l_0)}$. Thus, for example, we have a first syzygy $x_0^i f \otimes x_0 - x_0^{i+1} f \otimes 1$ between elements of $x_0^i I_{Y_i'}^{(k,l_0)} \otimes x_0$ and $x_0^{i+1} I_{Y_{i+1}'}^{(k,l_0)} \otimes 1$.

Finally, we define the differential maps $\phi_j : F_j \to F_{j-1}$. Mimicking [FHL], we set

$$\phi_0(s_{i,0,\ell,k} \otimes 1_R) = \phi_{i,0}(s_{i,0,\ell,k} \otimes x_0^i),$$
$$\phi_1(s_{i,1,\ell,k} \otimes 1_R) = \phi_{i,1}(s_{i,1,\ell,k} \otimes 1_R),$$
$$\phi_1(s_{i,0,\ell,k} \otimes 1_R) = s_{i,0,\ell} \otimes x_0 - h_{i,0}(s_{i,0,\ell,k} \otimes 1_R)$$

and, for $j > 1$,

$$\phi_j(s_{i,j,\ell,k} \otimes 1_R) = \phi_{i,j}(s_{i,j,\ell,k} \otimes 1_R),$$

and

$$\phi_j(s_{i,j-1,\ell,k} \otimes 1_R) = s_{i,j-1,\ell,k} \otimes x_0 - h_{i,j-1}(s_{i,j-1,\ell,k} \otimes 1_R) - \phi_{i,j-1}(s_{i,j-1,\ell,k} \otimes 1_R).$$

Proposition 4.4. *The sequence*

$$F_\bullet : \cdots \to F_j \to F_{j-1} \to \cdots \to F_0,$$

with differentials ϕ_j as defined above, is a complex with $\phi_0 \circ \phi_1 = 0$.

Proof. The proof is the same as given for Lemma 2.3 of [FHL]. It follows from commutativity of the diagram in Figure 1 and the fact for each i and j that $\phi'_{i,j-1}\phi'_{i,j} = 0$ (since $(G'_i)_\bullet$ is a resolution, hence a complex). $\square$

Theorem 4.5. *The complex $F_\bullet$ is a resolution of I_W with augmentation map ϕ_0.*

Proof. First of all, since B is in the image of ϕ_0 and since B generates I_W, ϕ_0 is surjective.

Now we have to prove that $\ker(\phi_{j-1}) = \mathrm{Im}(\phi_j)$, for each $j > 0$. By Proposition 4.4, we have $\mathrm{Im}(\phi_j) \subseteq \ker(\phi_{j-1})$. As for the other inclusion, the proof given in Lemma 2.4 of [FHL] works throughout with minor changes. $\square$

Remark 4.6. The result of Theorem 4.5 gives an explicit resolution $F_\bullet$ over R in cases where we have explicit resolutions of the ideals $I_{Y_i'}^{(k,l_0)}$ over R'. In such cases, although the free resolution in Theorem 4.5 is not minimal, we can in principle determine the graded Betti numbers for the minimal resolution from the graded Betti numbers for $F_\bullet$ since we would have explicit matrices for the differential maps, and so could tell how many columns have nonzero scalar entries. This determines how much the graded Betti numbers for $F_\bullet$ exceed those for a minimal resolution.

5. Fat points

In this section we apply the results of the previous sections to the case of fat points with all but one point having support in a hyperplane. In this case, $W = l_0 p_0 + \cdots + l_r p_r$ (whence $s = 0$ and $L_0 = p_0$), with $p_1, \ldots, p_r \in H_0$, for some hyperplane $H_0 \subset \mathbb{P}^n$, but $p_0 \notin H_0$ (note that taking $l_0 = 0$ reduces to the case of points contained in a hyperplane, considered in [FHL]).

We define the tth *truncation* of a graded R'-module $M = \oplus_{\ell \geq 0} M_\ell$ with homogeneous components M_ℓ to be

$$(M)_{\geq t} = \bigoplus_{\ell \geq t} M_\ell.$$

As usual, for simplicity we assume that H_0 is defined by $x_0 = 0$ and that $I_{p_0} = (x_1, \ldots, x_n)$. Then

$$I_{Y_i'}^{(0,l_0)} = I_{Y_i'} \cap (x_1, \ldots, x_n)^{l_0} = \bigoplus_{\ell \geq l_0} (I_{Y_i'})_\ell = (I_{Y_i'})_{\geq l_0}$$

is nothing but the l_0th truncation of $I_{Y'_i}$.

