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A UNIFIED EXISTENCE THEORY FOR EVOLUTION EQUATIONS AND SYSTEMS UNDER NONLOCAL CONDITIONS

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Abstract. We investigate the effect of nonlocal conditions expressed by linear continuous mappings over the hypotheses which guarantee the existence of global mild solutions for functional-differential equations in a Banach space. A progressive transition from the Volterra integral operator associated to the Cauchy problem, to Fredholm type operators appears when the support of the nonlocal condition increases from zero to the entire interval of the problem. The results are extended to systems of equations in a such way that the system nonlinearities behave independently as much as possible and the support of the nonlocal condition may differ from one variable to another.

Keywords: functional-differential equation; evolution system; nonlocal Cauchy problem; mild solution; measure of noncompactness; spectral radius of a matrix.
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1. Introduction

This paper deals with the Cauchy problem for functional-differential evolution equations in a Banach space X, with a nonlocal condition expressed by a linear mapping

$$\left\{ \begin{array}{l} u'(t) = A(t)\,u(t) + \Phi(u)(t), \quad \text{for a.a. } t \in [0,a] \\ u(0) = F(u). \end{array} \right. \tag{1.1}$$

Here $\{A(t)\}_{t\in[0,a]}$ is a family of densely defined linear operators (not necessarily bounded or closed) in the Banach space X generating an evolution operator, Φ is a nonlinear mapping, and F is linear.

Containing the general functional term Φ, our equation is more general than the most studied one given by

$$\Phi(u)(t) = g(t, u_t),$$

where the function $u_t(s) = u(t+s)$, for $s \in [-r, 0]$, $r > 0$, $t \in [0,a]$ stands for the memory in lots of models for processes with aftereffect (see, e.g. [26]). In particular, it covers evolution equations which are perturbed by a superposition operator Φ,

$$\Phi(u)(t) = f(t, u(t)), \quad t \in [0,a] \tag{1.2}$$

associated to some function $f : [0,a] \times X \to X$, integro-differential equations and equations with modified argument.

In the mathematical modeling of real processes from physics, chemistry or biology, the nonlocal conditions can be seen as feedback controls by which the "sum" of the states of the process along its evolution equals the initial state. The mapping F expressing the nonlocal condition can be linear or nonlinear, of discrete or continuous type. For instance, as a linear mapping, it can be given by a finite sum of multi-point form

$$F(u) = \sum_{k=1}^{m} c_k u(t_k), \tag{1.3}$$

where $0 < t_1 < t_2 < \ldots < t_m \leq a$ and c_k are real numbers. More general, it can be expressed in terms of a Stieltjes integral

$$F(u) = \int_0^a u(t)\, d\phi(t).$$

Nonlocal problems with multi-point conditions and more general with linear and nonlinear nonlocal conditions were discussed in the literature by various approaches. We refer the reader to the papers [1], [4]-[11], [14], [18], [19], [21], [25], [27] and the references therein.

As it was first remarked in [6], it is important to take into consideration the *support* of the nonlocal condition, that is the minimal closed subinterval $[0, a_F]$ of $[0, a]$ with the property

$$F(u) = F(v) \text{ whenever } u = v \text{ on } [0, a_F]. \tag{1.4}$$

This means that the mapping F only depends on the restrictions of the functions from $C([0, a]; X)$, to the subinterval $[0, a_F]$. The case $a_F = 0$ recovers the classical Cauchy problem, while the case $a_F = a$ corresponds to a *global* nonlocal condition dissipated over the entire interval $[0, a]$ of the problem. When $0 < a_F < a$, we say that the nonlocal condition is *partial*. As we shall see, moving a_F from 0 to a, we realize a progressive transition from Volterra to Fredholm nature of the equivalent integral equation.

