Nonlinear elliptic inequalities with gradient terms on the Heisenberg group

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Abstract

In this paper we give sufficient conditions both for existence and for nonexistence of nontrivial nonnegative entire solutions of nonlinear elliptic inequalities with gradient terms on the Heisenberg group. The picture is completed with the presentation of a uniqueness result which is, as far as we know, the first attempt for general equations with gradient terms on the Heisenberg group.

Keywords: Nonlinear elliptic inequalities on the Heisenberg group, existence, nonexistence, uniqueness and qualitative properties of entire solutions

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Dedicated to Professor Enzo Mitidieri
on the occasion of his 60th birthday,
with great feelings of esteem and affection.

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1. Introduction

In this paper we first study existence and uniqueness of nonnegative nontrivial radial stationary entire solutions u of

$$\Delta^{\varphi}_{\mathbb{H}^m} u = f(u)\ell(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}), \tag{1.1}$$

where $\Delta^{\varphi}_{\mathbb{H}^m} u$ is the φ–Laplacian on the Heisenberg group $\mathbb{H}^m$, whose rigorous definition will be given in Section 2, and then for

$$\Delta^{\varphi}_{\mathbb{H}^m} u \geq f(u)\ell(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}) \tag{1.2}$$

Liouville type theorems, that is non–esistence of nonnegative nontrivial entire solutions u.

The operator $\Delta^{\varphi}_{\mathbb{H}^m}$ includes as main prototype the well known *Kohn–Spencer Laplacian in* $\mathbb{H}^m$. Moreover, f, ℓ and φ *satisfy throughout the paper*

$$f,\, \ell \in C(\mathbb{R}^+_0), \quad f > 0 \quad \textit{and} \quad \ell > 0 \ \textit{in}\ \mathbb{R}^+, \tag{$\mathscr{H}$}$$

$$\begin{gathered} \varphi \in C(\mathbb{R}^+_0) \cap C^1(\mathbb{R}^+), \quad \varphi(0) = 0, \quad \varphi' > 0 \quad \textit{in}\ \mathbb{R}^+, \\ \lim_{s\to\infty} \varphi(s) = \varphi(\infty) = \infty. \end{gathered} \tag{ϕ}$$

In particular, in the case of the p–Laplacian, that is when $\varphi(s) = s^{p-1}$, $p > 1$, we simply write $\Delta^{p}_{\mathbb{H}^m} u$.

Since 1957 it is well known that for semilinear coercive inequalities in the Euclidean setting, existence of solutions, as well as nonexistence, involves the Keller–Osserman condition, cfr. [15], [23]. For further generalization to quasilinear inequalities, possibly with singular of degenerate weights, we refer to [7]–[10], [21]–[22]. The first result in this direction, but in the Heisenberg group setting, can be found in [17, 2]. Recently, this has been extended to the Carnot groups in [1], adding further restrictions due to the presence of a new term which arises since the norm is not ∞–harmonic in that setting.

Since we are interested in nonnegative entire solutions of elliptic coercive inequalities in all the space, as in [10, 17, 2] we make use of an appropriate *generalized Keller–Osserman condition for inequality* (1.2). To this aim we also *assume throughout the paper that*

$$\int_{0^+} \frac{t\varphi'(t)}{\ell(t)}\,dt < \infty, \qquad \int^{\infty} \frac{t\varphi'(t)}{\ell(t)}\,dt = \infty, \tag{ϕL}$$

holds. Thus *the function* $K : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *given by*

$$K(s) = \int_0^s \frac{t\varphi'(t)}{\ell(t)}\,dt \tag{1.3}$$

is a C^1*–diffeomorphism from* $\mathbb{R}_0^+$ *to* $\mathbb{R}_0^+$, with

$$K'(s) = \frac{s\varphi'(s)}{\ell(s)} > 0 \qquad \text{in } \mathbb{R}^+, \tag{1.4}$$

thanks to (ϕ) and ($\mathscr{H}$). Consequently K has increasing inverse $K^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and denoting by $F(s) = \int_0^s f(t)\,dt$ we say that *the generalized Keller–Osserman condition holds for* (1.2) if

$$\int^{\infty} \frac{ds}{K^{-1}(F(s))} < \infty. \tag{KO}$$

If $\ell \equiv 1$, then K coincides with the function

$$H(s) = s\varphi(s) - \int_0^s \varphi(t)\,dt, \qquad s \geq 0,$$

which represents the Legendre trasform of $\Phi(s) = \int_0^s \varphi(t)\,dt$ for all $s \in \mathbb{R}_0^+$. Furthermore, in the case of the p–Laplacian, $H(s) = (p-1)s^p/p$, so that if $\ell \equiv 1$, then (KO) reduces to the well known Keller–Osserman condition for the p–Laplacian, that is $\int^{\infty} F(s)^{-1/p} ds < \infty$.

At this point we roughly recall that the nonexistence of entire solutions for coercive problems is connected with the validity of condition (KO), while the failure of (KO) gives existence of entire solutions. In particular, in the latter case Theorem 1.5 of [8], relative to the Euclidean case, shows that we can expect only unbounded solutions or equivalently *large solutions*. We are now in a position to extend and to generalize in several directions the core of Corollary 1.4 of [10], without requiring any monotonicity on ℓ.

Theorem 1.1. *Let $f(0) = 0$ and $\ell(0) > 0$ in $(\mathscr{H})$. Then* (1.1) *admits a nonnegative local radial stationary C^1 solution. If furthermore f is nondecreasing in $\mathbb{R}_0^+$ and*

$$\int^{\infty} \frac{dt}{K^{-1}(F(t))} = \infty \tag{$VsKO$}$$

holds, then (1.1) *possesses a nonnegative entire large radial stationary solution u of class $C^1(\mathbb{H}^m)$. Finally, if in addition*

$$\int_{0^+} \frac{dt}{K^{-1}(F(t))} = \infty \tag{1.5}$$

is valid, then $u > 0$ in $\mathbb{H}^m$.

The request of Theorem 1.1 are fairly natural and general. Theorem 1.1 can be applied not only in the p–Laplacian case, $\varphi(s) = s^{p-1}$, $p > 1$, but also in the generalized mean curvature case, $\varphi(s) = s(1 + s^2)^{(p-2)/2}$, $p \in (1, 2)$. For other elliptic operators we refer to [25] and [2].

The next result concerns uniqueness of radial stationary solutions of (1.1), as in Theorem 1.1 we do not require any monotonicity assumption on ℓ in $\mathbb{R}_0^+$.

Theorem 1.2. *Assume that f and ℓ are locally Lipschitz continuous in $\mathbb{R}_0^+$, that $\ell(0) > 0$ and finally that $\varphi^{-1} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_0^+)$. Then, for each fixed $u_0 > 0$ equation* (1.1) *admits a unique radial stationary solution u, with $u(O) = u_0$, where O is the natural origin in $\mathbb{H}^m$, in the open maximal ball B_R of $\mathbb{H}^m$.*

When φ is the p–Laplacian operator Theorem 1.2 is applicable if and only if $1 < p \leq 2$. The remaining case $p > 2$ seems to be fairly delicate. Theorem 1.2 is valid under general assumptions, so that in principle we cannot assert that the solution is entire. For existence of entire solutions we refer the interested reader to Theorem 4.2, which yields to the proof of Theorem 1.1.

In what follows we assume monotonicity on f. In particular in the next theorem we require strict monotonicity on f, similarly as in [8, 17, 1, 2]. Indeed, this assumption is due to the technique used, that is to an argument involving a comparison theorem.

For the first Liouville type theorem we assume that ℓ is *b–monotone nonincreasing on* $\mathbb{R}_0^+$, that is there exists $b \in (0,1]$ such that

$$\inf_{t\in[0,s]} \ell(t) \geq b\,\ell(s) \quad \text{ for all } s \in \mathbb{R}_0^+.$$

Clearly, if ℓ is monotone nonincreasing in $\mathbb{R}_0^+$, then ℓ is 1–monotone nonincreasing on the same set, furthermore the above condition allows a controlled oscillatory behavior of ℓ on $\mathbb{R}_0^+$. Similar results when ℓ is monotone nonincreasing can be found earlier in [10].

Theorem 1.3. *Suppose that f is strictly increasing in $\mathbb{R}_0^+$ and that ℓ is b–monotone nonincreasing in $\mathbb{R}_0^+$. Assume that there exist an exponent $\tau < 1$ and a constant $\theta \geq 1$ such that*

$$(\phi 1) \qquad s^\tau \varphi'(st) \leq \theta \varphi'(t) \quad \text{for all } s \in (0,1],\ t \in \mathbb{R}^+.$$

Then every nonnegative bounded C^1–solution u of (1.2) *is constant in $\mathbb{H}^m$.*

The restriction that the solutions are assumed bounded in Theorem 1.3 is essential. Indeed, the simple inequality $\Delta_{\mathbb{H}^m} u \geq \ell(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}) \cdot u$, with $\ell(s) = 4m/(s^2+1)$, admits the regular nonnegative unbounded entire solution $u(q) = w(|z|) = |z|^2 + 1$, $q = (z,t) \in \mathbb{H}^m$.

The restriction $(\phi 1)$ implies in particular that $\varphi(\infty) = \infty$, as required in the main assumption (ϕ). Furthermore, $(\phi 1)$ is satisfied with $\tau = 2 - p$ and $\theta = 1$ whenever φ is homogeneous, that is $\varphi(s) = s^{p-1}$, $p > 1$. Clearly, if φ' is nondecreasing in $\mathbb{R}^+$, again $(\phi 1)$ is automatic for every $\tau \in [0,1)$ and $\theta = 1$. Of course there are cases in which φ' is nonincreasing in $\mathbb{R}^+$ and $(\phi 1)$ holds, as for instance in the case of the generalized mean curvature operator, $\varphi(s) = s(1+s^2)^{(p-2)/2}$, $p \in (1,2)$, for which $(\phi 1)$ holds with $\tau = 2 - p \in (0,1)$. Finally, the exponent τ in $(\phi 1)$ can be negative only if $\varphi'(s) \to 0$ as $s \to 0^+$ and $\varphi'(s) \to \infty$ as $s \to \infty$, as for the p–Laplacian operator when $p > 2$.

Under the assumptions of Theorem 1.3, then $\ell(0) > 0$ by $(\mathscr{H})$ and the b–monotonicity. If furthermore $\ell(\infty) = \lim_{s\to\infty} \ell(s) > 0$, then the corresponding nonexistence results can be deduced from inequalities including no gradient terms, since $\Delta^{\varphi}_{\mathbb{H}^m} u \geq f(u)\ell(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}) \geq \ell(\infty) f(u)$. Thus the truly significant new case for Theorem 1.3 is when $\ell(\infty) = 0$.

For quasilinear elliptic inequalities of the type $\Delta^{\varphi}_{\mathbb{H}^m} u \geq f(u)$ we refer to the pioneering work of *Mitidieri* and *Pohozaev* in the Euclidean setting, see i.e. [19, 20, 21], and to recent contributions due to *D'Ambrosio* and *Mitidieri*, see for instance [5, 6] and the references therein. In [5, 6] the results are also obtained for a wide class of degenerate elliptic operators in the Heisenberg group. More recently, *D'Ambrosio*, *Farina*, *Mitidieri* and *Serrin* proved in [4] comparison principles, uniqueness, regularity and symmetry results for p–regular distributional solutions of quasilinear very weak elliptic equations of coercive type and for related inequalities. Finally, *D'Ambrosio* and *Mitidieri* presented in [7] Liouville theorems and applications to general systems, which include the celebrated Allen–Cahn equation, Ginzburg–Landau systems, Gross–Pitaevskii systems and Lichnerowicz type equations.

