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Nonlocal diffusion second order partial differential equations

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Abstract

The paper deals with a second order integro-partial differential equation in $\mathbb{R}^n$ with a nonlocal, degenerate diffusion term. Nonlocal conditions, such as the Cauchy multipoint and the weighted mean value problem, are investigated. The existence of periodic solutions is also studied. The dynamic is transformed into an abstract setting and the results comes from an approximation solvability method. It combines a Schauder degree argument with an Hartman-type inequality and it involves a Scorza-Dragoni type result. The compact embedding of a suitable Sobolev space in the corresponding Lebesgue space is the unique amount of compactness which is needed in this discussion. The solutions are located in bounded sets and they are limits of functions with values in finitely dimensional spaces.

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1 Introduction

This paper deals with the partial integro-differential equation

$$u_{tt} = cu_t + bu(t,\xi) + u(t,\xi)\int_\Omega k(\xi,\eta)u(t,\eta)d\eta + h(t,u(t,\xi)) \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is an open, bounded domain with C^1-boundary, b and c are given constants, $h\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $k \in W^{2,1}(\overline{\Omega} \times \overline{\Omega}; \mathbb{R})$.
If the integral term is replaced by the usual Laplace operator Δu, then according to the values of c and b, (1.1) becomes the damped wave equation or the telegraph equation or the Klein-Gordon equation and it is a model for many phenomena. For example, it governs the propagation of electro-magnetic waves in an electrically conducting medium, the motion of a string or a membrane with external damping, the evolution of visco-elastic fluids influenced by Maxwell theory and the heat propagation in a thermally conducting medium (see e.g. [15, 18] and the references therein).
The classical diffusion equation implies an infinite velocity propagation of information; namely, a change in temperature or concentration in some point of the domain is instantaneously felt everywhere. In recent years, to circumvent this drawback, many authors have proposed alternatives to describe heat and mass transfer. A continuous diffusion coefficient which vanishes when $u = 0$ is assumed in [12], to obtain a degenerate process with finite speed of propagation. Alternatively, a fractional diffusion term is considered in [9], instead of the standard Laplace operator.
Diffusion operators such as the integral contained in (1.1) introduce a memory effect in the equation and are able to capture long distance interactions into the process that occur in a number of applications; hence they are frequently preferable than the classical punctual diffusions such as Laplace operator. As a consequence, several investigations recently appeared, for first order dynamics, which include an integral diffusion term (see e.g. [14], [16] and references therein). As far as we know, this is the first paper investigating a nonlocal diffusion second order equation.
The presence of u as a non-constant diffusion coefficient (similarly as in [20]) means that the diffusion has a degenerate nature as in [12]. This is an expected behavior in several contexts. This term allows a super-linear growth of the right hand side of the equation, also often appearing in many applications.
The main aim of this paper is to start a theory on some important classes of solutions of (1.1). We assume the following conditions

(1) the partial derivative $\frac{\partial h}{\partial z}\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and there is a positive constant N, such that

$$\left| \frac{\partial h(t,z)}{\partial z} \right| \leq N \text{ for all } (t,z) \in [0,T] \times \mathbb{R}.$$

(2) $\max\{\sup_{(\xi,\eta)\in\Omega\times\Omega} |k(\xi,\eta)|, \sup_{(\xi,\eta)\in\Omega\times\Omega} \|Dk(\xi,\eta)\|_{\mathbb{R}^n}\} = K < \infty$,
where the symbol D stands for the derivative (i.e. the gradient) with respect to the variables in the vector $\xi \in \Omega$.

(3) $b \geq N + \sqrt{6\delta K|\Omega|}$ where $|\Omega|$ denotes the Lebesgue measure of Ω and $\delta = \max\limits_{t\in[0,T]} |h(t,0)|$.

Example 1 *The generalized mean curvature function* $h(t,z) = \frac{z^\beta}{(1+z^2)^\alpha}$ *verifies (1) when* $\beta \geq 1, \alpha > \frac{\beta-1}{2}$. *In this case* $\delta = 0$ *in (3).*

A classical kernel describing dispersion is the Laplace kernel $k(\xi,\eta) = \frac{1}{2d}e^{-\|\xi-\eta\|}d$, *for a given constant* d *(see [24]). It verifies (2) for any* $\Omega \subset \mathbb{R}^n$.

First we study the associated periodic problem

$$\begin{cases} u_{tt} = cu_t + bu(t,\xi) + u(t,\xi)\displaystyle\int_\Omega k(\xi,\eta)u(t,\eta)d\eta + h(t,u(t,\xi)) \\ u(0,\xi) = u(T,\xi) \\ u_t(0,\xi) = u_t(T,\xi). \end{cases} \tag{1.2}$$

The existence of double periodic solutions is a classical result for the telegraph or Klein-Gordon equation. On the contrary, just few results are known for periodic solutions having a periodic derivative with respect to t (see [1, 10, 17, 25, 26]).
Then we consider nonlocal boundary conditions such as the Cauchy multipoint problem

$$\begin{cases} u_{tt} = cu_t + bu(t,\xi) + u(t,\xi)\displaystyle\int_\Omega k(\xi,\eta)u(t,\eta)d\eta + h(t,u(t,\xi)) \\ u(0,\xi) = \displaystyle\sum_{i=1}^k \alpha_i u(t_i,\xi) \\ u(T,\xi) = \displaystyle\sum_{i=1}^k \beta_i u(t_i,\xi) \end{cases} \tag{1.3}$$

with $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \ldots, k$, where $0 \le t_1 < \cdots < t_k \le T$; and the mean value problem,

$$\begin{cases} u_{tt} = cu_t + bu(t,\xi) + u(t,\xi)\displaystyle\int_\Omega k(\xi,\eta)u(t,\eta)d\eta + h(t,u(t,\xi)) \\ u(0,\xi) = \dfrac{1}{T}\displaystyle\int_0^T p_1(t)u(t,\xi)\,dt \\ u(T,\xi) = \dfrac{1}{T}\displaystyle\int_0^T p_2(t)u(t,\xi)\,dt \end{cases} \tag{1.4}$$

with $p_1, p_2 \in L^1([0,T],\mathbb{R})$.
In the last two cases we also need, respectively, that:

(4) $\sum_{i=1}^k |\alpha_i| \le 1$ and $\sum_{i=1}^k |\beta_i| \le 1$.

(5) $\dfrac{\|p_1\|_{L^1([0,T],\mathbb{R})}}{T} \le 1$ and $\dfrac{\|p_2\|_{L^1([0,T],\mathbb{R})}}{T} \le 1$.

Notice that the homogeneous Dirichlet condition in the variable t, i.e. $u(0,\xi) = u(T,\xi) = 0$ is included in our discussion since it is, for instance, a special case of problem (1.3) when $\alpha_i = \beta_i = 0$ for all $i = 1,..,k$. In addition, by exchanging the role of t and x, for $\Omega = [0,B]$ and $c = 0$, the Dirichlet condition in x (see, e.g., [23]) can be studied as well, for a suitable equation. Finally we remark that, in all previous problems (1.2), (1.3) and (1.4), we can further assume that u satisfies the Dirichlet condition in x, i.e. that $u/\partial\Omega = 0$.
Here are our main results:

Theorem 2 *Under the assumptions (1)-(3) the problem* (1.2) *has at least a solution.*

Theorem 3 *Under the assumptions (1)-(4) the problem* (1.3) *has at least a solution.*

Theorem 4 *Under the assumptions (1)-(3) and (5) the problem* (1.4) *has at least a solution.*

We assume that $u(t,\cdot)$ belongs to the Sobolev space $H := W^{1,2}(\Omega)$ for all $t \in [0,T]$ and we reformulate equation (1.1) in Section 2 in this abstract context (see (2.1)). Since H is a separable Hilbert space, we can introduce an orthonormal basis (see Section 3 for its definition) and hence a sequence of finitely dimensional subspaces H_n.
Now we briefly account of the techniques for attaching, in Section 4.1, the abstract periodic problem (4.1). We first introduce, in (4.6), a sequence of approximating problems (P_n) with values in the finite dimensional space H_n. A suitable combination of Schauder degree arguments and Hartman-type inequalities allows to solve each (P_n). A Scorza-Dragoni type result (see Theorem 9) is involved, in this part, for getting the global continuity of the r.h.s. in the equation in (4.6). A limiting argument then leads to a solution of (4.6). The compact embedding of $W^{1,2}(\Omega)$ in the Banach space $L^2(\Omega)$ is the unique amount of compactness which is needed in this discussion. A similar technique, in Section 4.2, is introduced for solving the abstract nonlocal problem (4.14) which is the base for the study of (1.3) and (1.4). In conclusion, the main idea developed in this part is an approximation solvability method based on compact embedded Gel'fand triples with a Hilbert space and on Hartman-type conditions. As a consequence, the solutions are limits of functions with values in finite-dimensional subspaces. They are also localized in suitable bounded sets in the sense that $\|u(t,\cdot)\|_H \leq R$ for some constant R which does not depend on t. The proof of Theorems 1.2, 1.3 and 1.4 is in Section 5. When further assuming, in (1.2), (1.3) and (1.4), that $u/\partial\Omega = 0$, then we need to replace $W^{1,2}(\Omega)$ with its closed subspace $W_0^{1,2}(\Omega)$. Section 3 contains notation and some preliminary results.

Some non-local problems, treated with different techniques, recently appeared in [5], [6], [7] and [19]. They are all associated to first order dynamics; the first, in particular, is based on an approximation technique similar to the one used in this paper. To the best of our knowledge, this is the first application of this technique to the study of second order dynamics.