The support problem is even more interesting in case of a system of equations in n unknown functions $u_1, u_2, \ldots, u_n$, when a nonlocal condition is expressed by a linear mapping $F = F(u_1, u_2, \ldots, u_n)$. In this case, we may speak about the *support of F with respect to each of the variables.* The notion is introduced in this paper for the first time, and together with the vectorial method that is used, allows us to localize independently each component u_i of a solution $(u_1, u_2, \ldots, u_n)$.

In addition, as an other original feature of our study, the localization of a solution, and in case of systems, of each of the solution components, is realized in a *tube*, i.e. a set of the form

$$\{(t, u) : \; t \in [0, a], \; u \in X, \; |u| \leq R(t)\},$$

of a time-depending radius $R(t)$. In a physical interpretation, this means that the variation of a quantity $u(t)$ is allowed to be nonuniformly larger or smaller during the evolution, as prescribed by function $R(t)$.

We finish this introductory part by some notations and basic results. Throughout this paper, the norm of a Banach space X is denoted by $|.|$, the open and closed balls of X, of radius R centered at the origin, are denoted by $B(0, R)$, $\overline{B}(0, R)$, respectively;

the symbol $|.|_{\mathcal{L}(X,Y)}$ is used for the norm of a linear continuous mapping from X to Y, with the understanding that $\mathcal{L}(X,Y)$ is the space of all bounded linear operators from X to Y. Also, the norm on $L^p(b_1,b_2)$ $(1\le p\le\infty)$ is denoted by $|.|_{L^p(b_1,b_2)}$, and the symbol $|.|_{L^\infty(b_1,b_2)}$ is also used for the sup norm on $C[b_1,b_2]:=C([b_1,b_2];\mathbf{R})$. The notation $L^1_+(b_1,b_2)$ stands for the set of all nonnegative functions in $L^1(b_1,b_2)$. The open and closed balls of $C([0,a];X)$ of radius R centered at the origin are denoted by $B_C(0,R)$, $\overline{B}_C(0,R)$, respectively.

We recall that an operator $T:\Delta\to\mathcal{L}(X,X)$, where $\Delta=\{(t,s):0\le s\le t\le a\}$, is called an *evolution operator* if $T(t,s):X\to X$ is a bounded linear operator for every $(t,s)\in\Delta$, and the following conditions are satisfied:

(i) $T(s,s)=I$ (identity of X), $\quad T(t,r)T(r,s)=T(t,s)$ for $0\le s\le r\le t\le a$;

(ii) $(t,s)\mapsto T(t,s)$ is strongly continuous on Δ.

Note that, since T is strongly continuous on the compact set Δ, there exists a constant $M>0$ such that

$$|T(t,s)|_{\mathcal{L}(X,X)}\le M\,,\quad \text{for all }\ (t,s)\in\Delta. \tag{1.5}$$

By α we shall denote the *Kuratowski measure of noncompactness* on a Banach space X, i.e.

$$\alpha(D)=\inf\{\varepsilon>0: D \text{ admits a finite cover by sets of diameter } \le\varepsilon\}$$

for any bounded $D\subset X$. The symbol α_C will stand for the corresponding Kuratowski measure of noncompactness on $C([b_1,b_2];X)$. Recall (see [2], [3], [12], [17]) that for an equicontinuous set $D\subset C([b_1,b_2];X)$ with $D(t)$ bounded for each $t\in[b_1,b_2]$, one has

$$\alpha_C(D)=\max_{t\in[b_1,b_2]}\alpha(D(t)). \tag{1.6}$$

Also recall (see [15], [22]) that for a countable set $D\subset L^1(b_1,b_2;X)$ with $|u(t)|\le\eta(t)$ for a.a. $t\in[b_1,b_2]$ and every $u\in D$, where $\eta\in L^1_+(b_1,b_2)$, the function $t\mapsto\alpha(D(t))$ belongs to $L^1(b_1,b_2)$ and

$$\alpha\left(\left\{\int_{b_1}^{b_2}u(s)\,ds: u\in D\right\}\right)\le 2\int_{b_1}^{b_2}\alpha(D(s))\,ds. \tag{1.7}$$

The main tool of nonlinear functional analysis that we shall use is the Leray-Schauder type continuation theorem of Mönch [20] (see also [12], [23]) involving a compactness condition which in particular holds for condensing operators.