Recently, in [17, 2] results similar to Theorem 1.3 are given when ℓ is C–*monotone nondecreasing in* $\mathbb{R}^+_0$, that is there exists $C \geq 1$ such that

$$\sup_{t\in[0,s]} \ell(t) \leq C\,\ell(s) \quad \text{for all } s \in \mathbb{R}^+_0.$$

In the next result we extend Theorem 1.3–(i) of [17] from the p–Laplacian inequality in $\mathbb{H}^m$ to the $\Delta^{\varphi}_{\mathbb{H}^m}$ operator.

Theorem 1.4. *Suppose that f is also nondecreasing in $\mathbb{R}^+_0$, and that ℓ is also C–monotone nondecreasing in $\mathbb{R}^+_0$. If $(VsKO)$ holds, then there exists a nonnegative large solution $u \in C^1(\mathbb{H}^m)$ of inequality* (1.2).

Theorem 1.4 extends also the existence Theorem 6.1 of [2], where (ϕL) is replaced by a stronger condition. More details are given in Section 7.

Furthermore, we recall that the converse of Theorem 1.4, that is nonexistence of nonnegative entire solutions of inequality (1.2) when (KO) is valid, has been established in Theorem 1.1 of [17]. In particular, Theorem 1.1 of [17] is the generalization of Theorem 1.3–(ii) of [17] and is given under the further requests that $\ell(0) > 0$ and that f is strictly increasing in $\mathbb{R}^+_0$. These two conditions appear also in [8, 17, 1] and are used in the main proofs when a general solution u of (1.2) is compared with an appropriate radial stationary solution v of the reverse inequality, in order to overcome the difficulty at points in which $D_{\mathbb{H}^m} u = D_{\mathbb{H}^m} v = 0$. Lately, Theorem 1.1 of [17] has been further extended to the case $\ell(0) = 0$ in the nonexistence Theorems 5.1 and 5.2 of [2], but under more stringent conditions on the regularity of solutions due to the necessity of a deep analysis on the set where the horizontal gradient vanishes.

The paper is organized as follows. In Section 2 we recall some preliminary notions related to the operator $\Delta^{\varphi}_{\mathbb{H}^m}$ on Heisenberg group, as well as regularity properties of weak solutions. Section 3 deals with the radial version of $\Delta^{\varphi}_{\mathbb{H}^m}$. In Section 4 we prove Theorem 1.1, the main existence theorem of the paper, where no monotonicity assumptions on ℓ are required. Furthermore, in Section 5 we present a uniqueness result which is, as far as we know, the first attempt for general equations with gradient terms on the Heisenberg group $\mathbb{H}^m$. The proof of Theorem 1.3, which is a Liouville type result for bounded solutions of (1.2), is given in Section 6 under the nonincreasing b–monotonicity on ℓ. Finally, in Section 7 we give the proof of the existence Theorem 1.4 assuming the nondecreasing C–monotonicity on ℓ.

2. Preliminaries

Let $\mathbb{H}^m$ be the Heisenberg group of dimension $2m+1$, that is the Lie group whose underlying manifold is $\mathbb{R}^{2m+1}$ endowed with the non–Abelian group law

$$q \circ q' = \left(z + z', t + t' + 2\sum_{i=1}^{m} (y_i x_i' - x_i y_i') \right)$$

for all q, $q' \in \mathbb{H}^m$, with

$$q = (z,t) = (x_1, \dots, x_m, y_1, \dots, y_m, t), \quad q' = (z',t') = (x_1', \dots, x_m', y_1', \dots, y_m', t').$$

The vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial t}, \tag{2.1}$$

for $j = 1, \ldots, m$, constitute a basis for the real Lie algebra of left–invariant vector fields on $\mathbb{H}^m$. This basis satisfies the Heisenberg canonical commutation relations for position and momentum $[X_j, Y_k] = -4\delta_{jk}\partial/\partial t$, all other commutators being zero. A vector field in the span of $\{X_j, Y_j\}_{j=1}^m$ will be called horizontal. The *Kohn–Spencer Laplacian*, or equivalently the *horizontal Laplacian in* $\mathbb{H}^m$, is defined as follows

$$\Delta_{\mathbb{H}^m} = \sum_{j=1}^m (X_j^2 + Y_j^2) = \sum_{j=1}^m \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} \right) + 4|z|^2 \frac{\partial^2}{\partial t^2},$$

and $\Delta_{\mathbb{H}^m}$ is *hypoelliptic* according to the celebrated Theorem 1.1 due to *Hörmander* in [14].

In $\mathbb{H}^m$ the natural origin is denoted by $O = (0, 0)$. Define

$$r(q) = r(z, t) = (|z|^4 + t^2)^{1/4} \quad \text{for all } q = (z, t) \in \mathbb{H}^m,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^{2m}$. The *Korányi norm* is homogeneous of degree 1, with respect to the dilations $\delta_R : (z, t) \mapsto (Rz, R^2 t)$, $R > 0$. Indeed, for all $q = (z, t) \in \mathbb{H}^m$

$$r(\delta_R(q)) = r(Rz, R^2 t) = (|Rz|^4 + R^4 t^2)^{1/4} = Rr(q).$$

Hence, the *Korányi distance*, is

$$d(q, q') = r(q^{-1} \circ q') \quad \text{for all } (q, q') \in \mathbb{H}^m \times \mathbb{H}^m,$$

and the *Korányi open ball*, of radius R centered at q_0, is

$$B_R(q_0) = \{q \in \mathbb{H}^m \, : \, d(q, q_0) < R\}.$$

For simplicity B_R denotes the ball of radius R centered at $q_0 = O$.

Let $u \in C^1(\mathbb{H}^m)$ be fixed. The *horizontal gradient* $D_{\mathbb{H}^m} u$ is

$$D_{\mathbb{H}^m} u = \sum_{j=1}^m (X_j u) X_j + (Y_j u) Y_j.$$

Furthermore, if $f \in C^1(\mathbb{R})$, then $D_{\mathbb{H}^m} f(u) = f'(u) D_{\mathbb{H}^m} u$. The natural product

$$W \cdot Z = \sum_{j=1}^m w^j z^j + \widetilde{w}^j \widetilde{z}^j,$$

for $W = w^j X_j + \widetilde{w}^j Y_j$ and $Z = z^j X_j + \widetilde{z}^j Y_j$ produces $|D_{\mathbb{H}^m} u|^2_{\mathbb{H}^m} = D_{\mathbb{H}^m} u \cdot D_{\mathbb{H}^m} u$ for the horizontal vector field $D_{\mathbb{H}^m} u$. Moreover, if also $v \in C^1(\mathbb{H}^m)$ then the Cauchy–Schwarz inequality

$$|D_{\mathbb{H}^m} u \cdot D_{\mathbb{H}^m} v|_{\mathbb{H}^m} \le |D_{\mathbb{H}^m} u|_{\mathbb{H}^m} |D_{\mathbb{H}^m} v|_{\mathbb{H}^m}$$

continues to be valid.

The *density function*

$$\psi(z,t) = |D_{\mathbb{H}^m} r|^2_{\mathbb{H}^m} = \frac{|z|^2}{r^2(z,t)} \quad \text{for all } (z,t) \in \mathbb{H}^m, \text{ with } (z,t) \neq 0, \tag{2.2}$$

is homogeneous of degree 0, with respect to the dilatation δ_R. Clearly, ψ is bounded in $\mathbb{H}^m$, with $0 \le \psi \le 1$. Furthermore, direct calculation shows

$$\Delta_{\mathbb{H}^m} r = \frac{2m+1}{r} \psi \quad \text{in } \mathbb{H}^m \setminus \{O\},$$

for details we refer to Section 2.1 of [17].

Let now $W : \mathbb{H}^m \to \mathbb{H}^m$ be a horizontal vector field of class C^1, that is $W = (w^1, \cdots, w^m, \widetilde{w}^1, \cdots, \widetilde{w}^m, t)$, with w^j, $\widetilde{w}^j \in C^1(\mathbb{H}^m)$. Then *the horizontal divergence* for W is

$$\mathrm{div}_0 W = \sum_{j=1}^m [X_j(w^j) + Y_j(\widetilde{w}^j)].$$

If furthermore $g \in C^1(\mathbb{R})$, then the *Leibnitz formula* holds, namely

$$\mathrm{div}_0(gW) = g\,\mathrm{div}_0(W) + D_{\mathbb{H}^m} g \cdot W.$$

In particular, $\Delta_{\mathbb{H}^m} u = \mathrm{div}_0 D_{\mathbb{H}^m} u$ for each $u \in C^2(\mathbb{H}^m)$.

A well known generalization of the Kohn–Spencer Laplacian is the horizontal p–Laplacian on the Heisenberg group defined by

$$\Delta^p_{\mathbb{H}^m} u = \mathrm{div}_0(|D_{\mathbb{H}^m} u|^{p-2}_{\mathbb{H}^m} D_{\mathbb{H}^m} u), \qquad p \in (1, \infty).$$

From [3] and [16] we know that weak solutions of the equation $\Delta^p_{\mathbb{H}^m} u = 0$ satisfy Harnack inequality and, as a consequence, up to a modification on a set of Lebesgue measure zero, they are locally Hölder continuous of some exponent

$\alpha \in (0,1)$. However, in [12] Garofalo emphasized that the fundamental question whether the horizontal gradient $\nabla_{\mathbb{H}^m} u$ of such a weak solution is also continuous (or Hőlder continuous), with respect to the intrinsic distance attached to the vector fields $X_1, \ldots, X_m, Y_1, \ldots, Y_m$, is still open at the submission of this paper. Substantial progress in that direction can be found in [18].

Furthermore, $C^{1,\alpha}_{\mathrm{loc}}$ regularity has been proved for solutions with special symmetries in [12], for instance in the first Heisenberg group $\mathbb{H}^1$ he obtains such regularity for all weak solutions of the horizontal p–Laplacian, with $p \geq 2$ which are of the form $u(z,t) = u(|z|,t)$. For the case $1 < p < 2$ and other remarks we refer to [26].

A further generalization of the horizontal p–Laplacian is defined as follows

$$\Delta^{\varphi}_{\mathbb{H}^m} u = \mathrm{div}_0 (A(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}) D_{\mathbb{H}^m} u),$$

where $A(s) = \varphi(s)/s$ with φ satisfying (ϕ). Of course, the horizontal p–Laplacian follows by the choice $\varphi(s) = s^{p-1}$, $p > 1$ and $s \in \mathbb{R}^+_0$, so that $A(s) = s^{p-2}$ is defined in $\mathbb{R}^+$ and satisfies the required properties (ϕ).

As in [17] we write the A–Laplacian in Euclidean divergence form by making use of the following matrix $B = B(q)$, defined by

$$B(q) = B(z,t) = \begin{bmatrix} I_{2m} & 2y \\ & -2x \\ 2y^t - 2x^t & 4|z|^2 \end{bmatrix}, \tag{2.3}$$

where $x^t = (x_1, x_2, \ldots, x_m)$ and $y^t = (y_1, y_2, \ldots, y_m)$. Throughout the paper we denote by div, D, and $<,>$ respectively the ordinary Euclidean divergence, the gradient and the scalar product in $\mathbb{R}^{2m+1}$. Consequently, $BDu = D_{\mathbb{H}^m} u$, where BDv is the vector in $\mathbb{R}^{2m+1}$ whose components in the standard basis $\{\partial_{x_j}, \partial_{y_j}, \partial_t\}_{j=1}^m$ are given by the matrix multiplication B with the components of Du in the same basis. With this in mind we deduce the required expression

$$\Delta^{A}_{\mathbb{H}^m} u = \mathrm{div}_0 (A(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}) D_{\mathbb{H}^m} u) = \mathrm{div}(A(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}) BDu). \tag{2.4}$$

In particular

$$< Du, BDv >= D_{\mathbb{H}^m} u \cdot D_{\mathbb{H}^m} v.$$

If $\varphi(s) = s$, $s \in \mathbb{R}_0^+$, then (2.4) reduces to *the well known formula for the Kohn–Spencer Laplacian*, that is $\Delta^A_{\mathbb{H}^m} u = \Delta_{\mathbb{H}^m} u = \operatorname{div}(BDu)$.