2 Abstract formulation of the problem

In this section we transform the partial integro-differential equation (1.1) into an abstract ordinary differential equation in a suitable infinite dimensional framework. By means of this reformulation we will prove in Section 5 the existence of a solution, $u : [0,T]\times\Omega \to \mathbb{R}$, of (1.1), which is twice differentiable with respect to t, with an absolutely continuous derivative and a second derivative belonging to $L^1([0,T], W^{1,2}(\Omega,\mathbb{R}))$, such that at every value $t \in [0,T]$ the function $u(t,\cdot)$ belongs to the Sobolev space $W^{1,2}(\Omega,\mathbb{R})$.
The approximation solvability method we are going to show requires the introduction of an Hilbert space compactly embedded in a Banach space.
To this aim, let $H = W^{1,2}(\Omega,\mathbb{R})$ and $E = L^2(\Omega,\mathbb{R})$. It is well known that H is a separable Hilbert space which is compactly embedded in E. Denoting with $\|\cdot\|_2$ the norm in L^2, for every $w \in H$ we put

$$\|w\|_H = \sqrt{\int_\Omega \left(w^2(\xi) + \|Dw(\xi)\|^2_{\mathbb{R}^n}\right)\, d\xi} = \left(\|w\|_2^2 + \|Dw\|_2^2\right)^{\frac{1}{2}}.$$

Considering for each $t \in [0,T]$, the map $x : [0,T] \to H$ defined as $x(t) = u(t,\cdot)$, we can substitute (1.1) with the following problem

$$x''(t) = F(t, x(t), x'(t)), \text{ for a.e. } t \in [0,T], \tag{2.1}$$

where $F\colon [0,T]\times H\times H\to H,\ F(t,w,v)=cv+bw+\overline{g}(w)+\overline{h}(t,w)$ with

$$\begin{array}{llcllcl} \overline{g}\colon & H & \to & H, & \overline{g}(w)(\xi) & = & w(\xi)\displaystyle\int_\Omega k(\xi,\eta)w(\eta)\,d\eta, \\ \overline{h}\colon & [0,T]\times H & \to & H, & \overline{h}(t,w)(\xi) & = & h(t,w(\xi)). \end{array}$$

The map $F\colon [0,T]\times H\times H\to H$ is well defined. Indeed for $w\in H$

$$\|\overline{g}(w)\|_H^2=\|\overline{g}(w)\|_2^2+\|D\overline{g}(w)\|_2^2.$$

By condition (2) we have that

$$\|\overline{g}(w)\|_2^2=\int_\Omega \left|w(\xi)\left(\int_\Omega k(\xi,\eta)w(\eta)\,d\eta\right)\right|^2 d\xi\le K|\Omega|\|w\|_2^4.$$

Moreover,

$$D\overline{g}(w)(\xi)=Dw(\xi)\left(\int_\Omega k(\xi,\eta)w(\eta)\,d\eta\right)+w(\xi)\left(\int_\Omega Dk(\xi,\eta)w(\eta)\,d\eta\right),\ \text{for a.a.}\ \xi\in\Omega.$$

Hence, again by (2) it follows that $\overline{g}(w)\in H$.
To prove that the map $\overline{h}$ is well defined as well, first of all, notice that, if we denote with $h_2'(t,z)=\frac{\partial h(t,z)}{\partial z}$, from (1) it follows that

$$|h(t,z)|=|h(t,0)+h_2'(t,\eta)\,z|\le|h(t,0)|+N|z|, \tag{2.2}$$

for all $(t,z)\in[0,T]\times\mathbb{R}$, where η is a number between 0 and z. Moreover,

$$Dh(t,w(\xi))=h_2'(t,w(\xi))Dw(\xi)\ \text{for a.a.}\ \xi\in\Omega.$$

Thus, by (1), $\overline{h}(t,w)\in H$ for any $t\in[0,T]$ and $w\in H$.

3 Solution technique

We study problem (2.1) localizing the solutions as a by product of the approximation method. Precisely, we consider the abstract problem (2.1) in a separable Hilbert space $(H,\|\cdot\|)$.
Throughout the paper, I represents the real interval $[0,T]$. Given $A\subset H$, let $\overline{A}^H$ be the closure of A with respect to the norm of H, while B_H^R denotes the open ball $B^R=\{w\in H\colon \|w\|<R\}$. By H^ω and $\langle\cdot,\cdot\rangle$ we denote respectively the space H endowed with the weak topology and the inner product in H.
By $C^0(I,H), C^1(I,H)$ and $L^1(I,H)$ we mean respectively the Banach space of all continuous functions $x\colon I\to H$ with norm

$$\|x\|_0=\max_{t\in I}\|x(t)\|,$$

the Banach space of all functions $x\colon I\to H$ with norm

$$\|x\|_{C^1}=\max\{\|x\|_0,\|x'\|_0\}$$

and the Banach space of summable functions with norm

$$\|x\|_1=\int_0^T\|x(t)\|\,dt.$$

It is well known that $\{x_j\} \to x_0$ in $C^1(I,H)$ when $j \to \infty$ implies the point-wise convergence of $\{x_j\}$ to x_0 and of $\{x_j'\}$ to x_0'.
We recall the following characterization of weak convergence in the space of continuous functions.

Theorem 5 *(see [8]) A sequence of continuous functions $\{x_n\} \rightharpoonup x \in C(I;H)$ if and only if*

1. *there exists $N > 0$ such that, for every $n \in \mathbb{N}$ and $t \in I, \|x_n(t)\| \leq N$;*
2. *for every $t \in I, x_n(t) \rightharpoonup x(t)$.*

It follows that $\{x_n\} \rightharpoonup x \in C(I;H)$ implies that $\{x_n\} \rightharpoonup x \in L^1(I;H)$.
Let $S \subseteq \mathbb{R}$ be a measurable subset. A subset $A \subset L^1(S,H)$ is called uniformly integrable if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\Omega \subset S$ and $\mu(\Omega) < \delta$ implies

$$\left\| \int_\Omega f \, d\mu \right\| < \varepsilon \quad \text{for all } f \in A,$$

where μ is the Lebesgue measure on $\mathbb{R}$.
Consider the space of all functions $x : I \to H$ differentiable with an absolutely continuous derivative x' and a second derivative x'' belonging to $L^1(I,H)$. It is well known (see e.g. [4]) that this space can be identified with the Sobolev space $W^{2,1}(I,H)$ and the embedding $W^{2,1}(I,H) \hookrightarrow C^1(I,H)$ is continuous.
We look for classical solutions $x \in W^{2,1}(I,H)$, i.e. functions that satisfies the equation (2.1) for a.a. $t \in I$ and we obtain them by means of an approximation solvability method. To this aim we consider an orthonormal basis $\{e_i\}$ of H: denoting with H_n the subspace with basis $\{e_1, \cdots, e_n\}$, we approximate the original problem by a family of auxiliary problems by means of the natural projections $P_n \colon H \to H_n$ $(n \in \mathbb{N})$ (see (4.6) in Section 4). Then, by a limit argument, we obtain the existence of a solution for the original problem.
Some of the main properties of the projection P_n are contained in the following.

Lemma 6 *The projections $P_n : H \to H_n$ satisfies the following properties:*

(a) $P_n : H^\omega \to H_n$ is continuous;

(b) if $g_n \rightharpoonup g$ in H then $P_n g_n \rightharpoonup g$ in H.

Proof.
(a) We have that $P_n w = \sum_{k=1}^n \langle e_k, w \rangle e_k$ for every $w \in H$, thus by the definition of weak convergence $P_n w_j \to P_n w_0$ for $w_j \rightharpoonup w_0$.
(b) For every $w \in H$, due to $P_n w \to w$ in H, we have

$$\begin{aligned} \langle P_n g_n - g, w \rangle &= \langle P_n g_n - P_n g, w \rangle + \langle P_n g - g, w \rangle = \langle g_n - g, P_n w \rangle + \langle P_n g - g, w \rangle \\ &= \langle g_n - g, P_n w - w \rangle + \langle g_n - g, w \rangle + \langle P_n g - g, w \rangle \to 0 \end{aligned}$$

and we get the thesis. ■
We solve the family of approximating auxiliary problems using a continuation principle in a Banach space F. Namely,

Theorem 7 *(see e.g. [2]) Let Q be a closed, convex subset of a Banach space F with nonempty interior and $T \colon Q \times [0,1] \to F$ be a compact map having a closed graph such that $\mathcal{T}(Q,0) \subset \operatorname{int} Q$ and $\mathcal{T}(\cdot,\lambda)$ is fixed point free on the boundary of Q for all $\lambda \in [0,1)$. Then there exists $y \in F$ such that $y = \mathcal{T}(y,1)$.*

The continuation principle consists into associating to the problem to be solved a one-parameter family of linearized problems. One f the conditions for its application requires in particular that no linearized problem has solutions tangent to the boundary of a given non-empty closed bounded set. We apply the mentioned continuation principle in the Banach space $C^1(I, H_n)$. It involves a suitable set $Q_n \subseteq C^1(I, H_n)$ of candidate solutions of the approximating finite dimensional problem. We ensure that the candidate solutions are non tangent to the boundary of Q_n by means of Hartman-type conditions (see (4.3) below) and the following result based on Nagumo conditions.

Theorem 8 *(see [21, Lemma 2.1]) Let $\psi : [0, +\infty) \to [0, +\infty)$ be a continuous and non decreasing function, with*

$$\lim_{s\to\infty} \frac{s^2}{\psi(s)} ds = \infty, \tag{3.1}$$

and R be a positive constant. Then there exists a positive constant B such that if $x \in W^{2,1}(I, H)$ is such that $||x''(t)|| \leq \psi(||x'(t)||)$ for a.a. $t \in I$ and $||x(t)|| \leq R$ for every $t \in I$, it holds $||x'(t)|| \leq B$ for every $t \in I$.

In [21] the result is given for $x \in C^2(I, H)$. It is easy to prove (see, e.g., [3]) that the statement holds also for $x \in W^{2,1}(I, H)$.
For our purposes it is sufficient to assume strictly localized Hartman-type conditions. This restriction requires an approximation argument based on a Scorza-Dragoni type result. The following theorem is a special case of [13, Proposition 7.11].

Theorem 9 *Let $g : I \times \mathbb{R}^p \to \mathbb{R}^n$ be such that*

i) $g(\cdot, w)$ is measurable for every $w \in \mathbb{R}^p$;

ii) $g(t, \cdot)$ is continuous for a.e. $t \in I$.

Then there exists a decreasing sequence of open subsets $\{\theta_m\}$ of I such that for every $m \in \mathbb{N}, \mu(\theta_m) < \frac{1}{m}$ and g is continuous in $(I \backslash \theta_m) \times \mathbb{R}^p$.