Note for t large enough or for i small enough we get some simplifications. Of course, for $t < l_0$, we have $h((I_{Y'_i})_{\geq l_0}, t) = 0$, but for $t \geq l_0$, we have $h((I_{Y'_i})_{\geq l_0}, t) = h(I_{Y'_i}, t)$. If $i < d$, with d as in Lemma 3.4, then we have $\alpha((I_{Y'_i})_{\geq l_0}) > l_0$, hence also $\alpha(I_{Y'_i}) > l_0$. Thus $(I_{Y'_i})_{\geq l_0} = I_{Y'_i}$ and so $h((I_{Y'_i})_{\geq l_0}, t) = h(I_{Y'_i}, t)$ holds for all $t \geq 0$. We also have $(I_{Y'_{l'}})_{\geq l_0} = (I_{p_0})^{l_0}$.

The results of Theorem 3.1 can be written in terms of the l_0th truncations of the ideals $I_{Y'_i}$ as a direct sum of R'-modules as

$$I_W = ((I_{Y'_0})_{\geq l_0}) \oplus (x_0(I_{Y'_1})_{\geq l_0}) \oplus \cdots \oplus (x_0^{l'-1}(I_{Y'_{l'-1}})_{\geq l_0})) \oplus (x_0^{l'}(I_{Y'_{l'}})_{\geq l_0})) \oplus (x_0^{l'+1}(I_{Y'_{l'}})_{\geq l_0})) \oplus \cdots$$

and so

$$h(I_W, t) = \sum_{i=0}^{t-l_0} h((I_{Y'_i})_{\geq l_0}, t-i).$$

Although our resolutions are not in general minimal, we can in some cases say something about minimal resolutions. For example, in the case above by Lemma 2.3 of [FMN], we have

$$\beta_{i,j,\ell}((I_{Y'_i})_{\geq l_0}) = \beta_{i,j,\ell}(I_{Y'_i}),$$

for each $\ell > l_0 + j$.

For another example, say the points p_i are in $\mathbb{P}^2$, hence p_i for $i > 0$ colinear with p_0 not on that line. Since the codimension is 2, the minimal free resolution of I_W is of the form

$$0 \to F_1 \to F_0 \to I_W \to 0.$$

But by Theorem 4.1 we have a minimal set of homogeneous generators (hence we know the graded Betti numbers for F_0 and so F_0 itself up to graded isomorphism), and as above (i.e., by Theorem 3.1) we know the Hilbert function of I_W. From this we know the graded Betti numbers of F_1 and hence F_1 up to graded isomorphism.

6. Galaxies

As another application of our results we compute galactic Waldschmidt constants. In order to do this we need to prove a lemma. Let $X \subset \mathbb{P}^n$ be a set of c points regarded as a reduced subscheme. It is well known that $\operatorname{reg}(I_X) = \tau + 1$ where τ is the least degree t such that the points impose independent conditions on forms of degree t (i.e., such that $h(I_X, t) = \binom{t+n}{n} - c$).

Lemma 6.1. *Let $H \subset \mathbb{P}^n$ be a hyperplane and let $X \subset \mathbb{P}^n$ be a set of $c+1$ points regarded as a reduced subscheme, with exactly c of the points lying in H. Let $X' = X \cap H$, then* $\operatorname{reg}(I_{X'}) = \operatorname{reg}(I_X)$.

Proof. Choose coordinates such that $I_H = (x_0)$, where $K[\mathbb{P}^n] = K[x_0, \ldots, x_n]$ and so $K[H] = K[x_1, \ldots, x_n]$. Let $\tau' = \operatorname{reg}(I_{X'}) - 1$ and let $\tau = \operatorname{reg}(I_X) - 1$. Thus the points of X' impose independent conditions on forms of degree τ' in $K[H]$, and hence also in $K[x_0, \ldots, x_n]$. Let p be the point of X not in H; up to choice of coordinates we can regard p as being general, hence it imposes an additional independent condition. Thus $\tau \leq \tau'$. On the other hand, it follows from [FrL, Corollary 3.3] and from [DG, Proposition 2.1] that $\tau' \leq \tau$, hence $\tau = \tau'$, so $\operatorname{reg}(I_{X'}) = \operatorname{reg}(I_X)$. □

To state our result let $\mathcal{G} = \mathcal{G}(n, N, e, N)$. Let $G \subset \mathbb{P}^{n+N}$ be the reduced Galactic inclic scheme whose components are the elements of $\mathcal{G}$; i.e., G is the reduced scheme theoretic union of the N points of $\mathcal{G}$ and the $\binom{u}{e}$ e-wise intersections of the associated star configuration $S(n, e, u)$.