Multiplying (2.4) by $\phi \in C_0^\infty(\mathbb{H}^m)$, we get

$$\begin{aligned}\int_{\mathbb{R}^{2m+1}} \phi\, \Delta^\varphi_{\mathbb{H}^m} u = \int_{\mathbb{R}^{2m+1}} \phi \operatorname{div}(A(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}) BDu) \\ = -\int_{\mathbb{R}^{2m+1}} A(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}) < BDu, D\phi > \\ = -\int_{\mathbb{R}^{2m+1}} A(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}) D_{\mathbb{H}^m} u \cdot D_{\mathbb{H}^m} \phi.\end{aligned}$$

Hence *the weak formulation of* (1.2) is given by

$$-\int_{\mathbb{R}^{2m+1}} A(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m}) D_{\mathbb{H}^m} u \cdot D_{\mathbb{H}^m} \phi \geq \int_{\mathbb{R}^{2m+1}} f(u)\ell(|D_{\mathbb{H}^m} u|_{\mathbb{H}^m})\phi, \qquad (2.5)$$

for all $\phi \in C_0^\infty(\mathbb{H}^m)$, $\phi \geq 0$.

In conclusion, we say that $u \in C^1(\mathbb{H}^m)$ is an *entire (weak) classical solution* of (1.2) if (2.5) is satisfied for all $\phi \in C_0^\infty(\mathbb{H}^m)$, with $\phi \geq 0$.

Later we make use of the next comparison theorem given in Proposition 2.1 of [17], in the extended version stated in Proposition 4.2 of [2].

Proposition 2.1. *Let $\Omega \subset\subset \mathbb{H}^m$ be a relatively compact domain. If u and v are of class $C(\overline{\Omega}) \cap C^1(\Omega)$ and satisfy*

$$\begin{cases} \Delta^\varphi_{\mathbb{H}^m} u \geq \Delta^\varphi_{\mathbb{H}^m} v & \text{in} \quad \Omega, \\ u \leq v & \text{on } \partial\Omega, \end{cases} \qquad (2.6)$$

then $u \leq v$ in Ω.

Finally, we report the strong maximum principle given in Proposition 2.2 of [17].

Proposition 2.2. *Let $\Omega \subset \mathbb{H}^m$ be a domain and let φ satisfy $(\phi 1)$. Assume that u is a solution of class $C(\overline{\Omega}) \cap C^1(\Omega)$ of the inequality*

$$\Delta^\varphi_{\mathbb{H}^m} u \geq 0 \quad \text{in} \quad \Omega \qquad (2.7)$$

and that $u(q_M) = \sup_\Omega u = u^*$ for some $q_M \in \Omega$. Then $u \equiv u^*$ in Ω.

3. Radial version of the φ–Laplacian

Let υ be a *radial regular* function, that is for all $q = (z, t) \in \mathbb{H}^m$

$$\upsilon(q) = \alpha(r(q)), \qquad r(q) = r(z, t) = (|z|^4 + t^2)^{1/4}, \tag{3.1}$$

where $\alpha : \mathbb{R}_0^+ \to \mathbb{R}$, $\alpha \in C(\mathbb{R}_0^+) \cap C^2(\mathbb{R}^+)$. From (2.2), it results

$$|D_{\mathbb{H}^m} r|_{\mathbb{H}^m} = \psi^{1/2},$$

so that

$$|D_{\mathbb{H}^m} \upsilon(q)|_{\mathbb{H}^m} = |\alpha'(r)| \cdot |D_{\mathbb{H}^m} r|_{\mathbb{H}^m} = |\alpha'(r)|\psi^{1/2}. \tag{3.2}$$

Thus

$$\begin{aligned} \Delta^{\varphi}_{\mathbb{H}^m} \upsilon = \psi^{1/2} \Big[& \psi^{1/2} \varphi'(|\alpha'(r)|\psi^{1/2})\alpha''(r) \\ & + \frac{2m+1}{r} \operatorname{sgn}(\alpha'(r))\varphi(|\alpha'(r)|\psi^{1/2}) \Big], \end{aligned} \tag{3.3}$$

which is the *radial version of* $\Delta^{\varphi}_{\mathbb{H}^m} \upsilon$. As noted in [17], it is possible to shift the origin for the Korányi distance from O to any other point q_0, indeed if we denote with $\bar{r}(q) = d(q_0, q) = r(q_0^{-1} \circ q)$, direct calculation shows

$$[X_j(\bar{r})](q) = [X_j(r)](q_0^{-1} \circ q), \qquad [Y_j(\bar{r})](q) = [Y_j(r)](q_0^{-1} \circ q).$$

Hence *the invariance with respect to the left multiplication holds*, namely

$$\Delta^{\varphi}_{\mathbb{H}^m}(\alpha \circ \bar{r})(q) = \Delta^{\varphi}_{\mathbb{H}^m}(\alpha \circ r)(q_0^{-1} \circ q). \tag{3.4}$$

This property will be useful in what follows.

A further particular radial case of (1.2) is the subcase of *radial stationary solutions*, that is solutions of the form

$$\upsilon(q) = w(|z|), \qquad q = (z, t) \in \mathbb{H}^m, \tag{3.5}$$

where $w : \mathbb{R}_0^+ \to \mathbb{R}$, $w \in C(\mathbb{R}_0^+) \cap C^2(\mathbb{R}^+)$. This case is morally the case $t = 0$ of (3.1), with $r(q) = |z|$ and $|\cdot|$ the Euclidean norm in $\mathbb{R}^{2m}$. Consequently, the density function ψ, given in (2.2), is identically 1. In particular

$$D_{\mathbb{H}^m}|z| = \sum_{j=1}^{m} (X_j|z|)X_j + (Y_j|z|)Y_j = \sum_{j=1}^{m} \frac{\partial |z|}{\partial x_j} X_j + \frac{\partial |z|}{\partial y_j} Y_j = \frac{z}{|z|},$$

so that

$$\Delta_{\mathbb{H}^m}|z| = \sum_{j=1}^m X_j^2|z| + Y_j^2|z| = \sum_{j=1}^m \frac{\partial^2|z|}{\partial x_j^2} + \frac{\partial^2|z|}{\partial y_j^2}$$
$$= \sum_{j=1}^m \frac{1}{|z|} - \frac{x_j^2}{|z|^3} + \frac{1}{|z|} - \frac{y_j^2}{|z|^3} = \frac{2m-1}{|z|}.$$

In turn $|D_{\mathbb{H}^m}|z||_{\mathbb{H}^m} \equiv 1$, that is $\psi \equiv 1$. Consequently,

$$\Delta^{\varphi}_{\mathbb{H}^m} v = \varphi'(|w'(|z|)|)w''(|z|) + \frac{2m-1}{|z|}\mathrm{sgn}(w'(|z|))\varphi(|w'(|z|)|). \tag{3.6}$$

Hence, as noted above, *radial stationary functions in the Heisenberg group* $\mathbb{H}^m$ *behave as Euclidean radial functions in* $\mathbb{R}^{2m}$.

4. Proof of the existence Theorem 1.1

The next result can be proved using some of the main ideas of the proof of Proposition 3.1 in [10], see also Chapter 4 in [25], but with notable improvements in several directions. We recall in passing that ($\mathscr{H}$), (ϕ) *and* (ϕL) *are supposed to hold throughout the paper*, without further mentioning. We point out that no monotonicity assumptions are required on ℓ. For simplicity in notation we put $|z| = r$ in what follows. Furthermore we assume that $\wp$ is sufficiently smooth, just for simplicity. For the results of this section, the case $\varphi(\infty) < \infty$, not covered in this paper, could be treated as in Chapters 4 and 8 of [25], where $\ell \equiv 1$.

Theorem 4.1. *Assume that* $\wp \in C^1(\mathbb{R}_0^+)$, *with* $\wp$ *and* $\wp'$ *nondecreasing in* $\mathbb{R}_0^+$, *and* $\wp > 0$ *in* $\mathbb{R}^+$. *Suppose furthermore that* $f(0) = 0$ *and* $\ell(0) > 0$. *Then for all* $\eta > 0$ *and* r_0, $r_1 \in \mathbb{R}_0^+$, *with* $0 < r_0 < r_1$, *problem*

$$\begin{cases} [\wp\, A(|w'|)w']' = \wp f(w)\ell(|w'|) \quad \text{in } (r_0, r_1], \quad 0 < r_0 < r_1, \\ w \ge 0, \quad w' \ge 0, \quad w'(r_0) = 0, \\ w(r_1) = \eta, \quad w < \eta \quad \text{in } [r_0, r_1), \end{cases} \tag{4.1}$$

admits a C^1 *solution* w *in* $[r_0, r_1]$, *with the property that there exists* $s_1 \in [r_0, r_1)$ *such that* $w(r) \equiv w(r_0) \ge 0$ *in* $[r_0, s_1]$, $w' > 0$ *in* $(s_1, r_1]$ *and* w' *is differentiable* *in* $(s_1, r_1]$*, so that* w *satisfies the equation*

$$\frac{\varphi'(w')}{\ell(w')}\, w'' = \sigma f(w) - \frac{\wp'}{\wp} \cdot \frac{\varphi(w')}{\ell(w')} \tag{4.2}$$

in $(s_1, r_1]$.

If $w(r_0) = 0$*, then*

$$\int_{0^+} \frac{du}{K^{-1}(F(u))} < \infty. \tag{4.3}$$

If $r_0 = 0$ *the same conclusions hold provided that*

$$\limsup_{r \to 0^+} \frac{r\wp'(r)}{\wp(r)} < \infty. \tag{4.4}$$

PROOF. For the purpose of this proof, we shall redefine f so that $f(u) = f(\eta)$ for all $u \geq \eta$, and $f(u) = 0$ when $u \leq 0$. This will not affect the conclusion of the proposition, since clearly any ultimate solution w of (4.1), with $w \geq 0$, $w' \geq 0$ in $[r_0, r_1]$, satisfies $0 \leq w \leq \eta$.

We shall make use of the Leray–Schauder fixed point theorem. Denote by X the Banach space $X = C^1[r_0, r_1]$, endowed with the usual norm $\|w\| = \|w\|_\infty + \|w'\|_\infty$. Let $\mathcal{T}$ be the mapping from X to X, defined pointwise for all $w \in X$ and $r \in [r_0, r_1]$ by

$$\mathcal{T}[w](r) = \eta - \int_r^{r_1} \varphi^{-1}\left(\frac{1}{\wp(s)} \int_{r_0}^{s} \wp(\tau) f(w(\tau))\ell(|w'(\tau)|)d\tau\right) ds. \tag{4.5}$$

Clearly, $\mathcal{T}[w](r_1) = \eta$. Furthermore, for each $r \in (r_0, r_1]$

$$\mathcal{T}[w]'(r) = \varphi^{-1}\left(\frac{1}{\wp(r)} \int_{r_0}^{r} \wp(\tau) f(w(\tau))\ell(|w'(\tau)|)d\tau\right). \tag{4.6}$$

Obviously $\mathcal{T}[w]'$ is continuous and nonnegative in $(r_0, r_1]$, since $0 \leq f(w) \leq f_\eta$ for all $w \in X$, where $f_\eta = \max_{u \in [0,\eta]} f(u) > 0$, and $\ell > 0$ in $\mathbb{R}^+$ by $(\mathscr{H})$. As a matter of fact

$$\begin{aligned} 0 \leq \frac{1}{\wp(r)} \int_{r_0}^{r} \wp(\tau) f(w(\tau))\ell(|w'(\tau)|)d\tau &\leq f_\eta \max_{r \in [r_0, r_1]} \ell(|w'(r)|)(r - r_0) \\ &= C_w(r - r_0), \end{aligned}$$

with $C_w = f_\eta \max_{r \in [r_0, r_1]} \ell(|w'(r)|)$. Therefore $\mathcal{T}[w]'(r)$ approaches 0 as $r \to r_0^+$, and in turn $\mathcal{T}[w] \in X$, with $\mathcal{T}[w]'(r_0) = 0$.