Finally, notice that the assumption of the compactness of the operator T in Theorem 7 is quite a difficult assumption to check in infinite dimensional Banach spaces. We overcome this difficulty considering throughout the paper the Hilbert space H compactly embedded in a Banach space $(E, \|\cdot\|_E)$ with the relation of norms:

$$\|w\|_E \leq q\|w\| \text{ for all } w \in H, \tag{3.2}$$

for some $q > 0$.

4 Abstract existence results

To obtain existence results of solutions of equation (2.1) we consider at first an abstract equation associated to a periodic condition and to a nonlocal boundary condition.

4.1 Periodic boundary conditions

In this subsection we study the abstract periodic problem:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), \text{ for a.e. } t \in [0, T], \\ x(0) = x(T), \\ x'(0) = x'(T) \end{cases} \tag{4.1}$$

under the following assumptions:

$(F1)$ for every $x, y \in H$ the function $f(\cdot, x, y) : I \to H$ is measurable;

$(F2)$ for a.e. $t \in I$ the function $f(t, \cdot, \cdot) : H \times H \to H$ is continuous from $(H \times H)$ to H^ω;

$(F3)$ for a.e. $t \in I$ the function $f(t, \cdot, \cdot) : H \times H \to H$ is continuous in the topology of the space E;

$(F4)$ there exist a positive constant R and a function $\beta : [0, +\infty) \to [0, +\infty)$ continuous and non decreasing satisfying (3.1) such that

$$\|f(t, x, y)\| \le \beta(\|y\|) \text{ for a.e. } t \in I, \text{ every } x, y \in H \text{ with } \|x\| \le R. \tag{4.2}$$

By classical solutions of problem (4.1), we mean functions $x \in W^{2,1}(I, H)$ such that $x(0) = x(T), x'(0) = x'(T)$ and

$$x''(t) = f(t, x(t), x'(t)) \text{ for a.e. } t \in I.$$

Theorem 10 *Let conditions $(F1) - (F4)$ hold. In addition, assume that for a.e. $t \in I$, for all $x, y \in H$ with $\|x\| = R$ and $\langle x, y \rangle = 0$ it holds*

$$\langle x, f(t, x, y) \rangle + \|y\|^2 \ge 0. \tag{4.3}$$

Then problem (4.1) admits a solution with values in $\overline{B^R_H}$.

The proof of Theorem 10 is based on the following results of existence and characterization of the solution for finite dimensional linear problems.

Lemma 11 *Given a function $g \in L^1([0, T], \mathbb{R}^n \times \mathbb{R}^n)$ and the matrix $A = \begin{pmatrix} 0 & \mathcal{I} \\ \mathcal{I} & 0 \end{pmatrix}$, where $\mathcal{I}$ denotes the identity in the space $\mathbb{R}^n$, the problem*

$$\begin{cases} z'(t) + Az(t) = g(t) \textit{ for a.e. } t \in [0, T] \\ z(0) = z(T), \end{cases} \tag{4.4}$$

has a unique solution given by

$$z(t) = e^{At}(Id - e^{AT})^{-1} \int_0^T e^{A(T-s)} g(s) ds + \int_0^t e^{A(t-s)} g(s) ds, \tag{4.5}$$

where e^{At} denotes the semigroup generated by A and Id denotes the identity in the space $\mathbb{R}^n \times \mathbb{R}^n$.

Proof. By the definition of the matrix A, the associated homogeneous problem has only the trivial solution, then it follows that the linear operator $Id - e^{AT}$ is invertible and the claimed result follows easily. ■

The next result trivially follows.

Lemma 12 *Given a function $f \in L^1([0,T],\mathbb{R}^n)$, a function $x \in W^{2,1}([0,T];\mathbb{R}^n)$ is a solution of the second order problem*

$$\begin{cases} x''(t) - x(t) = f(t) \text{ for a.e. } t \in [0,T] \\ x(0) = x(T), \\ x'(0) = x'(T). \end{cases}$$

if and only if the vector valued function $z : [0,T] \to \mathbb{R}^n \times \mathbb{R}^n$, $z(t) = (x(t), x'(t))$ is a solution of the periodic first order problem (4.4) *with $g(t) = (0, f(t))$.*

Proof of Theorem 10.
First of all, let us consider the map $\psi : [0,+\infty) \to [0,+\infty)$ defined as $\psi(s) = 2\beta(s)+R+1$, where β and R are the map and the positive constant defined in condition (F4). Since β is a continuous and non decreasing function, the function ψ is continuous and non decreasing as well and there exists $\lim_{s\to\infty}\beta(s) = l \in (0,\infty]$. Therefore, by (F4)

$$\lim_{s\to\infty}\frac{s^2}{\psi(s)} = \lim_{s\to\infty}\frac{s^2}{\beta(s)(2+\frac{R+1}{\beta(s)})} = +\infty,$$

both when l is finite or infinite. Hence by Theorem 8 there exists a constant $B > 0$ such that for every $x \in W^{2,1}(I,H)$ with $||x''(t)|| \le \psi(||x'(t)||)$ for a.a. $t \in I$ and $||x(t)|| \le R$ for every $t \in I$, it holds $||x'(t)|| \le B$ for every $t \in I$.

From now on, the proof splits into several steps.

Step 1. *Introduction of a sequence of problems in a finite dimensional space.*
For every $n \in \mathbb{N}$, let us consider the problem

$$\begin{cases} x''(t) = P_n f(t, x(t), x'(t)), \text{ for a.e. } t \in I, \\ x(0) = x(T), \\ x'(0) = x'(T). \end{cases} \tag{4.6}$$

We shall prove that problem (4.6) has a solution in the closed, convex set with nonempty interior

$$Q_n = \{q \in C^1(I,H_n) : \|q(t)\| \le R, \|q'(t)\| \le 2B \text{ for every } t \in I\}, \tag{4.7}$$

where $R > 0$ and $B > 0$ are defined as above.
Fix $\epsilon \in (0,R)$. According to Urisohn lemma, there exists a continuous function $\mu : H \to [0,1]$ such that $\mu \equiv 0$ on $\{w \in H : \|w\| \le R-\epsilon \text{ or } \|w\| \ge R+\epsilon\}$ and $\mu \equiv 1$ on $\{w \in H : R - \frac{\epsilon}{2} \le \|w\| \le R + \frac{\epsilon}{2}\}$. It follows that $\phi : H \to H$ defined by

$$\phi(w) = \begin{cases} \mu(w)\frac{w}{\|w\|} & R-\epsilon \le \|w\| \le R+\epsilon \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

is well-defined, continuous, bounded on all H and $\|\phi(w)\| \le 1$ for every $w \in H$.
Since, by Lemma 6, $P_n : H^\omega \to H_n$ is continuous and f satisfies (F1)-(F4), it is easy to prove that $P_n f/_{H_n\times H_n} : I \times H_n \times H_n \to H_n$ satisfies (i)-(ii) of Theorem 9. Thus there

exists a decreasing sequence of open subset $\{\theta_m\}$ of I such that $\mu(\theta_m) < \frac{1}{m}$ and $P_n f$ is continuous on $(I\backslash\theta_m) \times H^n \times H^n$ for every $m \in \mathbb{N}$. Now, by means of the sequence $\{\theta_m\}$ we can construct a family of approximating problems of (4.6).

STEP 2. *Solvability of a sequence of traslated problems.*
For every $m \in \mathbb{N}$, let us consider the problem

$$\begin{cases} x''(t) - x(t) = -x(t) + P_n f(t, x(t), x'(t)) + \phi(x(t))(\beta(\|x'(t)\|)\chi_{\theta_m}(t) + \frac{1}{m}), \\ \hfill \text{for a.e. } t \in I, \\ x(0) = x(T), \\ x'(0) = x'(T). \end{cases} \tag{4.9}$$

Fix $q \in Q_n$ and $\lambda \in [0,1]$, by Lemma 12 the linear periodic problem

$$\begin{cases} x''(t) - x(t) = -\lambda q(t) + \lambda P_n f(t, q(t), q'(t)) + \phi(q(t))(\beta(\|q'(t)\|)\chi_{\theta_m}(t) + \frac{1}{m}), \\ \hfill \text{for a.e. } t \in I, \\ x(0) = x(T) \\ x'(0) = x'(T) \end{cases} \tag{4.10}$$

has a unique solution such that $z(t) = (x(t), x'(t))$, $t \in I$ verifies (4.5) with $g : I \to \mathbb{R}^n \times \mathbb{R}^n$ defined as

$$g(t) = \left(0, -\lambda q(t) + \lambda P_n f(t, q(t), q'(t)) + \phi(q(t)) \left(\beta(\|q'(t)\|)\chi_{\theta_m}(t) + \frac{1}{m}\right)\right). \tag{4.11}$$

We can then define the map $\mathcal{T}_n^m \colon Q_n \times [0,1] \to C^1(I, H_n)$ that associate to $(q, \lambda) \in Q_n \times [0,1]$ the solution $\mathcal{T}_n^m(q, \lambda)$ of (4.10). It is clear that x is a solution of (4.9) if and only if $x = \mathcal{T}_n^m(x, 1)$. We then apply Theorem 7 to prove the existence of fix points of $\mathcal{T}_n^m(\cdot, 1)$.
(a) We show that the multimap $\mathcal{T}_n^m$ has a closed graph in the space $Q_n \times [0,1] \times C^1(I, H_n)$. Assume that $(q_j, \lambda_j, x_j) \to (q_0, \lambda_0, x_0) \in Q_n \times [0,1] \times C^1(I, H_n)$, where $x_j = \mathcal{T}_n^m(q_j, \lambda_j)$, in particular, taking the limit as $j \to \infty$, we get $x_0(0) = x_0(T)$ and $x'(0) = x'(T)$. From (F2) and the continuity of ϕ, β and P_n it follows that, for a.e. $t \in I$, the sequence

$$x_j(t) - \lambda_j q_j(t) + \lambda_j P_n f(t, q_j(t), q_j'(t)) + \phi(q_j(t)) \left(\beta(\|q_j'(t)\|)\chi_{\theta_m}(t) + \frac{1}{m}\right)$$

converges to

$$x_0(t) - \lambda_0 q_0(t) + \lambda_0 P_n f(t, q_0(t), q_0'(t)) + \phi(q_0(t)) \left(\beta(\|q_0'(t)\|)\chi_{\theta_m}(t) + \frac{1}{m}\right).$$