In order to compute $\widehat{\alpha}(I_X)$, we will determine $\alpha((I_X)^{(j)})$ for an unbounded sequence of values of j. Our inductive procedure requires information about star configurations as a starting point. The following result is from [BH].

Theorem 6.2. *Let $1 \le e \le n < u$ be integers. Let $A \subset \mathbb{P}^n$ be the reduced scheme theoretic union of the linear varieties comprising the star configuration $S(n,e,u)$. Then for each integer $r > 0$ we have $\alpha(reA) = ru$, $\alpha(I_A) = u - e + 1$ and, if $e = n$, $\operatorname{reg}(I_A) = u - n + 1$.*

Finally we have

Theorem 6.3. *Let G be a reduced galactic inclic scheme as above.*

(a) *Then*

$$\frac{2}{\widehat{\alpha}(I_G)} \le \rho(I_G)$$

and if in addition $e = n$, then

$$\rho(I_G) \le \frac{u-n+1}{\widehat{\alpha}(I_G)}.$$

(b) *If K has characteristic 0, then*

$$\widehat{\alpha}(I_G) = \frac{N(u-e)+u}{N(u-e)+e}.$$

Proof of Theorem 6.3. Let $\mathcal{G} = \mathcal{G}(n, N, e, N; S(n,e,u), \mathcal{H})$ and let the points of the galactic halo $\mathcal{H}$ be $p_1, \ldots, p_N$. Let $G_0 = S(n,e,u) = A \subset \mathbb{P}^n$, $G_1 = A + p_1 \subset \mathbb{P}^{n+1}, \ldots, G = G_N = A + p_1 + \cdots + p_N \subset \mathbb{P}^{n+N}$.

(a) The bounds on $\rho(I_G)$ come from $\alpha(I_G)/\widehat{\alpha}(I_G) \le \rho(I_G)$ and, when $e = n$, $\rho(I_G) \le \operatorname{reg}(I_G)/\widehat{\alpha}(I_G)$ [BH]. Since G spans $\mathbb{P}^{n+N}$, we see $1 < \alpha(I_G)$, but G is contained in the span of $\mathbb{P}^n$ and N points, each of which is contained in a hyperplane in $\mathbb{P}^{n+N}$, so $\alpha(I_G) \le 2$; thus $\alpha(I_G) = 2$. And by Lemma 6.1 with $e = n$, $\operatorname{reg}(I_G) = \operatorname{reg}(I_{G_0})$, but $\operatorname{reg}(I_{G_0}) = u - n + 1$ by Theorem 6.2.

(b) Now define the following sequence: $a_0 = re$, $a_1 = ru$, and for $i \ge 0$, let $a_{i+2} = 2a_{i+1} - a_i$. It's easy to check that $a_i = iru - (i-1)re$. In what comes below, for each i we regard I_{G_i} as an ideal in $K[\mathbb{P}^{n+i}]$. We begin by noting that $a_1 = \alpha(I_{a_0 G_0})$. We will show by induction that $a_{i+1} = \alpha(I_{a_i G_i})$, and hence that $\alpha(I_{a_N G_N}) = (N+1)ru - Nre$, so $\widehat{\alpha}(I_{G_N}) = \lim_{r\to\infty}((N+1)ru - Nre)/(Nru - (N-1)re) = ((N+1)u - Ne)/(Nu - (N-1)e)$, as claimed.