Let w be a fixed point of $\mathcal{T}$ in X. We claim that $w(r_0) \geq 0$. Otherwise $w(r_0) < 0$, while $w(r_1) = \eta > 0$. Thus there exists a first point $s_1 \in (r_0, r_1)$ such that $w(r) < 0$ in $[r_0, s_1)$ and $w(s_1) = 0$. Consequently $f(w(r)) = 0$ in $[r_0, s_1]$ and so $w' \equiv 0$ for $r \in [r_0, s_1]$ by (4.6). Hence, $w(s_1) = w(r_0) < 0$ which is impossible, proving the claim. Therefore, $w \geq 0$ and $w' \geq 0$ in $[r_0, r_1]$ by (4.6). Moreover, we assert that $w < \eta$ in $[r_0, r_1)$. Indeed, from the fact that $f > 0$ in $(0, \eta]$ and $\ell > 0$ in $\mathbb{R}_0^+$, it follows that for all $r \in [r_0, r_1)$

$$\begin{aligned}\int_r^{r_1} &\varphi^{-1}\left(\frac{1}{\wp(s)}\int_{r_0}^s \wp(\tau)f(w(\tau))\ell(|w'(\tau)|)d\tau\right)ds \\ &\geq \int_{\max\{\tau_0, r\}}^{r_1} \varphi^{-1}\left(\frac{1}{\wp(s)}\int_{r_0}^s \wp(\tau)f(w(\tau))\ell(|w'(\tau)|)d\tau\right)ds > 0,\end{aligned}$$

where τ_0 is a point in $[r_0, r_1)$ such that $f(w(r)) > 0$ for all $r \in (\tau_0, r_1]$, which exists since $f \circ w \in C[r_0, r_1]$, $f(w(r_1)) = \eta > 0$ and $\ell(|w'(r)|) > 0$ for all $r \in [\tau_0, r_1]$. The assertion now follows from (4.5).

Define the homotopy $\mathcal{H} : X \times [0, 1] \to X$ by

$$\mathcal{H}[w, \sigma](r) = \sigma\eta - \int_r^{r_1} \varphi^{-1}\left(\frac{\sigma}{\wp(s)}\int_{r_0}^s \wp(\tau)f(w(\tau))\ell(|w'(\tau)|)d\tau\right)ds. \quad (4.7)$$

By the above argument, any fixed point $w_\sigma = \mathcal{H}[w_\sigma, \sigma]$ is in X and has the properties that $w_\sigma \geq 0$, $w'_\sigma \geq 0$ in $[r_0, r_1]$ and $w_\sigma(r_1) = \sigma\eta$. Additionally, by (4.6) we find that $\varphi(w'_\sigma)$ is of class $C^1[r_0, r_1]$, and then from (4.7) that w_σ is a classical distribution solution of the problem

$$\begin{cases} [\wp\, A(|w'_\sigma|)w'_\sigma]' = \sigma\wp f(w_\sigma)\ell(|w'_\sigma|) \quad \text{in } (r_0, r_1], \\ w'_\sigma(r_0) = 0, \quad w_\sigma(r_1) = \sigma\eta. \end{cases} \quad (4.8)$$

In turn, it is evident that any function w_1 which is a fixed point of $\mathcal{H}[w, 1]$ (that is $w_1 = \mathcal{H}[w_1, 1]$) is a nonnegative distribution solution of (4.1), with $w'_1(r_0) = 0$, $w_1 \geq 0$ and $w'_1 \geq 0$ in $[r_0, r_1]$, and $w_1 < \eta$ in $[r_0, r_1)$, as shown above.

We assert that such a fixed point $w = w_1$ exists, using the Browder version of the Leray–Schauder theorem (see Theorem 11.6 of [13]).

To begin with, obviously $\mathcal{H}[w, 0] \equiv 0$ for all $w \in X$, that is $\mathcal{H}[w, 0]$ maps X into the single point $w_0 = 0$ in X. (This is the first hypothesis required in the application of the Leray–Schauder theorem.) We next show that $\mathcal{H}$ is compact from $X \times [0,1]$ into X. First, $\mathcal{H}$ is continuous on $X \times [0,1]$. Indeed, let $(w_j, \sigma_j)_j \in X \times [0,1]$, with $w_j \to w$ in X, that is $w_j \to w$ and $w_j' \to w'$ uniformly in $[r_0, r_1]$, and $\sigma_j \to \sigma$. Clearly $\sigma_j f(w_j)\ell(|w_j'|) \to \sigma f(w)\ell(|w'|)$, since the modified function f is continuous in $\mathbb{R}$, and so $\mathcal{H}[w_j, \sigma_j] \to \mathcal{H}[w, \sigma]$ by (4.7) and the dominated convergence theorem, as required.

Next let $(w_k, \sigma_k)_k$ be a bounded sequence in $X \times [0,1]$, say $\|w_k'\|_\infty \le L$ for some $L > 0$ and for all $k \in \mathbb{N}$. Put $\ell_L = \max_{\tau\in[0,L]} \ell(\tau)$. It is clear from (4.7) that

$$\|\mathcal{H}[w_k, \sigma_k]'\|_\infty \le \varphi^{-1}(c), \quad c = f_\eta \ell_L (r_1 - r_0), \tag{4.9}$$

since φ^{-1} is strictly increasing in $\mathbb{R}^+$ by (ϕ) and $\wp$ is assumed to be nondecreasing in $\mathbb{R}_0^+$. Consequently, $(\mathcal{H}[w_k, \sigma_k])_k$ is equi–bounded in X and equi–Lipschitz continuous in $[r_0, r_1] \times [0,1]$. Define

$$\mathcal{I}_k(r_0, r) = \int_{r_0}^{r} \wp(\tau) f(w_k(\tau))\ell(|w_k'(\tau)|)d\tau \quad \text{and} \quad \mathcal{J}_k(r_0, r) = \frac{\mathcal{I}_k(r_0, r)}{\wp(r)}.$$

Then for all r, with $0 < r_0 \le r \le r_1$,

$$0 \le \mathcal{J}_k(r_0, r) \le c \quad \text{and} \quad \lim_{r\to r_0^+} \mathcal{J}_k(r_0, r) = 0,$$

where c is given in (4.9).

Now, fix $\varepsilon > 0$ and let $\delta = \delta(\varphi^{-1}, \varepsilon) > 0$ be the corresponding number of the uniform continuity of φ^{-1} in $[0, c]$. Take any r, s, with $0 < r_0 \le r < s \le r_1$ and $|s - r| < \delta/C$, where

$$C = f_\eta \ell_L \left(1 + \kappa\right), \quad \text{where } \kappa = \max_{t\in[r_0,r_1]} \frac{t\wp'(t)}{\wp(t)}.$$

This is possible since $r_0 > 0$ and $\wp(t) \ge \wp(r_0) > 0$. Now, for some $\xi \in (r, s)$

$$\frac{|\wp(s) - \wp(r)|}{\wp(s)}(r - r_0) = \frac{\wp'(\xi)|s - r|}{\wp(s)}\, s\, \frac{r - r_0}{s} \le \frac{s\wp'(s)}{\wp(s)}\, |s - r|,$$

since $\wp'$ is nondecreasing in $\mathbb{R}_0^+$. Therefore for all k

$$\Big|\sigma_k \mathcal{J}_k(r_0, r) - \sigma_k \mathcal{J}_k(r_0, s)\Big| \le \left|\frac{\wp(s) - \wp(r)}{\wp(r)\wp(s)}\mathcal{I}_k(r_0, r) - \frac{1}{\wp(s)}\mathcal{I}_k(r, s)\right|$$

$$
\begin{aligned}
&\leq \frac{1}{\wp(s)}\,|\mathcal{I}_k(r,s)| + \frac{|\wp(s)-\wp(r)|}{\wp(r)\wp(s)}\,|\mathcal{I}_k(r_0,r)| \\
&\leq f_\eta \ell_L \left(|s-r| + \frac{|\wp(s)-\wp(r)|}{\wp(r)\wp(s)} \int_{r_0}^{r} \wp(\tau)d\tau \right) \qquad (4.10)\\
&\leq f_\eta \ell_L \left(|s-r| + \frac{|\wp(s)-\wp(r)|}{\wp(s)} (r-r_0) \right) \\
&\leq f_\eta \ell_L \left(1 + \frac{s\wp'(s)}{\wp(s)} \right) |s-r| \leq C|s-r| < \delta.
\end{aligned}
$$

In conclusion, we have for all r, s, with $0 < r_0 \leq r < s \leq r_1$ and $|s-r| < \delta/C$

$$
\left|\mathcal{H}[w_k,\sigma_k]'(r) - \mathcal{H}[w_k,\sigma_k]'(s)\right| = \left|\varphi^{-1}\left(\sigma_k \mathcal{J}_k(r_0,r)\right) - \varphi^{-1}\left(\sigma_k \mathcal{J}_k(r_0,s)\right)\right| < \varepsilon,
$$

uniformly in k.

As an immediate consequence of the Ascoli–Arzelà theorem $\mathcal{H}$ then maps bounded sequences into relatively compact sequences in X, so $\mathcal{H}$ is compact.

To apply the Leray–Schauder theorem it is now enough to show that there is a constant $M > 0$ such that

$$
\|w\| \leq M \quad \text{for all } (w,\sigma) \in X \times [0,1], \text{ with } \quad \mathcal{H}[w,\sigma] = w. \tag{4.11}
$$

Let (w,σ) be a pair of type (4.11). But, as observed above, $w \geq 0$, $w' \geq 0$ in $[r_0, r_1]$, being $w = \mathcal{H}[w,\sigma]$, so that $\|w\|_\infty = w(r_1) \leq \sigma\eta \leq \eta$. We claim that there exists $s_1 = s_1(w,\eta)$, with $r_0 \leq s_1 < r_1$, such that $w' > 0$ in $(s_1, r_1]$ and $w' \equiv 0$ in $[r_0, s_1]$. Indeed, the set $W_+ = \{r \in [r_0, r_1] : w'(r) > 0\}$ is nonempty, being $0 \leq w(r_0) < \eta$ and $w(r_1) = \eta$, and (relatively) open in $[r_0, r_1]$, being $w \in C^1[r_0, r_1]$. Put $s_1 = \inf W_+$. Clearly $s_1 \in [r_0, r_1)$ and $w \equiv w(r_0)$ in $[r_0, s_1]$, since we already proved that $w \geq w(r_0)$ and $w' \geq 0$ in $[r_0, r_1]$. Now, for any fixed $r \in (s_1, r_1]$ there exists $s \in (s_1, r)$ such that $w'(s) > 0$ and integrating the equation in (4.8) on $[s, r]$ we get

$$
\int_s^r [\wp\, A(|w'|)w']' d\tau = \sigma \int_s^r \wp\, f(w)\, \ell(|w'|) d\tau \geq 0,
$$

that is $\wp(r)A(|w'(r)|)w'(r) \geq \wp(s)A(|w'(s)|)w'(s) > 0$. Hence, $w' > 0$ in $(s_1, r_1]$, $w'(s_1) = 0$ being $s_1 \geq r_0$ and $w'(r_0) = 0$. In particular, $w > w(r_0) \geq 0$ in $(s_1, r_1]$ and $w < \eta$ in $[r_0, r_1)$, as shown above.