Moreover, since convergent sequences are bounded, there exists a positive constant D such that $\|x_j(t)\| \leq D$ for every $t \in [0,T]$. Thus, according to the definition of Q_n and to (F4) we obtain that

$$\begin{aligned} &||x_j(t) - \lambda_j q_j(t) + \lambda P_n f(t, q_j(t), q_j'(t)) + \phi(q_j(t))(\beta(\|q_j'(t)\|)\chi_{\theta_m}(t) + \tfrac{1}{m})|| \leq \\ &D + R + 2\beta(2B) + 1. \end{aligned} \tag{4.12}$$

Therefore, Lebesgue's dominated convergence Theorem implies that the sequence

$$\begin{aligned} x_j'(t) = \; & x_j'(0) + \int_0^t [x_j(s) - \lambda_j q_j(s)] \, ds + \\ & \lambda_j \int_0^t \left[P_n f(s, q_j(s), q_j'(s)) + \phi(q_j(s)) \left(\beta(\|q_j'(s)\|)\chi_{\theta_m}(s) + \frac{1}{m}\right)\right] ds \end{aligned}$$

converges to

$$x_0'(0)+\int_0^t \left[x_0(s)-\lambda_0 q_0(s)+\lambda_0 P_n f(s,q_0(s),q_0'(s))+\phi(q_0(s))\left(\beta(\|q_0'(s)\|)\chi_{\theta_m}(s)+\frac{1}{m}\right)\right]ds$$

for every $t\in I$. Thus, the uniqueness of the limit yields, for every $t\in I$,

$$x_0'(t)=x_0'(0)+\int_0^t \left[x_0(s)-\lambda q_0(s)+\lambda_0 P_n f(s,q_0(s),q_0'(s))+\phi(q_0(s))\left(\beta\|q_0'(s)\|)\chi_{\theta_m}(s)+\frac{1}{m}\right)\right]ds,$$

i.e. that $x_0=\mathcal{T}_n^m(\lambda_0,q_0)$. We have thus proved the closure of the graph.

(b) We prove that $\mathcal{T}_n^m$ is a compact map, i.e. that $\mathcal{T}_n^m(Q_n\times[0,1])$ is relatively compact. Again by Lemma 12 for any $q\in Q_n$ and $\lambda\in[0,1]$ the unique solution $x=\mathcal{T}_n^m(q,\lambda)$ of (4.10) is such that $z(t)=(x(t),x'(t))$ satisfies (4.5) with g defined in (4.11). Thus, according to (F4), since $||A||=1$ we get that, for a.e. $t\in I$,

$$\begin{aligned}&\max\{||x(t)||,||x'(t)||\}\\&\le ||z(t)||\le [e^T[||(Id-e^{AT})^{-1}||)+1]e^T\left(RT+\beta(2B)T+\beta(2B)+T\right).\end{aligned}\tag{4.13}$$

Hence $\{x' : x\in\mathcal{T}_n^m\big(Q_n\times[0,1]\big)\}$ and $\mathcal{T}_n^m\big(Q_n\times[0,1]\big)$ are bounded in $C(I,H_n)$. From (4.10) it follows that the set $\{x'' : x\in\mathcal{T}_n^m\big(Q_n\times[0,1]\big)\}$ is bounded in $L^1(I,H_n)$, thus $\{x' : x\in\mathcal{T}_n^m\big(Q_n\times[0,1]\big)\}$ is equicontinuous. Moreover $\mathcal{T}_n^m\big(Q_n\times[0,1]\big)$ is equicontinuous too. By Ascoli-Arzelá Theorem we get the conclusion.

(c) We show that $T_n^m(Q_n,0)\subset int\ Q_n$.

Consider $q\in Q_n$ and let $x=\mathcal{T}_n^m(q,0)$. Then x is a solution of the periodic problem

$$\begin{cases}x''(t)-x(t)=\phi(q(t))(\beta(\|q'(t)\|)\chi_{\theta_m}(t)+\frac{1}{m}), \text{ for a.e. } t\in I,\\ x(0)=x(T)\\ x'(0)=x'(T).\end{cases}$$

Similarly as in (b), we can prove that, for a.e. $t\in I$,

$$\max\{||x(t)||,||x'(t)||\}\le||z(t)||\le[e^T[||(Id-e^{AT})^{-1}||)+1]e^T\left(\beta(2B)\frac{1}{m}+\frac{T}{m}\right).$$

Thus, we obtain that $\mathcal{T}_n^m(Q_n\times\{0\})\subset$ int Q_n for every $m\ge\overline{m}$, with $\overline{m}$ sufficiently large.

(d) We prove that $\mathcal{T}_n^m(\cdot,\lambda)$ is fixed point free on the boundary of Q_n for all $\lambda\in[0,1)$. Since we already showed that $\mathcal{T}_n^m(\cdot,0)$ has no fixed points on ∂Q_n, it remains to prove this property for $\mathcal{T}_n^m(\cdot,\lambda)$ with $\lambda\in(0,1)$.

We reason by contradiction assuming the existence of $(q,\lambda)\in\partial Q_n\times(0,1)$ such that $q=\mathcal{T}_n^m(q,\lambda)$. Then there exists $t_0\in[0,T]$ such that $\|q(t_0)\|=R$ or $\|q'(t_0)\|=2B$. Since for a.a. $t\in I$ and every $\lambda\in[0,1]$ we have

$$\begin{aligned}\|q''(t)\| &= \left\|(1-\lambda)q(t)+\lambda P_n f(t,q(t),q'(t))+\phi(q(t))\left(\beta(\|q'(t)\|)\chi_{\theta_m}(t)+\frac{1}{m}\right)\right\|\\ &\le R+2\beta(\|q'(t)\|)+1=\psi(\|q'(t)\|)\end{aligned}$$

with $\|q(t)\|\le R$ for every $t\in I$ and ψ satisfying (3.1), Theorem 8 implies that $\|q'(t)\|\le B$ for every $t\in I$. Hence it must hold $\|q(t_0)\|=R$. Consider now the function $v:[0,T]\to\mathbb{R}$ defined as $v(t)=\frac{1}{2}(\|q(t)\|^2-R^2)$. Then clearly $v\in C^1(I,\mathbb{R})$ and it has a local maximum point in t_0. If $t_0\in(0,T)$, then $v'(t_0)=0$. If $t_0\notin(0,T)$ since q satisfies the boundary conditions, then both 0 and T are local maximum points for v. Hence $v'(0) \le 0$ and $v'(T) \ge 0$. Since $v'(t) = \langle q(t), q'(t)\rangle$ for every t, we then get

$$0 \ge v'(0) = \langle q(0), q'(0)\rangle = \langle q(T), q'(T)\rangle = v'(T) \ge 0$$

i.e. $v'(0) = v'(T) = 0$. Therefore without loss of generality, we may assume $t_0 \in (0, T]$ and $\langle q(t_0), q'(t_0)\rangle = 0$. Moreover there exists $h > 0$ such that $v'(t_0 - h) \ge 0$ and $\|q(s)\| \ge R - \frac{\epsilon}{2}$ for every $s \in [t_0 - h, t_0]$. Therefore, for a.e. $t \in [t_0 - h, t_0]$, there exists

$$v''(t) = (1-\lambda)\|q(t)\|^2 + \|q'(t)\|^2 + \lambda\langle q(t), P_n f(t, q(t), q'(t))\rangle + \left(\chi_{\theta_m}(t)\beta(\|q'(t)\|) + \frac{1}{m}\right)\|q(t)\|.$$

Consequently,

$$\begin{aligned}
0 \ge\ & v'(t_0) - v'(t_0 - h) \\
=\ & \int_{t_0-h}^{t_0} [(1-\lambda)\|q(s)\|^2 + \|q'(s)\|^2 + \lambda\langle q(s), P_n f(s, q(s), q'(s))\rangle + \\
& (\chi_{\theta_m}(s)\beta(\|q'(s)\|) + \tfrac{1}{m})\|q(s)\|]ds \\
=\ & \int_{[t_0-h,t_0]\cap\theta_m} [(1-\lambda)\|q(s)\|^2 + \|q'(s)\|^2 + \lambda\langle q(s), P_n f(s, q(s), q'(s))\rangle + \\
& (\beta(\|q'(s)\|) + \tfrac{1}{m})\|q(s)\|]ds + \\
& \int_{[t_0-h,t_0]\setminus\theta_m} [(1-\lambda)\|q(s)\|^2 + \|q'(s)\|^2 + \lambda\langle q(s), P_n f(s, q(s), q'(s))\rangle + \tfrac{1}{m}\|q(s)\|]ds \\
\ge\ & \int_{[t_0-h,t_0]\cap\theta_m} [\|q(s)\|(-\|f(s, q(s), q'(s))\| + \beta(\|q'(s)\|)) + \tfrac{1}{m}\|q(s)\|]ds \\
& + \int_{[t_0-h,t_0]\setminus\theta_m} [\|q'(s)\|^2 + \lambda\langle q(s), P_n f(s, q(s), q'(s))\rangle + \tfrac{1}{m}\|q(s)\|]ds.
\end{aligned}$$

From condition (F4) we get

$$\begin{aligned}
0\ \ \ge\ & \int_{[t_0-h,t_0]\cap\theta_m} [\|q(s)\|(-\|f(s, q(s), q'(s))\| + \beta(\|q'(s)\|)) + \tfrac{1}{m}\|q(s)\|]ds \\
& + \int_{[t_0-h,t_0]\setminus\theta_m} [\|q'(s)\|^2 + \lambda\langle q(s), P_n f(s, q(s), q'(s))\rangle + \tfrac{1}{m}\|q(s)\|]ds \\
\ge\ & \tfrac{1}{m}\int_{[t_0-h,t_0]\cap\theta_m} \|q(s)\|ds \\
& + \int_{[t_0-h,t_0]\setminus\theta_m} [\|q'(s)\|^2 + \lambda\langle q(s), P_n f(s, q(s), q'(s))\rangle + \tfrac{1}{m}\|q(s)\|]ds \\
>\ & \int_{[t_0-h,t_0]\setminus\theta_m} [\|q'(s)\|^2 + \lambda\langle q(s), P_n f(s, q(s), q'(s))\rangle + \tfrac{1}{m}\|q(s)\|]ds.
\end{aligned}$$