Theorem 1.1. *Let U be an open subset of a Banach space X, and let $N:\overline{U}\to X$ be continuous. Assume that for some $u_0\in U$ the following conditions are satisfied:*

(a) *$N(u)-u_0\ne\lambda(u-u_0)$ on ∂U for all $\lambda>1$;*

(b) *if $C\subset\overline{U}$ is countable and $C\subset\overline{\operatorname{conv}}(\{u_0\}\cup N(C))$, then $\overline{C}$ is compact.*

Then N has a fixed point in $\overline{U}$.

Finally, for the last part of the paper devoted to systems, we recall that for a square matrix of nonnegative entries $H\in\mathcal{M}_{n\times n}(\mathbf{R}_+)$, the *spectral radius* $\rho(H)$ is the maximum modulus of the eigenvalues, and that the following statements are equivalent:

(i) $\rho(H)<1$;

(ii) $H^k\to0$ (zero matrix) as $k\to\infty$;

(iii) $I-H$ is nonsingular and the entries of $(I-H)^{-1}$ are nonnegative (I being the unit matrix of the same order).

Details can be found in [24].

2. Existence and localization of solutions for evolution equations

Compared to other papers on the existence of solutions for local or nonlocal problems, our approach is to find solutions in a 'ball' of a time-depending radius. Hence we are looking for solutions in the bounded closed subset of $C([0,a];X)$,

$$\overline{U}:=\{u\in C([0,a];X):|u(t)|\leq R(t)\ \text{ for all } t\in[0,a]\},$$

where $R\in C[0,a]$ is a given function with $R(t)>0$ for all $t\in[0,a]$, and

$$U:=\{u\in C([0,a];X):|u(t)|<R(t)\ \text{ for all } t\in[0,a]\}.$$

In this section, the linear part of the equation of problem (1.1) will satisfy the following property (see, e.g. [10]):

(A): $\{A(t)\}_{t\in[0,a]}$ is a family of linear not necessarily bounded operators ($A(t):D(A)\subset X\to X$, $t\in[0,a]$, $D(A)$ is a dense subset of X not depending on t) generating a continuous evolution operator $T:\Delta\to\mathcal{L}(X,X)$.

We shall assume that

(h1): $\Phi:\overline{U}\to L^1(0,a;X)$ is continuous;

(h2): $F:C([0,a];X)\to X$ is a linear continuous mapping such that the operator from X to X, $x\mapsto x-F(T(.,0)\,x)$ has an inverse B.

Note that, by (h2) and the definition of the evolution operator, the operator B is linear and bounded, i.e. $B\in\mathcal{L}(X,X)$ (see [13, Corollary 3.2.8]).

Remark 2.1. A sufficient condition for (h2) to hold is that the norm of the operator $FT(.,0)$ from X to X is less than one. Indeed, in this case, $FT(.,0)$ is a contractive mapping and consequently, the operator from X to X, $x\mapsto x-F(T(.,0)\,x)$ is invertible. In the particular case, where F is of discrete type, given by (1.3), one has $a_F=t_m$, and the norm of the $FT(.,0)$ is less than one if

$$M\sum_{k=1}^{m}|c_k|<1.$$

Under conditions (h1) and (h2), a mild solution of the problem (1.1) in $\overline{U}$ is a function $u\in\overline{U}$ such that

$$\begin{aligned} u(t) &= T(t,0)\,BF\left(\int_0^{\cdot}T(.,s)\,\Phi(u)(s)\,ds\right) \\ &\quad+\int_0^t T(t,s)\,\Phi(u)(s)\,ds,\quad \text{for all }\ t\in[0,a]. \end{aligned} \tag{2.1}$$