To show that $a_{i+1} = \alpha(I_{a_i G_i})$ we will apply Corollary 3.5. The W of Corollary 3.5 is G_i; $L_0 = p_i$ and the L_j, $j > 0$ are the components of A and the points $p_1, \ldots, p_{i-1}$; H_0 is the linear span of $\mathbb{P}^n$ and $p_1, \ldots, p_{i-1}$; $\mathbb{P}^n$ there is $\mathbb{P}^{n+i}$ here; and $l' = l_j = a_i$ for all j. Moreover, k in the corollary is 0, since L_0 is a point. (Here there are no H_j for $j > 0$. Here $l_0 = a_i$.) The result of the corollary is that $\alpha(a_i G_i) = a_i + d$, where d is the least j such that what is there called Y'_j has $\alpha((I_{Y'_j})^{(0,l_0)}) = l_0$. But $Y'_j = (a_i - j)G_{i-1}$ (as long as $a_i - j \ge 0$), and $(I_{Y'_j})^{(0,l_0)}$ is just the truncation of $I_{Y'_j}$ at degree l_0. Thus the least j such that $\alpha((I_{Y'_j})^{(0,l_0)}) = l_0$ is the least j such that $\alpha((a_i - j)G_{i-1}) \le a_i$. But $\alpha((a_i - j)G_{i-1}) = a_i$ for $j = a_i - a_{i-1}$ by induction, and $\alpha((a_{i-1})G_{i-1}) < \alpha((a_i - j)G_{i-1})$ for $j < a_i - a_{i-1}$ by Lemma 3.4, so $a_i - a_{i-1}$ is the least j. Thus $\alpha(I_{a_i G_i}) = a_i + (a_i - a_{i-1}) = a_{i+1}$, as claimed. □

In the case that $N = 1$, $e = n$ and $u = n+1$, we can, up to choice of coordinates, regard G as the coordinate vertices in $\mathbb{P}^{n+1}$. In this case, $u = n+1$ so the ideal I_G can be chosen to be a monomial ideal and, in characteristic 0, we recover the known facts that $\widehat{\alpha}(I_G) = \frac{n+2}{n+1}$ and $\rho(I_G) = \frac{2(n+1)}{n+2}$. We note that the bound $\frac{2}{\widehat{\alpha}(I_G)} \le \rho(I_G)$ is always better than the bound $1 \le \rho(I_G)$, and the bound $\rho(I_G) \le \frac{u-n+1}{\widehat{\alpha}(I_G)}$ is often better than the bound $\rho(I_G) \le n + N$ (for example, if the characteristic is 0 and $1 < n \le u \le 2n + N$, then we have $\frac{u-n+1}{\widehat{\alpha}(I_G)} < n + N$).

References

[BH] C. Bocci and B. Harbourne. *Comparing powers and symbolic power of ideals*, J. Algebraic Geom. 19 (2010) 399–417.

[BH2] C. Bocci and B. Harbourne. *The resurgence of ideals of points and the containment problem*, Proc. Amer. Math. Soc. 138 (2010) 1175–1190.

[CoCoA] J. Abbott and A.M. Bigatti and G. Lagorio. *CoCoA-5: a system for doing Computations in Commutative Algebra*, Available at http://cocoa.dima.unige.it.

[Ch] G. V. Chudnovsky. *Singular points on complex hypersurfaces and multidimensional Schwarz Lemma*, Séminaire de Théorie des Nombres, Paris 1979–80, Séminaire Delange-Pisot-Poitou, Progress in Math vol. 12, M-J Bertin, editor, Birkhäuser, Boston-Basel-Stutgart (1981).

[CHT] S. Cooper, B. Harbourne and Z. Teitler. *Combinatorial bounds on Hilbert functions of fat points in projective space*, J. Pure Appl. Algebra 215:9 (2011) 2165–2179.

[D] H. Derksen. *Hilbert series of subspace arrangements*, J. Pure Appl. Algebra 209:1 (2007) 91–98.

[DG] E. Davis, A. V. Geramita. *The Hilbert function of a special class of* 1*-dimensional Cohen-Macaulay graded algebras*, The Curves Seminar at Queen's, Vol. X, Queen's Papers in pure and Applied Mathematics, Vol.67 (1984), 1H-29H.

[DS] H. Derksen and J. Sidman. *A sharp bound for the Castelnuovo-Mumford regularity of subspace arrangements*, Adv. Math. 172 (2002), no. 2, 151–157.

[DHST] M. Dumnicki, B. Harbourne, T. Szemberg and H. Tutaj-Gasińska. *Linear subspaces, symbolic powers and Nagata type conjectures*, Advances Math. 252 (2014) 471–491.