Clearly, $t_0 \in \theta_m$ implies, for h sufficiently small, $[t_0 - h, t_0] \setminus \theta_m = \emptyset$ thus a contradiction. On the other hand, if $t_0 \notin \theta_m$, we consider the function $g : [0, T] \to \mathbb{R}$ defined as:

$$g(s) = \|q'(s)\|^2 + \lambda\langle q(s), P_n f(s, q(s), q'(s))\rangle + \frac{1}{m}\|q(s)\|.$$

Notice that the function $\ell : [0, 1] \to \mathbb{R}$

$$\ell(\lambda) = \lambda\langle q(t_0), f(t_0, q(t_0), q'(t_0))\rangle + \|q'(t_0)\|^2$$

is monotone on $[0,1]$, thus (4.3), $q \in Q_n$, $\|q(t_0)\| = R$ and $\langle q(t_0), q'(t_0)\rangle = 0$ imply that $g(t_0) \geq \frac{R}{m}$. Since, according to Theorem 9, g is continuous on $[0,T] \setminus \theta_m$, for h sufficiently small we have $g(t) > 0$ for every $t \in [t_0 - h, t_0] \setminus \theta_m$ and we get again a contradiction. Hence, Theorem 7 for any $m \in \mathbb{N}$ guarantees the existence of $x_m \in W^{2,1}(I, H_n) \cap Q_n$ solution of problem (4.9).

STEP 3. *Solvability of the sequence of finite dimensional problems.*
For any $m \in \mathbb{N}$ let x_m be the solution of problem (4.9) obtained in Step 2. Reasoning as in (b), inequality (4.13) implies that the sequence $\{x_m\}$ has a subsequence, still denoted as the sequence, that converges to $x \in Q_n$ in $C^1(I, H_n)$ and $x''_m \rightharpoonup x''$ in $L^1(I, H_n)$. Notice, moreover, that since ϕ is bounded, β is continuous, $\lim_{m\to\infty} \chi_{\theta_m}(t) = 0$ for every $t \notin \cap_{m=1}^{\infty}\theta_m$, and $\mu\left(\cap_{m=1}^{\infty}\theta_m\right) = 0$, it follows that

$$\phi(x_m(t))(\beta(x'_m(t))\chi_{\theta_m}(t) + 1/m) \to 0, \text{ for a.a. } t \in [0,T].$$

Consequently, a standard limit argument implies that, for every $n \in \mathbb{N}$, there exists $x_n \in Q_n$ solution of (4.6).

STEP 4. *Solvability of problem* (4.1).
For any $n \in \mathbb{N}$ let x_n be the solution of the problem (4.6) obtained in Step 3. According to (F4), we get that for a.e. $t \in I$ and every $n \in \mathbb{N}$, $\|f(t, x_n(t), x'_n(t))\| \leq \beta(2B)$. Thus, denoting by $f_n(t) = f(t, x_n(t), x'_n(t))$, we get that the sequence $\{f_n\}$ is bounded and uniformly integrable in $L^1(I,H)$ and the set $\{f_n(t)\}$ is bounded for a.e. $t \in I$. Hence by the reflexivity of the space H, the sequence $\{f_n\}$ is relatively weakly compact in $L^1(I,H)$ (see [11]). W.l.o.g. assume that $f_n \rightharpoonup f_0 \in L^1(I,H)$. Lemma 6 implies then that

$$x''_n = P_n f_n \rightharpoonup f_0 \in L^1(I,H).$$

Since $\|x'_n(t)\| \leq 2B$ for every $t \in I$, w.l.o.g. we can assume that

$$x'_n(0) \rightharpoonup \gamma_0 \in H.$$

Define $y_0(t) := \gamma_0 + \int_0^t f_0(s)\, ds,\ t \in I$. It is easy to see that

$$x'_n(t) = x'_n(0) + \int_0^t x''_n(s)\, ds \rightharpoonup y_0(t)$$

for all $t \in I$. Therefore, from Theorem 5, $x'_n \rightharpoonup y_0$ in $C(I,H)$. Consequently, $x'_n(0) = x'_n(T) \rightharpoonup y_0(T)$ i.e. $y_0(0) = y_0(T)$. Moreover y_0 is differentiable and $y'_0(t) = f_0(t)$ for a.a. $t \in I$. Now, notice that the weak convergence of x'_n to y_0 in $C(I,H)$ implies the weak convergence of x'_n in $L^1(I,H)$, and that $\|x_n(t)\| \leq R$ for every $t \in I$ and $n \in \mathbb{N}$. Hence, denoting by

$$x_0(t) = x_0(0) + \int_0^t y_0(s) ds,$$

for some suitable $x_0(0) \in H$, analogously to the above reasoning, we can prove that $x_n \rightharpoonup x_0$ in $C(I,H)$ with $x_0(0) = x_0(T)$. Therefore $x_0 \in W^{2,1}(I,H), x'_0(t) = y_0(t)$ for every $t \in I$ and $x''_0(t) = f_0(t)$ for a.a. $t \in I$.
By the Mazur Lemma, there exists a sequence of convex combinations $\overline{h}_n = \sum_{k=n}^{\infty} \theta_{nk} f_k$ which converges to x''_0 in $L^1(I,H)$, with $\theta_{nk} \geq 0, \sum_{k=n}^{\infty} \theta_{nk} = 1$ and for every n there exists $k_0(n)$ such that $\theta_{nk} = 0$ for $k > k_0(n)$. The convergence of $\overline{h}_n$ to x''_0 in $L^1(I,H)$ implies its convergence almost everywhere, i.e. $\overline{f}_n(t) \to x''_0(t)$ in H for a.a. $t \in I$, thus by the compact embedding (3.2) we obtain $\overline{f}_n(t) \to x''_0(t)$ in E for a.a. $t \in I$. Moreover, according to

Theorem 5 we have $x_n(t) \to x_0(t)$ and $x_n'(t) \to x_0'(t)$ in E for every $t \in I$ and from (F3) we get $f(t, x_n(t), x_n'(t)) \to f(t, x_0(t), x_0'(t))$ in E. Hence $\overline{f}_n(t) \to f(t, x_0(t), x_0'(t))$ for a.a. $t \in I$. The uniqueness of the limit implies $x_0''(t) = f(t, x_0(t), x_0'(t))$ a.e. in I and the theorem is proved. ■

4.2 Nonlocal boundary conditions

To study problems (1.3) and (1.4) we consider the following abstract problem:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), \text{ for a.e. } t \in [0, T], \\ x(0) = Mx, \\ x(T) = Lx \end{cases} \tag{4.14}$$

under the conditions (F1)-(F4) of Section 4.1 and we assume that:

(M) $M : C(I, H) \to H$ is a linear bounded operator with $\|M\| \leq 1$;

(L) $L : C(I, H) \to H$ is a linear bounded operator with $\|L\| \leq 1$.

Again by an approximation solvability method we find classical solutions of problem (4.14), i.e. functions $x \in W^{2,1}(I, H)$ such that $x(0) = Mx, x(T) = Lx$ and

$$x''(t) = f(t, x(t), x'(t)) \text{ for a.e. } t \in I.$$

Theorem 13 *Let conditions* $(F1) - (F4), (M), (L)$ *and* (4.3) *hold. Then problem* (4.14) *admits a solution with values in* $\overline{B_H^R}$.

Analogously to Lemma 11 it is easy to prove the following result useful in the proof of Theorem 13.

Lemma 14 *Given a function* $f \in L^1(I, \mathbb{R}^n)$ *and two vectors* $x_0, x_1 \in \mathbb{R}^n$ *the second order Dirichlet problem*

$$\begin{cases} x''(t) = f(t) \text{ for a.e. } t \in [0, T] \\ x(0) = x_0, \\ x(T) = x_1. \end{cases}$$

has a unique solution given by

$$x(t) = \left(1 - \frac{t}{T}\right) x_0 + \frac{t}{T} x_1 - \frac{t}{T} \int_0^T (T - r) f(r) dr + \int_0^t (t - r) f(r) dr. \tag{4.15}$$

Proof of Theorem 13. The proof is very similar to the one of Theorem 10, hence we enphasize only the differences. First of all, we will prove the existence for every $n \in \mathbb{N}$ of at least a solution of the following sequence of finite dimensional problems

$$\begin{cases} x''(t) = P_n f(t, x(t), x'(t)), \text{ for a.e. } t \in I, \\ x(0) = P_n Mx, \\ x(T) = P_n Lx. \end{cases} \tag{4.16}$$

in the set Q_n defined in (4.7), where the constant R and B are obtained exactly as in the proof of Theorem 10. To this aim we introduce the sequence of translated problems for $m \in \mathbb{N}$,

$$\begin{cases} x''(t) = P_n f(t, x(t), x'(t)) + \phi(x(t))(\beta(\|x'(t)\|)\chi_{\theta_m}(t) + \frac{1}{m}), \text{ for a.e. } t \in I, \\ x(0) = P_n Mx, \\ x(T) = P_n Lx, \end{cases} \tag{4.17}$$

where ϕ is defined in (4.8), the sequence $\{\theta_m\}$ is from Theorem 9 and β is the map in condition (F4).
By Lemma 14 for any $q \in Q_n$ and $\lambda \in [0,1]$ the linear Dirichlet problem

$$\begin{cases} x''(t) = \lambda P_n f(t, q(t), q'(t)) + \phi(q(t))(\beta(\|q'(t)\|)\chi_{\theta_m}(t) + \frac{1}{m}), \text{ for a.e. } t \in I, \\ x(0) = \lambda P_n Mq \\ x(T) = \lambda P_n Lq \end{cases} \tag{4.18}$$

has a unique solution given by

$$\begin{aligned} x(t) = \; & (1 - \tfrac{t}{T})\lambda P_n Mq + \tfrac{t}{T}\lambda P_n Lq \\ & - \frac{t}{T}\int_0^T (T-r)\left[\lambda P_n f(r, q(r), q'(r)) + \phi(q(r))\left(\beta(\|q'(r)\|)\chi_{\theta_m}(r) + \frac{1}{m}\right)\right] dr \\ + & \int_0^t (t-r)\left[\lambda P_n f(r, q(r), q'(r)) + \phi(q(r))\left(\beta(\|q'(r)\|)\chi_{\theta_m}(r) + \frac{1}{m}\right)\right] dr. \end{aligned}$$