From now on, we shall denote by $[0,a_F]$ the support of F. It is important to note that one has

$$F(v)=F(\chi_{a_F}(v)),$$

for all $v \in C([0,a];X)$, where the operator $\chi_{a_F} : C([0,a];X) \to C([0,a];X)$ is given by

$$\chi_{a_F}(v)(t) = \begin{cases} v(t) & \text{if } t \in [0,a_F] \\ v(a_F) & \text{if } t \in (a_F,a]. \end{cases}$$

We shall consider the integral operator $N : \overline{U} \to C([0,a];X)$ defined by

(2.2)

$$N(u)(t) = T(t,0)\,BF\left(\int_0^{\cdot} T(.,s)\Phi(u)(s)\,ds\right) + \int_0^t T(t,s)\Phi(u)(s)\,ds\,, \quad t \in [0,a].$$

Thus, any mild solution in $\overline{U}$ of (1.1) is a fixed point of N. Now Mönch's continuation theorem, Theorem 1.1, yields the following very general existence principle for the problem (1.1).

Theorem 2.1. *Assume that the conditions* (h1) *and* (h2) *hold. In addition assume*

- **(h3^0):** *if* $u = \lambda N(u)$ *for some* $u \in \overline{U}$ *and* $\lambda \in (0,1)$, *then* $|u(t)| < R(t)$ *for all* $t \in [0,a]$.
- **(h4^0):** *if* $C \subset \overline{U}$ *is countable and* $C \subset \overline{\operatorname{conv}}(\{0\} \cup N(C))$, *then* $\overline{C}$ *is compact in* $C([0,a];X)$.

Then (1.1) *has a mild solution in* $\overline{U}$.

To convert the general principle from Theorem 2.1 into applicable existence criteria, we have to find sufficient conditions for (h3^0), (h4^0) to hold. To this aim, we consider the operators $N_1, N_2 : \overline{U} \to C([0,a];X)$ given by

$$N_1(u)(t) = T(t,0)\,BF\left(\int_0^{\cdot} T(.,s)\Phi(u)(s)\,ds\right), \tag{2.3}$$

$$N_2(u)(t) = \int_0^t T(t,s)\Phi(u)(s)\,ds, \tag{2.4}$$

for every $t \in [0,a]$ and $u \in \overline{U}$, and for simplicity, we denote

$$|BF| = |BF|_{\mathcal{L}(C([0,a];X),X)}\,.$$

Lemma 2.1. *Assume that the conditions* (h1) *and* (h2) *hold. In addition assume that*

- **(h3):** *there exist* $\delta \in L^1_+(0,a)$ *and a continuous nondecreasing function* $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ *with* $\psi(s) > 0$ *for all* $s > 0$, *such that*

$$|\Phi(u)(t)| \le \delta(t)\,\psi(|u(t)|) \quad \textit{for a.a. } t \in [0,a] \textit{ and all } u \in \overline{U}, \tag{2.5}$$

$$r := M^2\,|BF|\,|\delta(.)\,\psi(R(.))|_{L^1(0,a_F)} < \min_{t\in[0,a]} R(t)\,, \tag{2.6}$$

where $[0,a_F]$ *is the support of* F, *and*

$$\int_r^{R(t)} \frac{d\tau}{\psi(\tau)} \ge M\,|\delta|_{L^1(0,t)} \quad \textit{for all } t \in [0,a]\,, \tag{2.7}$$

where M *is given by* (1.5).

Then the condition (h3^0) *is satisfied.*

Proof. Let $u = \lambda N(u)$ for some $u \in \overline{U}$ and $\lambda \in (0,1)$. Then, for each $t \in [0,a]$, by (1.5), (2.5) and (2.6), one has

$$\begin{aligned}
|u(t)| &\leq \lambda \left(|N_1(u)(t)| + |N_2(u)(t)|\right) \\
&\leq \lambda \left(M|BF(\chi_{a_F}(N_2(u)))| + M\int_0^t |\Phi(u)(s)|\,ds \right) \\
&\leq \lambda \left(M|BF||\chi_{a_F}(N_2(u))|_{C([0,a];X)} + M\int_0^t \delta(s)\psi(|u(s)|)\,ds \right) \\
&\leq \lambda \left(M^2|BF| \sup_{t\in[0,a_F]} |\delta(.)\psi(|u(.)|)|_{L^1(0,t)} + M|\delta(.)\psi(|u(.)|)|_{L^1(0,t)} \right) \\
&\leq \lambda \left(r + M|\delta(.)\psi(|u(.)|)|_{L^1(0,t)} \right) =: c(t). \qquad (2.8)
\end{aligned}$$