Moreover, w' is differentiable in $(s_1, r_1]$ and by the equation in (4.8)

$$[\wp\,\varphi(w')]' = \sigma\wp f(w)\ell(|w'|),$$

which is equivalent in $(s_1, r_1]$ to (4.2). By (4.2) and the fact that $\wp$ is nondecreasing in $\mathbb{R}_0^+$, we get at once that in $(s_1, r_1]$

$$\frac{\varphi'(w')}{\ell(w')}\, w'' \le f(w).$$

Multiplying by $w' > 0$, integrating on $[s_1, r]$, $r \in (s_1, r_1]$, we have

$$\begin{aligned} K(w'(r)) = \int_0^{w'(r)} \frac{s\varphi'(s)}{\ell(s)}\, ds &= \int_{s_1}^{r} \frac{w'\varphi'(w')}{\ell(w')}\, w'' ds \\ &\le F(w(r)) - F(w(s_1)) \le F(w(r)) \le F(\eta). \end{aligned} \tag{4.12}$$

Since $w \equiv w(r_0)$ in $[r_0, s_1]$, we have shown the important *a priori estimate* for w'

$$0 \le w'(r) \le K^{-1}\left(F(\eta)\right) = W \quad \text{for all } r \in [r_0, r_1]. \tag{4.13}$$

Hence, by (4.13) also $\|w'\|_\infty \le W$. Thus we can take $M = \eta + W$ in (4.11).

The Leray–Schauder theorem therefore implies that the mapping $\mathcal{T}[w] = \mathcal{H}[w, 1]$ has a fixed point $w \in X$, which is the required solution of (4.1), proving the assertion above.

If $w(r_0) = 0$, that is $w \equiv w(r_0) = 0$ in $[r_0, s_1]$, then (4.13) and integration on $[s_1, r_1]$ give

$$\int_0^{\eta} \frac{du}{K^{-1}(F(u))} = \int_{s_1}^{r_1} \frac{w'(r)dr}{K^{-1}(F(w(r)))} \le r_1 - s_1 < \infty,$$

that is (4.3) holds.

Finally, if $r_0 = 0$ and (4.4) holds, then we can proceed word by word as in the case $r_0 > 0$. The only change occurs at the end of (4.10) where now

$$\kappa = \sup_{t \in (0, r_1]} \frac{t\wp'(t)}{\wp(t)},$$

which is finite by (4.4).

In particular, we have shown under the assumptions of Theorem 4.1, with also (4.4) when $r_0 = 0$, that *for all* r_0, r_1, *with* $0 \le r_0 < r_1$, *problem* (4.1) *admits a classical maximal solution* w *in* $[r_0, R)$, *where* R *is defined by*

$$R = \sup\{\tau \ge r_1 \ : \ w \text{ can be defined in } [r_0, \tau] \text{ as a solution of (4.1)}\}.$$

Of course, $R > r_1$, by the use of the standard initial value problem theory, being $w(r_1) = \eta$, $w'(r_1) > 0$. Furthermore, *there exists* $s_1 \in [r_0, r_1)$ *such that* $w(r) \equiv w(r_0) \ge 0$ *in* $[r_0, s_1]$ *and*

$$w' > 0 \quad \textit{in} \quad (s_1, R). \tag{4.14}$$

In particular, when $r_0 = 0$, the function $v = v(|z|) = w(r)$, $r = |z|$, is a radial stationary solution of (1.1) when $\wp(r) = r^{2m-1}$ in the open ball B_R of $\mathbb{H}^m$.

Theorem 4.2. *Assume that* $\wp \in C^1(\mathbb{R}_0^+)$, *with* $\wp$ *and* $\wp'$ *nondecreasing in* $\mathbb{R}_0^+$, *and* $\wp > 0$ *in* $\mathbb{R}^+$. *Suppose furthermore that* $f(0) = 0$, $\ell(0) > 0$ *and* $(VsKO)$ *holds. Then any maximal solution* v, *constructed in Theorem* 4.1, *is a* C^1 *maximal solution of*

$$[\wp \ A(|v'|)v']' = \wp f(v)\ell(|v'|) \tag{4.15}$$

in (r_0, R), *and* v *has the property that* $R = \infty$. *If furthermore* (1.5) *holds and*

$$\limsup_{r\to\infty} \frac{1}{\wp(r)} \int_{r_1}^{r} \wp(s)ds = \infty, \tag{4.16}$$

then v *has also the property that* $v^* = \lim_{r\to\infty} v(r) = \infty$ *and* $v > 0$ *in* $I = (r_0, \infty)$, $v' > 0$ *in* I *and* $v \in C^2(I)$.

In particular, v *is a positive entire large radial stationary solution of* (1.1) *when* $r_0 = 0$ *and* $\wp(r) = r^{2m-1}$.

Proof. Let v be a classical maximal solution of (4.15) in $[r_0, R)$, constructed as in Theorem 4.1. We want to show that v is global, namely that $R = \infty$. Suppose by contradiction that $R < \infty$. We claim that, if $R < \infty$, then necessarily

$$\lim_{r\to R^-} v(r) = v^* = \infty, \tag{4.17}$$

where the existence of the limit is guaranteed by the monotonicity of v, that is by (4.14). To prove (4.17), assume by contradiction that the limit is finite, that is $v^* \in (\eta, \infty)$. Since $v' > 0$ in (s_1, R), from (4.15) it follows that $[\wp\, \varphi(v')]' > 0$ in (s_1, R), therefore the function $\wp\, \varphi(v')$ is monotone increasing and approaches a limit as $r \to R^-$. Consequently, being $\wp$ positive and continuous in $r = R$, also $\varphi(v'(r))$ approaches a limit as $r \to R^-$. In turn, since $\varphi : \mathbb{R}_0^+ \to [0, a)$, $0 < a \le \infty$, is a homeomorphism, then v' approaches a limit v'_R as $r \to R^-$, with $v'_R \in [0, a)$. As shown in (4.12)

$$K(v'(r)) \le \int_{s_1}^{r} f(v)v'\, ds \le F(v(r)) \le F(v^*). \tag{4.18}$$

By the invertibility of K and the definition of v we have $0 \le v'(r) \le V^*$ for all $r \in [r_0, R)$, where $V^* = K^{-1}(F(v^*))$. It follows at once that $v'_R < \infty$, contradicting the maximality of R. Hence the claim (4.17).

Now we prove that if $v^* = \infty$, then $R = \infty$, obtaining the required contradiction. By (4.18), as noted above, $K(v'(r)) \le F(v(r))$ in (s_1, R). Consequently, $v'(r) \le K^{-1}(F(v(r)))$ in $[s_1, R)$, and by integration on $[s_1, r]$, with $r \in (s_1, R)$, we obtain

$$\int_{v(s_1)}^{v(r)} \frac{ds}{K^{-1}(F(s))} = \int_{s_1}^{r} \frac{v'(s)}{K^{-1}(F(v(s)))}\, ds \le R - s_1.$$

By $(VsKO)$ and the fact that $v^* = \infty$, we get a contradiction by letting $r \to R^-$, because the left hand side term goes to infinity. In conclusion the case $R < \infty$ cannot occur, and so $R = \infty$, as stated.

Now we prove the second part of the theorem, namely that $v^* = \infty$, under conditions (1.5) and (4.16). Assume by contradiction that $v^* < \infty$. By (4.14) and (4.18) we have $0 < v'(r) \le V^*$ for all $r \in I = (r_0, \infty)$, where $V^* = K^{-1}(F(v^*))$, as defined above. Furthermore, $\ell_* = \min_{s \in [0, V^*]} \ell(s) > 0$ by $(\mathscr{H})$ and the assumption $\ell(0) > 0$.

Moreover, (4.15) is valid in $I = (r_0, \infty)$, since $v(r_1) = \eta$ by (4.1). Now, $v > \eta$ in (r_1, ∞) by (4.14), f is nondecreasing in $\mathbb{R}_0^+$ and $f(\eta) > 0$ by $(\mathscr{H})$, so that $[\wp\, A(|v'|)v']' \ge c\, \wp$ in $[r_1, \infty)$, where $c = f(\eta)\ell_* > 0$. Thus, using that

$0 < v'(r) \le V^* < \infty$ in (r_1, ∞) and integrating on $[r_1, r]$ for all $r > r_1$, we get

$$\varphi(V^*) \ge \varphi(v'(r)) \ge \frac{\wp(r_1)}{\wp(r)}\varphi(v'(r_1)) + \frac{c}{\wp(r)}\int_{r_1}^{r} \wp(s)ds \ge \frac{c}{\wp(r)}\int_{r_1}^{r} \wp(s)ds$$

by (ϕ). By letting $r \to \infty$, assumption (4.16) gives the obvious contradiction $\varphi(V^*) = \infty$. Therefore, $v^* = \infty$, as stated.

Since v solves (4.1), clearly $v(r_0) \ge 0$, but the case $v(r_0) = 0$ cannot occur by Theorem 4.1 thanks to assumption (1.5). Since $v' \ge 0$ in $[r_0, \infty)$, it then follows that $v > 0$ in $[r_0, \infty)$. Integrating (4.15) in $[r_0, r]$, by $(\mathscr{H})$ and being $\ell(0) > 0$, we get

$$\wp(r)\varphi(v'(r)) = \int_{r_0}^{r} \wp(s)f(v)\ell(v')ds > 0.$$

Thus (ϕ) yields that $v'(r) > 0$ for all $r > r_0$ and

$$v'(r) = \varphi^{-1}\left(\frac{1}{\wp(r)}\int_{r_0}^{r} \wp(s)f(v)\ell(v')ds\right).$$

Hence v' is differentiable in I, with

$$v'' = \frac{\ell(v')}{\varphi'(v')}\left[f(v) - \frac{\wp'}{\wp}\frac{\varphi(v')}{\ell(v')}\right] \quad \text{in } I. \tag{4.19}$$

In particular, $v \in C^2(I)$.

The last part of the theorem is just a consequence of the fact that $\wp(r) = r^{2m-1}$ verifies (4.4) and (4.16), taking $r_0 = 0$ in Theorem 4.1. Thus the maximal solution $v = v(r)$, $r = |z|$, is a positive entire large radial stationary solution of (1.1).

Proof of Theorem 1.1. It is enough to apply Theorems 4.1 and 4.2, with $r_0 = 0$ and $\wp(r) = r^{2m-1}$, $m \ge 1$, to the radial stationary version of (1.1).

5. Qualitative properties and uniqueness

We now turn to the radial stationary equation of (1.1) and assume throughout the section that $(\mathscr{H})$ *and* (ϕ) *hold, with* $\ell(0) > 0$, without further mentioning.

Proposition 5.1. *Problem*

$$\begin{aligned} &[r^{2m-1}A(|v'|)v']' = r^{2m-1}f(v)\ell(|v'|) \qquad \textit{in } \mathbb{R}^+, \\ &v(0) = v_0 > 0, \qquad v'(0) = 0. \end{aligned} \tag{5.1}$$

has a solution on some interval $[0, r_0]$*,* $r_0 > 0$*.*

Proof. Any local solution of (5.1), for small $r > 0$, must be a fixed point of the operator

$$\mathcal{T}[v](r) = v_0 + \int_0^r \varphi^{-1}\left(\frac{1}{s^{2m-1}}\int_0^s \tau^{2m-1} f(v(\tau))\ell(|v'(\tau)|)d\tau\right)ds. \tag{5.2}$$

Fix $\varepsilon > 0$ so small that $[v_0-\varepsilon, v_0+\varepsilon] \subset \mathbb{R}^+$, so that by $(\mathscr{H})$

$$\begin{aligned} 0 < i = \min_{[v_0-\varepsilon,v_0+\varepsilon]} f(u) \ &\le \max_{[v_0-\varepsilon,v_0+\varepsilon]} f(u) = M < \infty, \\ 0 < l = \min_{[0,\varepsilon]} \ell(t) \ &\le \max_{[0,\varepsilon]} \ell(t) = L < \infty. \end{aligned}$$

Let $r_0 = r_0(\varepsilon)$ be so small that

$$r_0\varphi^{-1}(r_0LM) + \varphi^{-1}(r_0LM) \le \varepsilon. \tag{5.3}$$

This can be done since $\varphi^{-1}(0) = 0$ by (ϕ). Denote by $C^1[0, r_0]$ the usual Banach space of real functions of class C^1 in $[0, r_0]$, endowed with the norm $u \mapsto \|u\| = \|u\|_\infty + \|u'\|_\infty$. Put $v_0(r) \equiv v_0 \in C^1[0, r_0]$ and let

$$C = \{v \in C^1[0, r_0] \ : \ \|v - v_0\| \le \varepsilon\},$$

that is $v \in C$ if and only if $\|v-v_0\|_\infty + \|v'\|_\infty \le \varepsilon$. Clearly C is the closed ball in $C^1[0, r_0]$ of center v_0 and radius $\varepsilon > 0$, so that C is closed, convex and bounded in $C^1[0, r_0]$. If $v \in C$ then $v([0, r_0]) \subset [v_0-\varepsilon, v_0+\varepsilon]$ and $v'([0, r_0]) \subset [-\varepsilon, \varepsilon]$, and in turn $0 < f(v(r)) \le M$ and $0 < \ell(|v'(r)|) \le L$ for all $r \in [0, r_0]$. Furthermore,

$$0 \le \int_0^s \left(\frac{\tau}{s}\right)^{2m-1} f(v(\tau))\ell(|v'(\tau)|)d\tau \le \int_0^s f(v(\tau))\ell(|v'(\tau)|)d\tau, \quad 0 < s \le r_0,$$

where the last integral approaches 0 as $s \to 0^+$ by $(\mathscr{H})$. Thus the operator $\mathcal{T}$ in (5.2) is well defined.