Thus, denoting it by $\mathcal{T}_n^m(q, \lambda) \in W^{2,1}(I, H_n)$, we can define the map $\mathcal{T}_n^m \colon Q_n \times [0,1] \to C^1(I, H_n)$. It is clear that x is a solution of (4.18) if and only if $x \in \mathcal{T}_n^m(x, 1)$. We shall apply Theorem 7 to prove the solvability of (4.18).
(a) We show that the multimap $\mathcal{T}_n^m$ has closed graph in the space $Q_n \times [0,1] \times C^1(I, H_n)$. Assume that $(q_j, \lambda_j, x_j) \to (q_0, \lambda_0, x_0) \in Q_n \times [0,1] \times C^1(I, H_n)$, where $x_j = \mathcal{T}_n^m(q_j, \lambda_j)$. Then $x_j(0) = \lambda_j P_n Mq_j$ and $x_j(T) = \lambda_j P_n Lq_j$ and passing to the limit when $j \to \infty$ we obtain $x_0(0) = \lambda_0 P_n Mq_0$ and $x_0(T) = \lambda_0 P_n Lq_0$, for the continuity of P_n, M and L.
From (F2) and the continuity of ϕ, β and P_n it follows that, for a.e. $t \in I$, the sequence

$$\lambda_j P_n f(t, q_j(t), q_j'(t)) + \phi(q_j(t))\left(\beta(\|q_j'(t)\|)\chi_{\theta_m}(t) + \frac{1}{m}\right)$$

converges to

$$\lambda_0 P_n f(t, q_0(t), q_0'(t)) + \phi(q_0(t))\left(\beta(\|q_0'(t)\|)\chi_{\theta_m}(t) + \frac{1}{m}\right).$$

According to the definition of Q_n and (F4) we obtain that for every $j \in \mathbb{N}$

$$\left\|\lambda_j P_n f(t, q_j(t), q_j'(t)) + \phi(q_j(t))\left(\beta(\|q_j'(t)\|)\chi_{\theta_m}(t) + \frac{1}{m}\right)\right\| \leq 2\beta(2B) + 1. \tag{4.19}$$

Therefore, Lebesgue's dominated convergence Theorem implies that the sequence $\{x_j(t)\}$ converges to

$$\begin{aligned} & \gamma_0(t) = (1 - \tfrac{t}{T})\lambda_0 P_n Mq_0 + \tfrac{t}{T}\lambda_0 P_n Lq_0 \\ & - \frac{t}{T}\int_0^T (T-r)\left[\lambda P_n f(r, q_0(r), q_0'(r)) + \phi(q_0(r))\left(\beta(\|q_0'(r)\|)\chi_{\theta_m}(r) + \frac{1}{m}\right)\right] dr \\ & + \int_0^t (t-r)\left[\lambda P_n f(r, q_0(r), q_0'(r)) + \phi(q_0(r))\left(\beta(\|q_0'(r)\|)\chi_{\theta_m}(r) + \frac{1}{m}\right)\right] dr \end{aligned}$$

for every $t \in I$. Thus, the uniqueness of the limit yields, for every $t \in I$, $x_0(t) = \gamma_0(t)$, i.e. that $x_0 \in \mathcal{T}_n^m(\lambda_0, q_0)$. We have thus proved the closure of the graph.

(b) Now, we show that $\mathcal{T}_n^m$ is a compact map, i.e. that $\mathcal{T}_n^m(Q_n \times [0,1])$ is relatively compact. From (4.15), we get that

$$\begin{aligned} x'(t) = \; & -\tfrac{1}{T}\lambda P_n M q + \tfrac{1}{T}\lambda P_n L q \\ & -\frac{1}{T}\int_0^T (T-r)\left[\lambda P_n f(r, q(r), q'(r)) + \phi(q(r))\left(\beta(\|q'(r)\|)\chi_{\theta_m}(r) + \frac{1}{m}\right)\right] dr \\ & + \int_0^t (t-r)\left[\lambda P_n f(r, q(r), q'(r)) + \phi(q(r))\left(\beta(\|q'(r)\|)\chi_{\theta_m}(r) + \frac{1}{m}\right)\right] dr. \end{aligned}$$

Thus, from (4.19) it follows that the set $\{x'' : x \in \mathcal{T}_n^m(Q_n \times [0,1])\}$ is bounded in $L^1(I, H_n)$ and

$$||x'(t)|| \le \frac{2}{T}R + (2\beta(2B) + 1)T(T+1),$$

implying that also $\{x' : x \in \mathcal{T}_n^m(Q_n \times [0,1])\}$ is bounded in $C(I, H_n)$. Similarly we can prove that $\mathcal{T}_n^m(Q_n \times [0,1])$ is bounded in $C(I, H_n)$ and the thesis follows from the Ascoli-Arzelá Theorem as in (b) of Theorem 10.

(c) We show that $\mathcal{T}_n^m(Q_n, 0) \subset int\ Q_n$.

Consider $q \in Q_n$ and consider $x = \mathcal{T}_n^m(q, 0)$. Then x is a solution of the Cauchy problem

$$\begin{cases} x''(t) = \phi(q(t))(\beta(\|q'(t)\|)\chi_{\theta_m}(t) + \frac{1}{m}), \text{ for a.e. } t \in I, \\ x(0) = 0 \\ x(T) = 0. \end{cases}$$

Then, according to (4.15), we get that

$$\begin{aligned} x(t) = \; & -\frac{t}{T}\int_0^T (T-r)\phi(q(r))\left(\beta(\|q'(r)\|)\chi_{\theta_m}(r) + \frac{1}{m}\right) dr \\ & + \int_0^t (t-s)\phi(q(r))\left(\beta(\|q'(r)\|)\chi_{\theta_m}(r) + \frac{1}{m}\right) dr, \end{aligned}$$

thus (F4) implies for a.e. $t \in I, ||x(t)|| \le 2\left(\beta(2B)\frac{T}{m} + \frac{T^2}{m}\right)$. We obtain that $\mathcal{T}_n^m(Q_n \times \{0\}) \subset Q_n$ for m sufficiently big.

(d) We prove that $\mathcal{T}_n^m(\cdot, \lambda)$ has no fixed points on ∂Q_n for every $\lambda \in (0,1)$.

We reason by contradiction assuming the existence of $(q, \lambda) \in \partial Q_n \times (0,1)$ such that $q \in \mathcal{T}_n^m(q, \lambda)$. Then there exists $t_0 \in [0, T]$ such that $\|q(t_0)\| = R$ or $\|q'(t_0)\| = 2B$. Similarly as in Theorem 10 we can prove that $\|q'(t)\| \le B$ for every $t \in I$. Hence it must hold $\|q(t_0)\| = R$. If $t_0 = 0$, then, by condition (M), $R = \|q(0)\| \le \lambda\|P_n\|\|M\|\|q\|_0 < R$, a contradiction. If $t_0 = T$, we get the same contradiction by (L), hence $t_0 \in (0, T)$ and from now on the proof follows as in (d) of Theorem 10.

Thus, Theorem 7 ensures for any $m \in \mathbb{N}$ the existence of $x_m \in W^{2,1}(I, H_n) \cap Q_n$ solution of problem (4.18). Again as in the proof of Theorem 10 a standard limiting argument implies that, for every $n \in \mathbb{N}$, there exists $x_n \in Q_n$ solution of (4.16). Moreover, it is possible to prove that $f_n \rightharpoonup f_0 \in L^1(I, H)$ with $f_n(t) = f(t, x_n(t), x_n'(t))$, $x_n' \rightharpoonup y_0$ in $C(I, H)$ with $y_0(t) := \gamma_0 + \int_0^t f_0(s)\, ds$, $t \in I$, and $x_n \rightharpoonup x_0$ in $C(I, H)$ with $x_0(t) = \delta_0 + \int_0^t y_0(s) ds$. Therefore $x_0 \in W^{2,1}(I, H), x_0'(t) = y_0(t)$ for every $t \in I$ and $x_0''(t) = f_0(t)$ for a.a. $t \in I$. Moreover (M) implies $x_n(0) = Mx_n \rightharpoonup Mx_0$, and $x_n(0) \rightharpoonup x_0(0)$ yields $x_0(0) = Mx_0$. Similarly (L) implies $x_0(T) = Lx_0$. Finally, the conclusion of the proof is exactly as the one of Theorem 10. ■

5 Proofs of Theorems 2, 3 and 4

We write problem (1.2) in an abstract form as

$$\begin{cases} x''(t) = F(t, x(t), x'(t)), \text{ for a.e. } t \in [0, T], \\ x(0) = x(T), \\ x'(0) = x'(T) \end{cases} \tag{5.1}$$

and problems (1.3) and (1.4) as

$$\begin{cases} x''(t) = F(t, x(t), x'(t)), \text{ for a.e. } t \in [0, T], \\ x(0) = Mx, \\ x(T) = Lx \end{cases} \tag{5.2}$$

where the map $F : [0, T] \times H \times H \to H$ is defined in Section 2 and the operators $M : C(I, H) \to H$ and $L : C(I, H) \to H$ are defined respectively by

$$Mx = \sum_{i=1}^{k} \alpha_i x(t_i), \quad Lx = \sum_{i=1}^{k} \beta_i x(t_i),$$

with $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $0 < t_1 < \cdots < t_n \leq T$ for (1.3) and

$$Mx = \frac{1}{T}\int_0^T p_1(t)x(t)\, dt, \quad Lx = \frac{1}{T}\int_0^T p_2(t)x(t)\, dt,$$

with $p_1, p_2 \in L^1(I, \mathbb{R})$ for (1.4).
Easily assumptions (4) and (5) imply conditions (M) and (L) respectively. In this section we will prove that the map F satisfies the conditions (F1)-(F4) and (4.3). Thus, applying Theorems 10 and 13 we obtain the existence of a solution respectively of (5.1) and of (5.2). Hence, as a consequence, of (1.2), (1.3) and (1.4).
We start proving condition (F3).
Let $w_n \overset{E}{\to} w_0$. We have

$$\begin{aligned} &\|\overline{g}(w_n) - \overline{g}(w_0)\|_2^2 = \int_\Omega \left| w_n(\xi) \left(\int_\Omega k(\xi,\eta) w_n(\eta)\, d\eta \right) - w_0(\xi) \left(\int_\Omega k(\xi,\eta) w_0(\eta)\, d\eta \right) \right|^2 d\xi \\ &\leq 2 \int_\Omega |w_n(\xi)|^2 \left(\int_\Omega |k(\xi,\eta)|\, |(w_n(\eta) - w_0(\eta))|\, d\eta \right)^2 d\xi \\ &+ 2 \int_\Omega |w_n(\xi) - w_0(\xi)|^2 \left(\int_\Omega |k(\xi,\eta)|\, |w_0(\eta)|\, d\eta \right)^2 d\xi \\ &\leq 2K^2 |\Omega| \|w_n - w_0\|_2^2 (\|w_n\|_2^2 + \|w_0\|_2^2). \end{aligned}$$