We show that

$$c(t) < R(t) \quad \text{for every} \quad t \in [0,a]. \tag{2.9}$$

First we note that, by (2.6), $c(0) < R(0)$. Then, suppose by contradiction that there exists $t^* \in (0,a]$ such that $c(t^*) \geq R(t^*)$; therefore, we may find an interval $[0,b] \subset [0,a]$ with

$$c(t) < R(t) \quad \text{for every} \quad t \in [0,b), \quad c(b) = R(b).$$

By using (2.8) and (h3), we have

$$c'(t) = \lambda M\delta(t)\psi(|u(t)|) \leq \lambda M\delta(t)\psi(c(t)), \quad \text{for a.a.} \quad t \in [0,b].$$

This implies

$$\int_0^b \frac{c'(s)}{\psi(c(s))}\,ds \leq \lambda M \int_0^b \delta(s)\,ds. \tag{2.10}$$

Since $c(0) = \lambda r \leq r$, we have

$$\int_0^b \frac{c'(s)}{\psi(c(s))}\,ds = \int_{c(0)}^{c(b)} \frac{d\tau}{\psi(\tau)} = \int_{\lambda r}^{R(b)} \frac{d\tau}{\psi(\tau)} \geq \int_r^{R(b)} \frac{d\tau}{\psi(\tau)};$$

so by (2.10) we deduce

$$\int_r^{R(b)} \frac{d\tau}{\psi(\tau)} \leq \lambda M|\delta|_{L^1(0,b)}.$$

Then, if $|\delta|_{L^1(0,b)} > 0$, we obtain

$$\int_r^{R(b)} \frac{d\tau}{\psi(\tau)} \leq \lambda M\,|\delta|_{L^1(0,b)} < M\,|\delta|_{L^1(0,b)},$$

which contradicts (2.7). Note that in our case $c(b) = R(b)$, the equality $|\delta|_{L^1(0,b)} = 0$ is not possible, since otherwise $c(b) = \lambda r < R(b)$, which is impossible. Therefore $c(t) < R(t)$ for every $t \in [0,a]$, whence $|u(t)| < R(t)$ for all $t \in [0,a]$, as desired. □

Remark 2.2. In particular, if $R(t) = R$ (positive constant) for every $t \in [0,a]$, than $\overline{U} = \overline{B}_C(0,R)$ and the conditions (2.6), (2.7) read as follows:

$$r := M^2\,|BF|\,\psi(R)\,|\delta|_{L^1(0,a_F)} < R, \tag{2.11}$$

$$\int_r^R \frac{d\tau}{\psi(\tau)} \geq M\,|\delta|_{L^1(0,a)}\,. \tag{2.12}$$

In order to prove the next result, in correspondence to the function R, we introduce the undergraph of $2R$,

$$V_R = \left\{(t,s) \in \mathbf{R}^2 \,:\, 0 \leq s \leq 2R(t)\,,\ \ 0 \leq t \leq a\right\}$$

and we say that a function $\omega : V_R \to \mathbf{R}_+$ is L^1-*Carathéodory on the undergraph* V_R if

- **(ω1):** $\omega(.,s)$ is measurable on $\{t \in [0,a] \,:\, 2R(t) \geq s\}$ for every $s \in \left[0, 2\,|R|_{L^\infty(0,a)}\right]$;
- **(ω2):** $\omega(t,.)$ is continuous on $[0,2R(t)]$, for a.a. $t \in [0,a]$;
- **(ω3):** there exists $\eta \in L^1_+(0,a)$ such that $\omega(t,s) \leq \eta(t)$, for all $s \in [0,2R(t)]$ and a.a. $t \in [0,a]$.