[ELS] L. Ein, R. Lazarsfeld, and K.E. Smith. *Uniform behavior of symbolic powers of ideals*, Invent. Math. 144 (2001), 241–252.

[EV] Hd. Esnault and E. Viehweg. *Sur une minoration du degré d'hypersurfaces s'annulant en certains points*, Math. Ann. 263 (1983), no. 1, 75–86.

[Fa] G. Fatabbi. *On the resolution of ideals of fat points*, J. Algebra 242 (2001), 92–108.

[FHL] G. Fatabbi, B. Harbourne and A. Lorenzini. *Resolutions of ideals of fat points with support in a hyperplane*, Proc. Amer. Math. Soc. 134 (2006) 3475–3483.

[FaL] G. Fatabbi and A. Lorenzini. *On the graded resolution of ideals of a few general fat points of* $\mathbb{P}^n$, J. Pure Appl. Algebra 198 (2005) 123–150.

[FrL] S. Franceschini and A. Lorenzini. *Fat points of* P^n *whose support is contained in a linear proper subspace*, J. Pure Appl. Algebra 160 (2001), no. 2–3, 169–182.

[Fr] C. A. Francisco. *Resolution of small sets of fat points*, J. Pure Appl. Algebra 203 (2005) 220–236.

[FMN] C. A. Francisco, J.C. Migliore and U. Nagel. *On the componentwise linearity and the minimal free resolution of a tetrahedral curve*, J. Algebra 299 (2006) 535–569.

[GHM] A. Geramita, B. Harbourne and J. Migliore. *Star configurations in* $\mathbb{P}^n$, J. Algebra 376 (2013) 279–299.

[GMS] A. V. Geramita, J. Migliore and L. Sabourin. *On the first infinitesimal neighborhood of a linear configuration of points in* $\mathbb{P}^2$, J. Algebra 298 (2006) 563–611.

[GO] A. V. Geramita and F. Orecchia. *On the Cohen-Macaulay type of* s *lines in* $\mathbf{A}^{n+1}$, J. Algebra 70 (1981) 116–140.

[GHV1] E. Guardo, B. Harbourne and A. Van Tuyl. *Asymptotic resurgences for ideals of positive dimensional subschemes of projective space*, Advances Math. 246 (2013) 114–127.

[GHV2] E. Guardo, B. Harbourne and A. Van Tuyl. *Fat lines in* $\mathbb{P}^3$*: powers versus symbolic powers*, Journal Alg. 390 (2013) 221–230.

[HaHu] B. Harbourne and C. Huneke. *Are symbolic powers highly evolved?*, J. Ramanujan Math. Soc. 28, No.3 (Special Issue-2013) 311–330.

[HH] R. Hartshorne and A. Hirschowitz. *Droites en position générale dans l'espace projectif*, volume 961 of Lecture Notes in Mathematics (1982), 169–188.

[HS] A. Hirschowitz and C. Simpson. *Minimal resolution of the ideal of a general arrangement of a large number of points in* $\mathbb{P}^n$, Invent. Math. 126 (1996), 467–503.

[HoHu] M. Hochster and C. Huneke. *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. 147:2 (2002) 349–369.

[La] R. Lazarsfeld. *Positivity in Algebraic Geometry I.-II.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vols. 48-49, Springer Verlag, Berlin, 2004.

[L] A. Lorenzini. *The minimal resolution conjecture*, J. Algebra 156 (1993) 5–35.

[M] S. Mayes. *The asymptotic behaviour of symbolic generic initial systems of points in general position*, J. Pure Appl. Algebra 218:3 (2014) 381–390.

[S] J.M. Steele. *Probability theory and combinatorial optimization*, CBMS-NSF Regional Conference Series in Applied Mathematics, **69**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1997).

[V] G. Valla. *Betti numbers of some monomial ideals*, Proc. Amer. Math. Soc. 133 (2005) 57–63.

[W] M. Waldschmidt. *Propriétés arithmétiques de fonctions de plusieurs variables II*, In Séminaire P. Lelong (Analyse), 1975–76, Lecture Notes Math. 578, Springer-Verlag, 1977, 108–135.

Dip. di Matematica e Informatica, Università di Perugia, via Vanvitelli 1, 06123 Perugia, Italy
E-mail address: giuliana.fatabbi@unipg.it
E-mail address: anna.lorenzini@unipg.it

Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130 USA
E-mail address: brianharbourne@unl.edu