Hence $\overline{g}$ is $E - E$ continuous from the boundedness of $\{w_n\}$. Moreover we have

$$\begin{aligned} \|\overline{h}(t, w_n) - \overline{h}(t, w_0)\|_2^2 = & \int_\Omega |h(t, w_n(\xi)) - h(t, w_0(\xi))|^2\, d\xi \\ = & \int_\Omega |h_2'(t, \eta_n(\xi))|^2 \cdot |(w_n(\xi) - w_0(\xi))|^2\, d\xi \\ \leq & \; N^2 \|w_n - w_0\|_2^2, \end{aligned}$$

where $\eta_n(\xi)$ is a number between $w_n(\xi)$ and $w_0(\xi)$. Hence $\overline{h}(t, w_n) \stackrel{E}{\to} \overline{h}(t, w_0)$ and then $\overline{h}(t, \cdot)$ is $E - E$ continuous.
So $(w, v) \longmapsto F(t, w, v)$ is continuous from $E \times E$ into E, for each $t \in [0, T]$ and condition (F3) is satisfied.

Now we prove condition (F2).
Let $w_n \stackrel{H}{\to} w_0$. We have that

$$\begin{aligned}
&\int_\Omega \left\| w_n(\xi) \left(\int_\Omega Dk(\xi,\eta) w_n(\eta)\, d\eta \right) + Dw_n(\xi) \left(\int_\Omega k(\xi,\eta) w_n(\eta)\, d\eta \right) \right.\\
&\left. - Dw_0(\xi) \left(\int_\Omega k(\xi,\eta) w_0(\eta)\, d\eta \right) - w_0(\xi) \left(\int_\Omega Dk(\xi,\eta) w_0(\eta)\, d\eta \right) \right\|^2_{\mathbb{R}^n} d\xi \\
&\leq 4 \int_\Omega |w_n(\xi) - w_0(\xi)|^2 \left(\int_\Omega \|Dk(\xi,\eta)\|_{\mathbb{R}^n} |w_n(\eta)|\, d\eta \right)^2 d\xi \\
&+4 \int_\Omega |w_0(\xi)|^2 \left(\int_\Omega \|Dk(\xi,\eta)\|_{\mathbb{R}^n} |w_n(\eta) - w_0(\eta)|\, d\eta \right)^2 d\xi \\
&+4 \int_\Omega \|Dw_n(\xi) - Dw_0(\xi)\|^2_{\mathbb{R}^n} \left(\int_\Omega |k(\xi,\eta)| |w_n(\eta)|\, d\eta \right)^2 d\xi \\
&+4 \int_\Omega \|Dw_0(\xi)\|^2_{\mathbb{R}^n} \left(\int_\Omega |k(\xi,\eta)| |w_n(\eta) - w_0(\eta)|\, d\eta \right)^2 d\xi \\
&\leq 4K^2 |\Omega| \|w_n - w_0\|^2_H (\|w_n\|^2_H + \|w_0\|^2_H).
\end{aligned}$$

Hence $\{D\overline{g}(w_n)\}$ converges to $D\overline{g}(w_0)$ in L^2. Moreover, as before, $\|\overline{g}(w_n) - \overline{g}(w_0)\|_2 \to 0$ and hence $\|\overline{g}(w_n) - \overline{g}(w_0)\|_H \to 0$. Thus, $\overline{g}(w_n) \stackrel{H}{\to} \overline{g}(w_0)$.
Now, let $t \in [0, T]$. To prove the $H - H$ continuity of the map $\overline{h}(t, \cdot)$ we assume by contradiction that there exists a sequence $\{\widetilde{w}_n\}$ and $\overline{\varepsilon} > 0$ such that $\widetilde{w}_n \stackrel{H}{\to} w_0$ and $\|\overline{h}(t, \widetilde{w}_n) - \overline{h}(t, w_0)\|_H > \overline{\varepsilon}$ for any $n \in \mathbb{N}$. Thus we have

$$\begin{aligned}
&\overline{\varepsilon}^2 < \|\overline{h}(t, \widetilde{w}_n) - \overline{h}(t, w_0)\|^2_H = \\
&\qquad \int_\Omega \left(|h(t, \widetilde{w}_n(\xi)) - h(t, \widetilde{w}_0(\xi))|^2 + \|Dh(t, \widetilde{w}_n(\xi)) - Dh(t, w_0(\xi))\|^2_{\mathbb{R}^n} \right) d\xi.
\end{aligned}$$

For the continuity in E of the map $\overline{h}(t, \cdot)$, w.l.o.g. has to be

$$\int_\Omega \|Dh(t, \widetilde{w}_n(\xi)) - Dh(t, w_0(\xi))\|^2_{\mathbb{R}^n} d\xi > \overline{\varepsilon}^2 \quad \forall n \in \mathbb{N}. \tag{5.3}$$

By the convergence of $\{\widetilde{w}_n\}$ to w_0 in H there exists a subsequence $\{\widetilde{w}_{n_k}\}$ such that $\widetilde{w}_{n_k}(\xi) \to w_0(\xi)$ for a.e. $\xi \in \Omega$. We have the following estimate

$$\begin{aligned}
\int_\Omega \|Dh(t, \widetilde{w}_{n_k}(\xi)) - Dh(t, w_0(\xi))\|^2_{\mathbb{R}^n} d\xi &= \int_\Omega \|h_2'(t, \widetilde{w}_{n_k}(\xi)) D\widetilde{w}_{n_k}(\xi) - h_2'(t, w_0(\xi)) Dw_0(\xi)\|^2_{\mathbb{R}^n} d\xi \\
&\leq 2 \int_\Omega |h_2'(t, \widetilde{w}_{n_k}(\xi))|^2 \|D\widetilde{w}_{n_k}(\xi) - Dw_0(\xi)\|^2_{\mathbb{R}^n} d\xi \\
&\quad +2 \int_\Omega |h_2'(t, \widetilde{w}_{n_k}(\xi)) - h_2'(t, w_0(\xi))|^2 \|Dw_0(\xi)\|^2_{\mathbb{R}^n} d\xi
\end{aligned}$$

By the continuity of the map h_2' it follows $h_2'(t, \widetilde{w}_{n_k}(\xi)) \to h_2'(t, w_0(\xi))$ for a.e. $\xi \in \Omega$. Moreover by hypothesis (1) we have

$$|(h_2'(t, \widetilde{w}_{n_k}(\xi)) - h_2'(t, w_0)(\xi))|^2 \|Dw_0(\xi)\|^2_{\mathbb{R}^n} \leq 4N^2 \|Dw_0(\xi)\|_{\mathbb{R}^n}$$

and

$$|h_2'(t,\widetilde{w}_{n_k}(\xi))|^2 \|D\widetilde{w}_{n_k}(\xi) - Dw_0(\xi)\|^2_{\mathbb{R}^n} \leq N^2 \|D\widetilde{w}_{n_k}(\xi) - Dw_0(\xi)\|^2_{\mathbb{R}^n}.$$

Thus by the convergence of $\{\widetilde{w}_n\}$ to w_0 in H and by the Lebesgue's Convergence Theorem

$$\int_\Omega \|Dh(t,\widetilde{w}_{n_k}(\xi)) - Dh(t,w_0(\xi))\|^2_{\mathbb{R}^n}\, d\xi \ \longrightarrow 0 \text{ as } k \to \infty,$$

obtaining a contradiction with (5.3). Hence for any sequence $\{w_n\}$ such that $w_n \overset{H}{\to} w_0$ it follows $\overline{h}(t,w_n) \overset{H}{\to} \overline{h}(t,w_0)$.
Hence $(w,v) \longmapsto F(t,w,v)$ is continuous from $H \times H$ into H^ω, for each $t \in [0,T]$ and condition (F2) is satisfied.

To verify condition $(F1)$ we will prove that $\overline{h}(\cdot,w)$ is continuous, for every $w \in H$. In fact, let $t_0 \in [0,T]$ and $\{t_n\} \subset [0,T]$ be such that $t_n \to t_0$. According to (1) we obtain that $h(t_n,w(\xi)) \to h(t_0,w(\xi))$ and $Dh(t_n,w(\xi)) = h_2'(t_n,w(\xi))Dw(\xi) \to h_2'(t_0,w(\xi))Dw(\xi) = Dh(t_0,w(\xi))$ for a.a. $\xi \in \Omega$. As a consequence of (2.2) and by the boundedness of $|h(\cdot,0)|$ in $[0,T]$, the previous convergences are also dominated in E, implying that $\overline{h}(t_n,w) \overset{H}{\to} \overline{h}(t_0,w)$. Therefore, $\overline{h}(\cdot,w)$ is continuous, and hence, it is measurable.