We show that $\mathcal{T} : C \to C$ and that $\mathcal{T}$ is compact. Indeed for $v \in C$ we have

$$\begin{aligned}\|\mathcal{T}[v] - v_0\|_\infty &= \int_0^{r_0} \varphi^{-1}\left(\int_0^s \left(\frac{\tau}{s}\right)^{2m-1} f(v(\tau))\ell(|v'(\tau)|)d\tau\right) ds \\ &\le r_0\varphi^{-1}(r_0LM) \\ \|\mathcal{T}[v]'\|_\infty &\le \varphi^{-1}\left(\int_0^{r_0} f(v(\tau))\ell(|v'(\tau)|)d\tau\right) \le \varphi^{-1}(r_0LM).\end{aligned}$$

Thus $\mathcal{T}[v] \in C$ and so $\mathcal{T}(C) \subset C$ by (5.3). Let $(v_k)_k$ be a sequence in C and fix r, t be two points in $[0, r_0]$. Then

$$\begin{aligned}|\mathcal{T}[v_k](r) - \mathcal{T}[v_k](t)| &= \left|\int_r^t \varphi^{-1}\left(\int_0^s \left(\frac{\tau}{s}\right)^{2m-1} f(v_k(\tau))\ell(|v_k'(\tau)|)d\tau\right) ds\right| \\ &\le \varphi^{-1}(LM)|r-t|.\end{aligned}$$

Furthermore, as in (4.10), we compute

$$\left|\frac{\mathcal{I}_k(r)}{r^{2m-1}} - \frac{\mathcal{I}_k(t)}{t^{2m-1}}\right| \le LM\Big(|r-t| + (2m-1)\,|r-t|\Big) = 2mLM\,|r-t|\,,$$

where as in Theorem 4.1

$$\mathcal{I}_k(r) = \int_0^r \tau^{2m-1} f(v_k(\tau))\ell(|v_k'(\tau)|)d\tau. \tag{5.4}$$

Now for all $\sigma > 0$ there exists $\delta = \delta(\varphi^{-1}, \sigma) > 0$, thanks to the uniform continuity of φ^{-1} in $[0, r_0LM]$, such that for all r, $t \in [0, r_0]$, with $|r - t| < \delta/2mLM$, we have for all k

$$|\mathcal{T}[v_k]'(r) - \mathcal{T}[v_k]'(t)| = \left|\varphi^{-1}\left(\frac{\mathcal{I}_k(r)}{r^{2m-1}}\right) - \varphi^{-1}\left(\frac{\mathcal{I}_k(t)}{t^{2m-1}}\right)\right| \le \sigma.$$

Therefore, by the Ascoli–Arzelà theorem $\mathcal{T}$ maps bounded sequences into relatively compact sequences, with limit points in C, since C is closed.

Finally $\mathcal{T}$ is continuous, because if $v \in C$ and $(v_k)_k \subset C$ are such that $\|v_k - v\|$ tends to 0 as $k \to \infty$, then by the Lebesgue dominated convergence theorem, we can pass under the sign of integrals twice in (5.2), and so $\mathcal{T}[v_k]$ tends to $\mathcal{T}[v]$ pointwise in $[0, r_0]$ as $k \to \infty$. By the above argument, it is obvious that $\|\mathcal{T}[v_k] - \mathcal{T}[v]\| \to 0$ as $k \to \infty$ as claimed.

By the Schauder fixed point theorem, $\mathcal{T}$ possesses a fixed point v in C. Clearly, $v \in C^1[0, r_0]$ by the representation formula (5.2), that is

$$v(r) = v_0 + \int_0^r \varphi^{-1}\left(\int_0^s \left(\frac{\tau}{s}\right)^{2m-1} f(v(\tau))\ell(|v'(\tau)|)d\tau\right) ds, \tag{5.5}$$

as desired.

Once it is known that a solution v of (5.1) exists, then v necessarily obeys to (5.5). In particular, problem (5.1) admits a classical maximal solution v in $[0, R)$, where R is defined by

$$R = \sup\{r \geq r_0 \ : \ v \text{ can be defined in } [0, r] \text{ as a solution of (5.1)}\}.$$

Of course, $R > r_0$, by the use of the standard initial value problem theory, being $v(r_0) > 0$, $v'(r_0) > 0$. Furthermore, the solution $v = v(r) = v(|z|)$, $r = |z|$, is a radial stationary solution of (1.1) in the open ball B_R of $\mathbb{H}^m$.

Proof of Theorem 1.2. Let v_1 and v_2 be two C^1 solutions of (5.1), and $[0, \tilde{R})$ be the maximal interval in which both v_1 and v_2 exist. Assume by contradiction that there exists $\rho_0 \in (0, \tilde{R})$ such that $v_1(\rho_0) \neq v_2(\rho_0)$. Let R, with $\rho_0 < R < \tilde{R}$, be fixed. Then $v_1' > 0$ and $v_2' > 0$ in $(0, R]$ by (5.5). Put $V = \max\{v_1(R),\, v_2(R)\}$ and

$$V' = \max\left\{ \max_{r \in [0,R]} v_1'(r),\ \max_{r \in [0,R]} v_2'(r) \right\}.$$

We denote by L and $L_{\varphi^{-1}}$ the Lipschitz constants of ℓ and φ^{-1} in $[0, V']$, respectively, and by M the Lipschitz constant of f in $[v_0, V]$. Set

$$f_1 = \max_{t \in [v_0, V]} f(t), \qquad l_1 = \max_{t \in [0, V']} \ell(t).$$

Fix $r \in [0, R]$. Then

$$\begin{aligned}
|f(v_1)\ell(v_1') - f(v_2)\ell(v_2')| &\leq \ell(v_1')|f(v_1) - f(v_2)| + f(v_2)|\ell(v_1') - \ell(v_2')| \\
&\leq l_1 M |v_1 - v_2| + f_1 L |v_1' - v_2'| \\
&\leq l_1 M \int_0^r |v_1' - v_2'|\, ds + f_1 L |v_1' - v_2'|.
\end{aligned} \tag{5.6}$$

Choose $\delta > 0$ so small that

$$L_{\varphi^{-1}} c_\delta < 1 \quad \text{where} \quad c_\delta = \frac{l_1 M \delta^2}{2} + f_1 L \delta. \tag{5.7}$$

Since, for all $r \in (0, \delta]$

$$\left| \frac{\mathcal{I}_1(r)}{r^{2m-1}} - \frac{\mathcal{I}_2(r)}{r^{2m-1}} \right| \leq \int_0^r \left(\frac{s}{r}\right)^{2m-1} |f(v_1)\ell(v_1') - f(v_2)\ell(v_2')| ds$$

$$
\begin{aligned}
&\leq \int_0^r |f(v_1)\ell(v_1') - f(v_2)\ell(v_2')| ds \\
&\leq l_1 M \int_0^r ds \int_0^s |v_1' - v_2'|\, d\tau + f_1 L \int_0^r |v_1' - v_2'|\, ds \\
&\leq \frac{l_1 M \delta^2}{2} \max_{r\in[0,\delta]} |v_1'(r) - v_2'(r)| + f_1 L\delta \max_{r\in[0,\delta]} |v_1'(r) - v_2'(r)| \\
&= c_\delta \max_{r\in[0,\delta]} |v_1'(r) - v_2'(r)|,
\end{aligned}
$$

then

$$
\begin{aligned}
|v_1'(r) - v_2'(r)| &= \left| \varphi^{-1}\left(\frac{\mathcal{I}_1(r)}{r^{2m-1}}\right) - \varphi^{-1}\left(\frac{\mathcal{I}_2(r)}{r^{2m-1}}\right) \right| \\
&\leq L_{\varphi^{-1}} \left| \frac{\mathcal{I}_1(r)}{r^{2m-1}} - \frac{\mathcal{I}_2(r)}{r^{2m-1}} \right| \\
&\leq L_{\varphi^{-1}} c_\delta \max_{r\in[0,\delta]} |v_1'(r) - v_2'(r)|,
\end{aligned}
$$

Therefore

$$
\max_{r\in[0,\delta]} |v_1'(r) - v_2'(r)| \leq L_{\varphi^{-1}} c_\delta \max_{r\in[0,\delta]} |v_1'(r) - v_2'(r)|, \tag{5.8}
$$

so that $v_1' \equiv v_2'$ on $[0,\delta]$ by (5.7). Hence, $v_1 \equiv v_2$ on $[0,\delta]$, since $v_1(0) = v_2(0) = v_0$. Repeating the argument a finite number of times, being $[0,R]$ compact, we get that $v_1 \equiv v_2$ on $[0,R]$. This is impossible since $\rho_0 \in [0,R]$ and completes the proof.

Remark 5.2. Clearly Theorem 1.2 can be applied both in the p–Laplacian case, $\varphi(s) = s^{p-1}$ when $p \in (1,2]$ and in the generalized mean curvature case, $\varphi(s) = s(1+s^2)^{(p-2)/2}$, $p \in (1,2)$. Finally, Theorem 1.2 cannot be applied in the p–Laplacian case when $p > 2$, since φ^{-1} fails to be of class $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_0^+)$.

6. Nonexistence under nonincreasing b–monotonicity on ℓ

We recall that conditions (ϕ), (ϕL) and $(\mathscr{H})$ are assumed throughout the paper.