Moreover, we shall assume the following property

(h4): there exists a function $\omega : V_R \to \mathbf{R}_+$ which is L^1-Carathéodory on the undergraph V_R and such that for each countable set $C \subset \overline{U}$,

$$\alpha(\Phi(C)(t)) \leq \omega(t, \alpha(C(t)))\,, \quad \text{for a.a. } t \in [0,a] \tag{2.13}$$

and that the unique solution $\varphi \in C[0,a]$ with graph$(\varphi) \subset V_R$ of the inequality

$$\begin{aligned} \varphi(t) \ \ \leq \ \ & 2M^2\,|BF| \int_0^{a_F} \omega(s,\varphi(s))\; ds \\ & + 2M \int_0^t \omega(s,\varphi(s))\; ds, \quad \text{for all } \ t \in [0,a] \end{aligned} \tag{2.14}$$

is $\varphi \equiv 0$.

Note that the condition (h4) is well posed; indeed, if $C \subset \overline{U}$, then

$$\alpha(C(t)) \leq \alpha(B(0,R(t))) = 2R(t), \quad \text{for all } \ t \in [0,a].$$

Hence, $(t, \alpha(C(t))) \in V_R$, for every $t \in [0,a]$.

Remark 2.3 (the Kamke function of a nonlocal problem)**.** In the case of the classical Cauchy problem, when $A(t) = 0$ for every $t \in [0,a]$ and $F = 0$ (equivalently, when $a_F = 0$), the inequality (2.14) reduces to

$$\varphi(t) \leq 2\int_0^t \omega(s,\varphi(s))\, ds, \quad \textit{for all } \ t \in [0,a]$$

and the condition required in (h4) means that ω is a *Kamke function of the Cauchy problem.* By analogy, in the case of our nonlocal problem (1.1), the function ω in (h4) can be called a *Kamke function of the nonlocal initial value problem.*

Lemma 2.2. *Assume the conditions* (h1), (h2), (h4) *and*

(h3'): *there exist* $\delta \in L^1_+(0,a)$ *and a continuous nondecreasing function* $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ *with* $\psi(s) > 0$ *for all* $s > 0$, *such that* (2.5) *holds.*

Then the condition (h4^0) *is satisfied.*

Proof. Let $C \subset \overline{U}$ be countable with

$$C \subset \overline{\text{conv}}\left(\{0\} \cup N\left(C\right)\right), \tag{2.15}$$

where N is given by (2.2). First we show that C is equicontinuous. For this, it is enough to prove the equicontinuity of the set $N\left(C\right)$. First of all, we have that $N_2\left(C\right)$ is equicontinuous. In fact, for any fixed $\varepsilon > 0$, in correspondence to $\varepsilon/6M$, there exists $\eta(\varepsilon/6M) > 0$ such that for every measurable set $\mathcal{M}$ with $\lambda(\mathcal{M}) < \eta(\varepsilon/6M)$ (where λ denotes the Lebesgue measure on $[0,a]$) one has $\int_{\mathcal{M}} \delta(s)\psi(R(s))\, ds < \varepsilon/6M$, where $\delta(.)\psi(R(.)) \in L^1_+(0,a)$ (see (h3')). Let us fix $\gamma > 0$ with $\gamma < \eta(\varepsilon/6M)$. For any $u \in C$ and $t, \overline{t} \in [0,a]$ with $0 < t - \overline{t} < \gamma$, by using (1.5) and hypothesis (h3'), we have