Now let $\Theta \subset H$ be bounded, $w \in \Theta$ and $t \in [0,T]$. We have that

$$\begin{aligned}
\|F(t,w,v)\|^2_H \ &= \ \int_\Omega \left| cv(\xi) + bw(\xi) + w(\xi)\left(\int_\Omega k(\xi,\eta)w(\eta)\,d\eta\right) + h(t,w(\xi))\right|^2 d\xi \\
&\quad + \int_\Omega \Big\| cDv(\xi) + bDw(\xi) + w(\xi)\left(\int_\Omega Dk(\xi,\eta)w(\eta)\,d\eta\right) \\
&\quad + Dw(\xi)\int_\Omega k(\xi,\eta)w(\eta)\,d\eta) + h_2'(t,w(\xi))Dw(\xi)\Big\|^2_{\mathbb{R}^n} d\xi \\
&\leq \ 5c^2\|v\|^2_H + 5b^2\|w\|^2_H + 9K^2|\Omega|\|w\|^4_H + 8\delta^2|\Omega| + 8N^2\|w\|^2_H
\end{aligned}$$

Hence

$$\|F(t,w,v)\|_H \leq \sqrt{5c^2\|v\|^2_H + 5b^2\|w\|^2_H + 9K^2|\Omega|\|w\|^4_H + 8\delta^2|\Omega| + 8N^2\|w\|^2_H},$$

so condition (F4) is satisfied.

To prove condition (4.3) notice first that we have the following equality,

$$\begin{aligned}
\langle w, F(t,w,v)\rangle &= c\langle w,v\rangle + b\|w\|_H^2 + \int_\Omega w(\xi)\left(w(\xi)\int_\Omega k(\xi,\eta)w(\eta)\,d\eta + h(t,w(\xi))\right)d\xi \\
&\quad + \int_\Omega \left\langle Dw(\xi), w(\xi)\left(\int_\Omega Dk(\xi,\eta)w(\eta)\,d\eta\right)\right\rangle_{\mathbb{R}^n} d\xi \\
&\quad + \int_\Omega \left\langle Dw(\xi), Dw(\xi)\int_\Omega k(\xi,\eta)w(\eta)\,d\eta\right\rangle_{\mathbb{R}^n} d\xi \\
&\quad + \int_\Omega \langle Dw(\xi), h_2'(t,w(\xi))Dw(\xi)\rangle_{\mathbb{R}^n}\, d\xi = \\
&= c\langle w,v\rangle + b\|w\|_H^2 + \int_\Omega w(\xi)h(t,w(\xi))\,d\xi + \int_\Omega \langle Dw(\xi), h_2'(t,w(\xi))Dw(\xi)\rangle_{\mathbb{R}^n} d\xi \\
&\quad + \int_\Omega (w(\xi))^2\left(\int_\Omega k(\xi,\eta)w(\eta)\,d\eta\right)d\xi \\
&\quad + \int_\Omega \left\langle Dw(\xi), w(\xi)\int_\Omega Dk(\xi,\eta)w(\eta)\,d\eta\right\rangle_{\mathbb{R}^n} d\xi \\
&\quad + \int_\Omega \|Dw(\xi)\|_{\mathbb{R}^n}^2\left(\int_\Omega k(\xi,\eta)w(\eta)\,d\eta\right)d\xi.
\end{aligned}$$

By virtue of (1) and (2.2) the following estimation is true

$$\begin{aligned}
\langle w, F(t,w,v)\rangle &\geq c\langle w,v\rangle + b\|w\|_H^2 - \int_\Omega |w(\xi)|\left(|h(t,0)| + N|w(\xi)|\right)d\xi \\
&\quad - N\int_\Omega \big\|Dw(\xi)\big\|_{\mathbb{R}^n}^2 d\xi - K\left(\int_\Omega |w(\xi)|^2\,d\xi\right)\left(\int_\Omega |w(\eta)|\,d\eta\right) \\
&\quad - K\left(\int_\Omega \|Dw(\xi)\|_{\mathbb{R}^n}|w(\xi)|\,d\xi\right)\left(\int_\Omega |w(\eta)|\,d\eta\right) \\
&\quad - K\left(\int_\Omega \|Dw(\xi)\|_{\mathbb{R}^n}^2\,d\xi\right)\left(\int_\Omega |w(\eta)|\,d\eta\right) \\
&\geq c\langle w,v\rangle + \big(b-N\big)\|w\|_H^2 - \delta|\Omega|^{1/2}\,\|w\|_2 - K|\Omega|^{1/2}\|w\|_2\|w\|_H^2 \\
&\quad - \frac{1}{2}K|\Omega|^{1/2}\|w\|_2\|w\|_H^2 \\
&\geq c\langle w,v\rangle - \frac{3}{2}K|\Omega|^{1/2}\|w\|_H^3 + (b-N)\|w\|_H^2 - \delta|\Omega|^{1/2}\,\|w\|_H \\
&\geq c\langle w,v\rangle + \left(-\frac{3}{2}K|\Omega|^{1/2}\|w\|_H^2 + (b-N)\|w\|_H - \delta|\Omega|^{1/2}\right)\|w\|_H \geq c\langle w,v\rangle,
\end{aligned}$$

provided $R_1 \leq \|w\|_H \leq R_2$ where

$$R_{1,2} = \frac{b-N \pm \sqrt{(b-N)^2 - 6\delta|\Omega|K}}{3K|\Omega|^{1/2}}.$$

Thus condition (4.3) is satisfied for $w \in H$ with $R_1 \leq \|w\| \leq R_2$ and every v with $\langle w,v\rangle = 0$.

Thus applying respectively Theorems 10 and 13 we obtain the claimed result.

Remark 15 *Notice that in the case $b = N + \sqrt{6\delta K|\Omega|}$ it holds $R_1 = R_2$ and we can reach a conclusion only for $\|w\| = R_1 = R_2$.*

References

[1] Agarwal R.P., Chen J., O'Regan D.: *Periodic solutions for a resonant system of coupled telegraph-wave equations*, Dyn. Cont. Discr. Imp. Syst. B **15** (2008), 891–902.

[2] Andres J., Gorniewicz L.: Topological Fixed Point Principles for Boundary Value Problems, Kluwer Academic Publishers, Dordrecht, 2003.

[3] Andres J., Malaguti L., Pavlačková M.: *Hartman-type conditions for multivalued Dirichlet problems in abstract spaces*, Aims Proc. 2014, to appear.

[4] Barbu V.: Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff International Publishing, Leyden, 1976.

[5] Benedetti I., Loi N., Malaguti L., Obukhovskii V.: *An approximation solvability method for nonlocal differential problems in Hilbert spaces*, Comm. in Cont. Math., to appear.

[6] Benedetti I., Malaguti L., Taddei V.: *Nonlocal semilinear evolution equations without strong compactness: theory and applications*, Bound. Val. Probl., **2013:60** (2013), 1–18.

[7] Benedetti I., Taddei V., Väth M.: *Evolution problems with nonlinear nonlocal boundary conditions*, J. of Dyn. Diff. Eq., **25** no.2 (2013), 477–503.

[8] Bochner S., Taylor A.E.: *Linear functionals on certain spaces of abstractly-valued functions*, Ann. Math. **39** (1938), 913–944.

[9] Carrillo J.A., Vazquez J.L.: *Some free bundary problems involving non-local diffusion and aggregation*, Phil. Trans. R. Soc. A **373** no. 2050 (2015), 16 pp.

[10] De Araujo G.M., Guzman R.B., De Menezes S.B.: *Periodic solutions for nonlinear telegraph equation via elliptic regularization*, Comp. Appl. Math. **28** no. 2 (2009), 135–155.

[11] Diestel J., Ruess W.M., Schachermayer W.: *Weak compactness in $L^1(\mu, X)$,* Proc. Amer. Math. Soc. **118** (1993), 447–453.

[12] Gilding B.H., Kersner R.: *The Characterization of reaction-convection-diffusion processes by travelling waves*, J. Diff. Eq., **124** (1996), 27–79.

[13] Hu S. , Papageorgiou N.S., Handbook of multivalued analysis, Vol. I: Theory, Mathematics and its Applications, 419, Kluwer Academic Publishers, Dordrecht, 1997.

[14] Hutson V., Shen W., Vickers, G. T. *Spectral theory for nonlocal dispersal with periodic or almost-periodic time dependence.* Rocky Mountain J. Math. **38** (2008), 1147–1175.

[15] Jang T.S.: *A new solution procedure for the nonlinear telegraph equation*, Comm. Nonlin. Sci. Numer. Simul. **29** (2015), 307–326.

[16] Jin Y., Zhao X.Q.: *Spatial dynamics of a periodic population model with dispersal*, Nonlin. **22** (2009), 1167–1189.

[17] Kmit I.: *Fredholm solvability of a periodic Neumann problem for a linear telegraph equation*, Ukr. Math. J. **65** no. 3 (2013), 423-434.

[18] Li W., Zhang H.: *Positive doubly periodic solutions of telegraph equations with delays*, Bound. Val. Probl. **2015** (2015), 12 pp.

[19] Malaguti L., Rubbioni P.: *Nonsmooth feedback controls of nonlocal dispersal models*, Nonlin., **29** (2016) 823–850.

[20] McKibben M., Discovering evolution equations with applications, vol. 1. CRC Press, 2011.

[21] Schmitt K., Thompson R.C.: *Boundary value problems for infinite systems of second-order differential equations*, J . Diff. Eq. **18** (1975), 277–295.

[22] Paicu A., Vrabie I.I.: *A class of nonlinear evolution equations subjected to nonlocal initial conditions*, Nonlin. Anal. **72** (2010), 4091–4100.

[23] Rashidinia J., Mohammadi R.: *Tension spline approach for the numerical solution of nonlinear Klein-Gordon equation*, Comp. Phys. Comm. **181** (2010), 78–91.

[24] Samia Y., Lutscher F.: *Coexistence and spread of competitors in heterogeneous landscapes*, Bull. Math. Biol. **72** no. 8 (2010), 2089-2112.

[25] Yuan W., Xiong W., Jianming W.: *All meromorphic solutions of an auxiliary ordinary differential equation and its applications*, Acta Math. Sci. Ser. B **35** no. 5 (2015), 1241–1250.

[26] Wang F., An Y.: *Existence and multiplicity results of positive doubly periodic solutions for nonlinear telegraph sustem*, J. Math. An. Appl. **349** (2009), 30–42.