Lemma 6.1. *Assume that $(\phi 1)$ holds. Let ℓ be b–nonincreasing in $\mathbb{R}^+$ and f nondecreasing in $\mathbb{R}_0^+$. Fix*

$$
0 < \varepsilon < \eta < a < \infty, \quad \textit{and} \quad 0 < r_0 < r_1 < \infty.
$$

Then, there exist a finite radius $R > r_1$ and a strictly increasing, convex function $\alpha : [r_0, R) \longrightarrow [\varepsilon, a)$, $\alpha \in C^2[r_0, R)$, such that for every $q \in \mathbb{H}^m$ the radial function $v = \alpha \circ d_q$ satisfies

$$\begin{cases} \Delta^{\varphi}_{\mathbb{H}^m} v \le f(v)\ell(|D_{\mathbb{H}^m} v|_{\mathbb{H}^m}) & \text{in } B_R(q) \setminus \overline{B_{r_0}(q)} \\ v = \varepsilon \quad \text{on } \partial B_{r_0}(q), \\ v = a \quad \text{on } \partial B_R(q), \\ \varepsilon \le v \le \eta \quad \text{on } B_{r_1}(q) \setminus B_{r_0}(q). \end{cases} \tag{6.1}$$

Proof. Fix ε, η, a, r_0 and r_1 as in the statement. Let $\sigma \in (0, 1]$ be a parameter to be determined later and choose $R_\sigma > r_0$ such that

$$R_\sigma - r_0 = \int_\varepsilon^a \frac{ds}{K^{-1}(\sigma F(s))}. \tag{6.2}$$

Clearly R_σ is uniquely determined and *finite*, being a finite. Moreover, since the right hand side diverges as $\sigma \to 0^+$, there exists σ so small that $R = R_\sigma > r_1$. We implicitly define the function α_σ for all $r \in [r_0, R)$ by

$$R = r + \int_{\alpha_\sigma(r)}^a \frac{ds}{K^{-1}(\sigma F(s))}.$$

By construction, $\alpha_\sigma(r_0) = \varepsilon$ by (6.2). Moreover, since $K^{-1}(\sigma F) > 0$ and the integral in (6.2) is finite, then $\alpha_\sigma(r) \uparrow a$ as $r \to R^-$. A first differentiation yields

$$\alpha'_\sigma = K^{-1}(\sigma F(\alpha_\sigma)).$$

Hence α_σ is monotone increasing and $\sigma F(\alpha_\sigma) = K(\alpha'_\sigma)$ in $[r_0, R)$. Differentiating once more we get

$$\sigma f(\alpha_\sigma)\alpha'_\sigma = K'(\alpha'_\sigma)\alpha''_\sigma = \frac{\alpha'_\sigma \varphi'(\alpha'_\sigma)}{\ell(\alpha'_\sigma)}\,\alpha''_\sigma.$$

Thus α_σ is strictly convex, being $\alpha_\sigma > 0$ and $\alpha''_\sigma > 0$ by $(\mathscr{H})$, so that

$$[\varphi(\alpha'_\sigma)]' = \varphi'(\alpha'_\sigma)\alpha''_\sigma = \sigma f(\alpha_\sigma)\ell(\alpha'_\sigma). \tag{6.3}$$

Now set $v = \alpha \circ d_q$, so that v is a radial function in $\mathbb{H}^m$ and $v \in C^2_H$ radial function on $B_{R_\sigma}(q) \setminus B_{r_0}(q)$, where $v \in C^2_H$ means that the horizontal gradient of v is well defined and continuous. For further details we refer to [2] and [11].

We claim that there exists $\sigma \in (0,1]$, σ sufficiently small and independent of q, such that v is the required solution of (6.2). For simplicity in what follows we write α in place of α_σ. Hence, considering ($\mathscr{H}$), the positivity of α', the radial expression (3.3), together with (3.2), and (ϕ1) with $s = \psi^{\frac{1}{2}} \in (0,1]$ by (2.2), we have

$$\begin{aligned}
\frac{\Delta^{\varphi}_{\mathbb{H}^m} v}{f(v)\ell(|D_{\mathbb{H}^m} v|_{\mathbb{H}^m})} &= \frac{\psi\varphi'(\alpha'(r)\psi^{\frac{1}{2}})\alpha''(r)}{f(\alpha)\ell(\alpha'(r)\psi^{\frac{1}{2}})} + \frac{2m+1}{r} \cdot \frac{\psi^{\frac{1}{2}}\varphi(\alpha'(r)\psi^{\frac{1}{2}})}{f(\alpha)\ell(\alpha'(r)\psi^{\frac{1}{2}})} \\
&\le \frac{\theta}{b} \cdot \psi^{1-\tau/2} \cdot \frac{\varphi'(\alpha'(r))\alpha''(r)}{f(\alpha)\ell(\alpha'(r))} + \frac{2m+1}{r} \cdot \frac{\psi^{\frac{1}{2}}\varphi(\alpha'(r)\psi^{\frac{1}{2}})}{b\, f(\alpha)\ell(\alpha'(r))} \\
&\le \frac{\theta}{b}\sigma + \frac{2m+1}{r} \cdot \frac{\psi^{\frac{1}{2}}\varphi(\alpha'(r)\psi^{\frac{1}{2}})}{b\, f(\alpha)\ell(\alpha'(r))},
\end{aligned}$$

where in the last two inequalities we have used that $\ell(\alpha'(r)\psi^{\frac{1}{2}}) \ge b\ell(\alpha'(r))$ by the nonincreasing b–monotonicity of ℓ, (6.3) and the fact that $1-\tau/2 > 0$ being $\tau < 1$ by (ϕ1). Furthermore, $s^{\tau-1}\varphi(st) \le \theta\varphi(t)$ for all $s \in (0,1]$ and $t \in \mathbb{R}^+_0$, integrating (ϕ1) with respect to the variable t. Hence,

$$\frac{\psi^{\frac{1}{2}}\varphi(\alpha'(r)\psi^{\frac{1}{2}})}{f(\alpha)\ell(\alpha'(r))} \le \psi^{1-\tau/2} \cdot \frac{\theta\,\varphi(\alpha'(r))}{f(\alpha)\ell(\alpha'(r))} \le \frac{\theta\,\varphi(\alpha'(r))}{f(\alpha)\ell(\alpha'(r))}$$

as above.

On the other hand, (6.3) and an integration over $[r_0, r]$, $r_0 < r < R$, yield

$$\varphi(\alpha'(r)) = \varphi(\alpha'(r_0)) + \sigma \int_{r_0}^{r} f(\alpha(s))\ell(\alpha'(s))ds.$$

In turn, using the monotonicity of f and the b–monotonicity of ℓ we deduce

$$\begin{aligned}
\frac{\varphi(\alpha'(r))}{f(\alpha)\ell(\alpha'(r))} &= \frac{\varphi(\alpha'(r_0))}{f(\alpha)\ell(\alpha'(r))} + \frac{\sigma}{f(\alpha)\ell(\alpha'(r))}\int_{r_0}^{r} f(\alpha(s))\ell(\alpha'(s))ds \\
&\le \frac{\varphi(\alpha'(r_0))}{bf(\alpha(r_0))\ell(\alpha'(R))} + \sigma\frac{f(\alpha(r))\int_{r_0}^{r}\ell(\alpha'(s))\,ds}{bf(\alpha(r))\ell(\alpha'(R))} \\
&\le \frac{\varphi(\alpha'(r_0))}{bf(\alpha(r_0))\ell(\alpha'(R))} + \sigma\frac{\ell(0)}{b^2\ell(\alpha'(R))}(r-r_0).
\end{aligned}$$

Combining all the above estimates we get for all r with $r_0 < r < R$

$$\begin{aligned}
\frac{\Delta^{\varphi}_{\mathbb{H}^m} v}{f(v)\ell(|D_{\mathbb{H}^m} v|_{\mathbb{H}^m})} &\le \frac{\theta\sigma}{b} + \frac{2m+1}{br}\left[\frac{\varphi(\alpha'(r_0))}{bf(\alpha(r_0))\ell(\alpha'(R))} + \sigma\frac{\ell(0)}{b^2\ell(\alpha'(R))}(r-r_0)\right] \\
&\le \frac{\sigma}{b}\left[\theta + \frac{2m+1}{b^2}\frac{\ell(0)}{\ell(\alpha'(R))}\right] + \frac{2m+1}{b^2 r_0}\frac{\varphi(\alpha'(r_0))}{f(\alpha(r_0))\ell(\alpha'(R))}
\end{aligned}$$

Since $K(0) = 0$ and $\alpha(r_0) = \varepsilon$, by (6.3) we have $\alpha'(r_0) = K^{-1}(\sigma F(\varepsilon)) \to 0$ as $\sigma \to 0$. We take σ so small, say $\sigma \leq \bar{\sigma}$, in order to satisfy

$$\frac{\sigma}{b}\left[\theta + \frac{2m+1}{b^2}\frac{\ell(0)}{\ell(\alpha'(R))}\right] + \frac{2m+1}{b^2 r_0}\frac{\varphi(\alpha'(r_0))}{f(\alpha(r_0))\ell(\alpha'(R))} \leq 1.$$

This can be done, since $\alpha'(R) = K^{-1}(\sigma F(\alpha(R))) = K^{-1}(\sigma F(a)) \to 0$ as $\sigma \to 0^+$ and $\ell(0) > 0$.

In turn the claim is proved being v a radial solution of

$$\Delta^{\varphi}_{\mathbb{H}^m} v \leq f(v)\ell(|D_{\mathbb{H}^m} v|_{\mathbb{H}^m})$$

in $B_R(q) \setminus \overline{B_{r_0}(q)}$, with $r_0 < R < \infty$, by $(\mathscr{H})$.

It remains to show that $\varepsilon \leq v \leq \eta \quad$ on $B_{r_1}(q) \setminus B_{r_0}(q)$. To this aim we observe that by the monotonicity of α it is enough to verify that $\alpha(r_1) = \alpha_\sigma(r_1) \leq \eta$ for a certain σ, even smaller if necessary. Hence, from the trivial identity

$$\begin{aligned}\int_{\alpha(r_1)}^{a} \frac{ds}{K^{-1}(\sigma F(s))} &= R - r_1 = (R - r_0) + (r_0 - r_1)\\ &= \int_{\varepsilon}^{a} \frac{ds}{K^{-1}(\sigma F(s))} + r_0 - r_1\end{aligned}$$

and the fact that $\alpha(r_1) > \varepsilon$, we deduce

$$\int_{\varepsilon}^{\alpha(r_1)} \frac{ds}{K^{-1}(\sigma F(s))} = r_1 - r_0.$$

On the other hand, taking $\sigma > 0$ so small that $\int_{\varepsilon}^{\eta} ds/K^{-1}(\sigma F(s)) > r_1 - r_0$, then $\alpha(r_1) \leq \eta$. This completes the proof of the lemma.

Proof of Theorem 1.3. Let u be a nonnegative bounded entire solution of (1.2). We denote $u^* = \sup_{\mathbb{H}^m} u(q)$. Assume by contradiction that $u \not\equiv u^*$. By the strong maximum principle, Proposition 2.2 as given in [17], we have $u < u^*$ on $\mathbb{H}^m$. Choose $r_0 > 0$ and define

$$u_0^* = \sup_{\overline{B_{r_0}}} u < u^*.$$

We now choose $\eta > 0$ so small that $u^* - u_0^* > 2\eta$. Next take $\tilde{q} \in \Omega_{r_0} = \mathbb{H}^m \setminus \overline{B_{r_0}}$, such that $u(\tilde{q}) > u^* - \eta$. Take also ε and a in such a way that $0 < \varepsilon < \eta$ and

$a > 2\eta + \varepsilon$, obviously $a > \eta$. Put $r_1 = r(\tilde{q})$ so that $r_1 > r_0$. For such a choice of r_0, r_1, a, ε, η by Lemma 6.1 we can construct the radial function $v(q) = \alpha(r(q))$ on $B_R \setminus B_{r_0}$, with α and $R > r_1$, which is a solution of (6.1).