$$\begin{aligned}
&\left|N_2\left(u\right)\left(t\right) - N_2\left(u\right)\left(\overline{t}\right)\right| \\
= \; & \left|\int_0^{\overline{t}} T(t,s)\Phi(u)(s)\, ds + \int_{\overline{t}}^t T(t,s)\Phi(u)(s)\, ds - \int_0^{\overline{t}} T(\overline{t},s)\Phi(u)(s)\, ds\right| \\
\leq \; & \int_0^{\overline{t}} \left|T(t,s) - T(\overline{t},s)\right|_{\mathcal{L}(X,X)} \left|\Phi(u)(s)\right| ds + M\int_{\overline{t}}^t \left|\Phi(u)(s)\right| ds \\
\leq \; & \int_0^{\overline{t}} \left|T(t,s) - T(\overline{t},s)\right|_{\mathcal{L}(X,X)} \delta(s)\psi(R(s))\, ds + M\int_{\overline{t}}^t \delta(s)\psi(R(s))\, ds \\
\leq \; & \int_0^{\overline{t}-\gamma} \left|T(t,s) - T(\overline{t},s)\right|_{\mathcal{L}(X,X)} \delta(s)\psi(R(s))\, ds + 2M\int_{\overline{t}-\gamma}^{\overline{t}} \delta(s)\psi(R(s))\, ds \\
& +M\int_{\overline{t}}^t \delta(s)\psi(R(s))\, ds \\
\leq \; & \int_0^{\overline{t}-\gamma} \left|T(t,s) - T(\overline{t},s)\right|_{\mathcal{L}(X,X)} \delta(s)\psi(R(s))\, ds + \varepsilon/3 + \varepsilon/6.
\end{aligned} \tag{2.16}$$

Let $H := \int_0^a \delta(s)\psi(R(s))\, ds$. By the uniform continuity of the evolution operator T, there exists $\eta(\varepsilon/3H) > 0$ which can be chosen with $\eta(\varepsilon/3H) \leq \gamma$, such that if $0 < t - \overline{t} < \eta(\varepsilon/3H)$, $s \in [0,\overline{t}]$, then $|T(t,s) - T(\overline{t},s)|_{\mathcal{L}(X,X)} < \varepsilon/3H$. So (2.16) yields

$$\left|N_2(u)(t) - N_2(u)(\overline{t})\right| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/6 < \varepsilon.$$

Hence $N_2(C)$ is equicontinuous.

To prove that $N_1\left(C\right)$ is equicontinuous, first observe that by (2.4), the map N_1 in (2.3) can be written as

$$N_1\left(u\right)\left(t\right) = T\left(t,0\right) BF\left(N_2(u)\right),$$

for all $t \in [0,a]$ and $u \in \overline{U}$. Denote

$$\widetilde{M} = M\left|BF\right| |\delta(.)\psi(R(.))|_{L^1(0,a)}. \tag{2.17}$$

By the continuity of the evolution operator T, we have that for every $\varepsilon > 0$, there exists $\eta(\varepsilon/\widetilde{M}) > 0$ such that for every $t, \overline{t} \in [0,a]$ with $|t - \overline{t}| < \eta(\varepsilon/\widetilde{M})$, we have

$$|T(t,0) - T(\overline{t},0)|_{\mathcal{L}(X,X)} < \varepsilon/\widetilde{M}. \tag{2.18}$$

Assuming without less of generality that $t > \bar{t}$, according to (2.4), (h3'), (2.17) and (2.18), for every $u \in C$, we have the following estimation

$$\begin{aligned}
& |N_1(u)(t) - N_1(u)(\bar{t})| \\
= \;& |[T(t,0) - T(\bar{t},0)]BF(N_2(u))| \\
\leq \;& |T(t,0) - T(\bar{t},0)|_{\mathcal{L}(X,X)}|BF||N_2(u)|_{C([0,a];X)} \\
\leq \;& |T(t,0) - T(\bar{t},0)|_{\mathcal{L}(X,X)}M|BF| \sup_{t\in[0,a]} \int_0^t |\Phi(u)(s)|\, ds \\
\leq \;& |T(t,0) - T(\bar{t},0)|_{\mathcal{L}(X,X)}M|BF| \int_0^a \delta(s)\psi(R(s))\, ds \\
= \;& |T(t,0) - T(\bar{t},0)|_{\mathcal{L}(X,X)}\widetilde{M} \, < \, \varepsilon.
\end{aligned}$$