Being $v(\tilde{q}) \leq \eta$, it follows that

$$u(\tilde{q}) - v(\tilde{q}) > u^* - \eta - v(\tilde{q}) > u^* - \eta - \eta = u^* - 2\eta.$$

Since $u(q) - v(q) \leq u_0^* - \varepsilon < u^* - 2\eta - \varepsilon$ for all $q \in \partial B_{r_0}$ and

$$u(q) - v(q) \leq u^* - a < u^* - 2\eta - \varepsilon \quad \text{for all } q \in \partial B_R,$$

we deduce that the function $u - v$ attains a positive maximum μ on $B_R \setminus \overline{B_{r_0}}$. Let Γ_μ be a connected component of the set

$$\{q \in B_R \setminus \overline{B_{r_0}} : u(q) - v(q) = \mu\}.$$

For any $\xi \in \Gamma_\mu$, we have

$$u(\xi) > v(\xi), \quad |D_{\mathbb{H}^m} u(\xi)|_{\mathbb{H}^m} = |D_{\mathbb{H}^m} v(\xi)|_{\mathbb{H}^m}.$$

As a consequence in Γ_μ

$$\Delta_{\mathbb{H}^m}^{\varphi} u(\xi) \geq f(u(\xi))\ell(|D_{\mathbb{H}^m} u(\xi)|_{\mathbb{H}^m}) > f(v(\xi))\ell(|D_{\mathbb{H}^m} v(\xi)|_{\mathbb{H}^m}) \geq \Delta_{\mathbb{H}^m}^{\varphi} v(\xi),$$

since $f(u(\xi)) > f(v(\xi))$, by the strict monotonicity of f and since $\ell > 0$ in $\mathbb{R}_0^+$ by assumption. Hence by the C^1 regularity of u and v, in a sufficiently small neighborhood $\mathcal{N}$ of Γ_μ, the functions u and v satisfy

$$\Delta_{\mathbb{H}^m}^{\varphi} u \geq \Delta_{\mathbb{H}^m}^{\varphi} v \tag{6.4}$$

weakly in $\mathcal{N}$. Fix now a point $\xi \in \Gamma_\mu$, and for any $\varrho \in (0, \mu)$, denote by $\Omega_{\xi,\varrho}$ the connected component containing ξ of the set

$$\{q \in B_R \setminus \overline{B_{r_0}} : u(q) > v(q) + \varrho\}.$$

Let us now choose ϱ so close to μ that $\overline{\Omega_{\xi,\varrho}} \subset \mathcal{N}$. This can be shown by a compactness argument, for further details we refer to the proof of Theorem 4.3 of [2, page 702]. On $\partial\Omega_{\xi,\varrho}$ we have $u(q) = v(q) + \varrho$. Since $v(q) + \varrho$ solves

$$\Delta_{\mathbb{H}^m}^{A}(v + \varrho) = \Delta_{\mathbb{H}^m}^{\varphi} v \leq f(v)\ell(|D_{\mathbb{H}^m} v|_{\mathbb{H}^m}) \leq f(v + \varrho)\ell(|D_{\mathbb{H}^m}(v + \varrho)|_{\mathbb{H}^m}),$$

thanks to the monotonicity of f and the fact that ℓ is nonnegative in $\mathbb{R}_0^+$, we get by Proposition 2.1, namely Proposition 4.2 of [2], that

$$u(q) \le v(q) + \varrho.$$

But $u(\xi) = v(\xi) + \mu$. This contradicts the fact that $\xi \in \Omega_{\xi,\varrho}$ and shows that $u \equiv c$, where c is a nonnegative constant.

7. Existence under nondecreasing C–monotonicity on ℓ

In this section we extend to the $\Delta^{\varphi}_{\mathbb{H}^m}$ operator Theorem 1.3–(i) of [17] given for the p–Laplacian in the Heisenberg group as well as the existence Theorem 6.1 of [2].

In particular, in [2], the proof of Theorem 6.1, relative to the existence of entire large solutions of (1.2), uses the same main argument developed in [17]. We are planning to adapt the same construction in our context. It should be pointed out that Theorem 6.1 of [2] is proved under stronger conditions than (ϕL), namely assuming

$$\int_{0^+} \frac{\varphi'(t)}{\ell(t)}\,dt < \infty, \qquad \int^{\infty} \frac{\varphi'(t)}{\ell(t)}\,dt = \infty.$$

Proof of Theorem 1.4. Let $(VsKO)$ hold. We are going to construct a large entire radial stationary C^1 solution $u = u(|z|)$ of inequality (1.2), that is u is of the form (3.5).

First, let us define implicitly the function w on $\mathbb{R}_0^+$ by setting

$$r = \int_1^{w(r)} \frac{ds}{K^{-1}(F(s))}. \tag{7.1}$$

Hence, w is well defined, $w(0) = 1$ and $w(r) > 1$ for all $r > 0$ because of the positivity of the left hand side of (7.1) and of the function $K^{-1} \circ F$ in $\mathbb{R}^+$. Clearly, $w(r) \to \infty$ for $r \to \infty$ by $(VsKO)$. Differentiating (7.1) in $\mathbb{R}^+$, we obtain

$$w'(r) = K^{-1}(F(w(r))) > 0, \tag{7.2}$$

so that $K(w') = F(w)$ and differentiating again

$$K'(w')w'' = f(w)w',$$

that is in $\mathbb{R}^+$ by (1.4) and $(\mathscr{H})$,

$$w''\varphi'(w') = f(w)\ell(w'). \tag{7.3}$$

Fix $\rho > 0$ and define $A_\rho = \{(z,t) \in \mathbb{H}^m : |z| < \rho\}$. Let u_1 be the radial stationary function defined on $\mathbb{H}^m \setminus A_\rho$ by the formula

$$u_1(z,t) = w(|z|), \quad |z| = r, \quad \text{in } \mathbb{H}^m \setminus A_\rho.$$

Of course, $|D_{\mathbb{H}^m} u_1|_{\mathbb{H}^m} = w'$ by (3.2), being $\psi \equiv 1$ and $w' > 0$. Using (3.6), (ϕ) and (7.3), we see that u_1 satisfies

$$\Delta^{\varphi}_{\mathbb{H}^m} u_1 = \varphi'(w')w'' + \frac{2m-1}{|z|}\varphi(w') \geq f(u_1)\ell(|D_{\mathbb{H}^m} u_1|_{\mathbb{H}^m})$$

in $\mathbb{H}^m \setminus A_\rho$. Hence u_1 is a large radial stationary C^1 solution of (1.2) in $\mathbb{H}^m \setminus A_\rho$.

To produce a solution of (1.2) in A_ρ, fix $v_0 > 0$, $\Theta > 0$ which are numbers to be chosen later. Put

$$v(r) = v_0 + \frac{1}{\Theta}\int_0^{r\Theta} \varphi^{-1}(\tau)d\tau, \tag{7.4}$$

obviously v is well defined since $\varphi^{-1}(0) = 0$ and by (ϕ). Define

$$u_2(z,t) = v(|z|), \quad |z| = r, \quad \text{in} \;\; A_\rho.$$

From

$$v'(r) = \varphi^{-1}(r\Theta), \quad r = |z|, \tag{7.5}$$

we have $v'(0) = 0$, and so the function u_2 is of class C^1 in $\mathbb{H}^m$ with $D_{\mathbb{H}^m} u_2(0) = 0$. Using (3.6) along v, we get

$$\Delta^{\varphi}_{\mathbb{H}^m} u_2 = \varphi'(v')v'' + \frac{2m-1}{|z|}\varphi(v') = \Theta + \frac{2m-1}{|z|}\Theta|z| = 2m\Theta, \tag{7.6}$$

since $\varphi(v'(|z|)) = \Theta|z|$ by (7.5). If

$$2m\Theta \geq Cf(v(\rho))\ell(v'(\rho)), \tag{7.7}$$

where C is the constant of the C–monotonicity of ℓ, then by virtue of v', $v''>0$ in $\mathbb{R}^+$, the monotonicity of f and the C-monotonicity of ℓ, we obtain

$$\Delta^{\varphi}_{\mathbb{H}^m} u_2 \geq f(v(|z|))\ell(v'(|z|)) = f(u_2)\ell(|D_{\mathbb{H}^m} u_2|_{\mathbb{H}^m})$$

in A_ρ. In turn, assuming the validity of (7.7), we get that u_2 is a solution of inequality (1.2) in A_ρ.

The next step is to join u_1, u_2 so that the resulting function is C^1. To this aim we choose the positive parameters ρ, Θ, v_0 in such a way that (7.7) and

$$v(\rho) = w(\rho), \qquad v'(\rho) = w'(\rho) \tag{7.8}$$

are verified. In other words, by (7.2) and (7.4) we need to prove that the following conditions hold

$$(i) \quad v_0 + \frac{1}{\Theta}\int_0^{\rho\Theta} \varphi^{-1}(\tau)d\tau = w(\rho), \qquad (ii) \quad \varphi^{-1}(\rho\Theta) = K^{-1}(F(w(\rho))),$$
$$(iii) \quad 2m\Theta \geq Cf(v(\rho))\ell(v'(\rho)).$$

Let $w(\rho)=\mu$. Then by (7.1) we have $\mu>1$. Furthermore, by performing the change of variables $t=\varphi^{-1}(\tau)$ in the integral of (i) so that $d\tau=\varphi'(t)dt$ and $v'(\rho)=\varphi^{-1}(\rho\Theta)$ by (7.5), we have to verify

$$(i) \quad v_0 + \frac{1}{\Theta}\int_0^{K^{-1}(F(\mu))} t\varphi'(t)dt = \mu, \qquad (ii) \quad \rho\,\Theta = \varphi(K^{-1}(F(\mu))),$$
$$(iii) \quad \Theta \geq \frac{C}{2m} f(\mu)\ell(K^{-1}(F(\mu))).$$

Toward this aim, let μ be such that $1<\mu\leq 2$ and define

$$\rho = \int_1^\mu \frac{ds}{K^{-1}(F(s))} > 0. \tag{7.9}$$

Since $K^{-1}\circ F$ is monotone increasing in $\mathbb{R}^+_0$ and positive in $\mathbb{R}^+$, then

$$\frac{\mu-1}{K^{-1}(F(2))} \leq \rho \leq \frac{\mu-1}{K^{-1}(F(1))}, \tag{7.10}$$

being $1<\mu\leq 2$. Consequently $\rho\to 0$ as $\mu\to 1^+$. Thus we can choose μ so close to 1 that

$$\rho \leq \min\left\{\frac{1}{K^{-1}(F(2))}, \frac{2m\ \varphi(K^{-1}(F(1)))}{C^2 f(2)\ell(K^{-1}(F(2)))}\right\} \tag{7.11}$$

With this choice of ρ we immediately obtain that Θ defined in (ii), satisfies (iii). Indeed, by (ii) and (7.11),

$$\begin{aligned}\Theta = \frac{\varphi(K^{-1}(F(\mu)))}{\rho} &\geq \frac{C^2 f(2)\ell(K^{-1}(F(2)))}{2m} \cdot \frac{\varphi(K^{-1}(F(\mu)))}{\varphi(K^{-1}(F(1)))} \\ &\geq \frac{Cf(\mu)\ell(K^{-1}(F(\mu)))}{2m},\end{aligned}$$

where in the last inequality we have used that $\ell(K^{-1}(F(\mu))) \leq C\ell(K^{-1}(F(2)))$ by the nondecreasing C–monotonicity of ℓ, and the increasing monotonicity of f and of $\varphi \circ K^{-1} \circ F$.

Now it remains to prove the validity of (i). First observe that (ii) yields

$$\begin{aligned}\frac{1}{\Theta}\int_0^{K^{-1}(F(\mu))} t\varphi'(t)dt &= \frac{\rho}{\varphi(K^{-1}(F(\mu)))}\int_0^{K^{-1}(F(\mu))} t\varphi'(t)dt \\ &\leq \frac{\rho K^{-1}(F(\mu))}{\varphi(K^{-1}(F(\mu)))}\int_0^{K^{-1}(F(\mu))} \varphi'(t)dt \\ &= \rho K^{-1}(F(\mu)) < \rho K^{-1}(F(2)),\end{aligned}$$

being $\varphi' > 0$ in $\mathbb{R}^+$, $K^{-1} \circ F$ strictly increasing in $\mathbb{R}_0^+$ and $1 < \mu \leq 2$. In particular, by (7.11) and the above inequality, it follows

$$\frac{1}{\Theta}\int_0^{K^{-1}(F(\mu))} t\varphi'(t)dt < 1,$$

so that it is possible to choose $v_0 > 0$ in such a way that (i) holds, precisely

$$v_0 = \mu - \frac{1}{\Theta}\int_0^{K^{-1}(F(\mu))} t\varphi'(t)dt > 0,$$

being $1 < \mu \leq 2$.

Hence, we can conclude that, if μ is close enough to 1, the function

$$u(z) = \begin{cases} u_1(z) & \text{in} \quad \mathbb{H}^m \setminus A_\rho, \\ u_2(z) & \text{in} \quad A_\rho \end{cases}$$

is a large radial stationary C^1 solution of (1.2).

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