So $N_1(C)$ is equicontinuous. Hence, by (2.2), (2.3) and (2.4), we have the equicontinuity of $N(C)$. Therefore, by (2.15), the set C is equicontinuous too. Furthermore, for every fixed $t \in [0,a]$, the set $C(t)$ is relatively compact in X. Indeed, C is bounded in $C([0,a];X)$ and

$$\begin{aligned}
\alpha\left(C\left(t\right)\right) &\leq \alpha\left(\overline{\operatorname{conv}}\left(\{0\}\cup N\left(C\right)\left(t\right)\right)\right) = \alpha\left(N\left(C\right)\left(t\right)\right) \\
&\leq \alpha\left(N_1\left(C\right)\left(t\right)\right) + \alpha\left(N_2\left(C\right)\left(t\right)\right).
\end{aligned} \tag{2.19}$$

According to (1.7) and (h4), we have

$$\begin{aligned}
\alpha(N_2(C)(t)) &\leq 2\int_0^t \alpha(T(t,s)\Phi(C)(s))\, ds \\
&\leq 2M\int_0^t \omega(s,\alpha(C(s)))\, ds.
\end{aligned} \tag{2.20}$$

In addition, using the linearity of the mapping BF and (1.6), we deduce that

$$\begin{aligned}
\alpha\left(N_1(C)(t)\right) &\leq M|BF|\alpha_C\left(\chi_{a_F}\left(N_2(C)\right)\right) \\
&= M|BF| \max_{t\in[0,a_F]} \alpha(N_2(C)(t)) \\
&\leq 2M^2\left|BF\right| \int_0^{a_F} \omega\left(s,\alpha\left(C\left(s\right)\right)\right)\, ds.
\end{aligned} \tag{2.21}$$

Now (2.19), (2.20) and (2.21) give

$$\alpha\left(C\left(t\right)\right) \leq 2M^2\left|BF\right| \int_0^{a_F} \omega\left(s,\alpha\left(C\left(s\right)\right)\right)\, ds + 2M\int_0^t \omega(s,\alpha(C(s)))\, ds.$$

Hence the function

$$\varphi\left(t\right) = \alpha\left(C\left(t\right)\right), \quad \text{for all } \; t\in[0,a]$$

solves (2.14). In addition φ is continuous on $[0,a]$ and its graph is contained in V. Consequently, $\varphi \equiv 0$, that is $\alpha\left(C\left(t\right)\right) = 0$ for all $t \in [0,a]$. Thus $C\left(t\right)$ is relatively compact in X for each $t \in [0,a]$, as desired. □

Now Theorem 2.1 and Lemmas 2.1 and 2.2 yield the main existence result for (1.1).

Theorem 2.2. *Assume that the conditions* (h1)-(h4) *are satisfied. Then* (1.1) *has a mild solution in* $\overline{U}$.

In the setting of Remark 2.2, from Theorem 2.2 we deduce the following result.

Corollary 2.1 (case of time-independent radius)**.** *Assume that the conditions* (h1)*,* (h2) *and* (h4) *hold, where* $\overline{U} = \overline{B}_C(0,R)$, $R > 0$ *and* $V_R = \{(t,s) \in \mathbf{R}^2 : 0 \le s \le 2R, 0 \le t \le a\}$*. In addition assume that*

(h3*): *there exist* $\delta \in L^1_+(0,a)$ *and a continuous nondecreasing map function* $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ *with* $\psi(s) > 0$ *for all* $s > 0$*, such that*

$$|\Phi(u)(t)| \le \delta(t)\psi(|u(t)|), \quad \text{for a.a. } t \in [0,a] \quad \text{and all } u \in \overline{B}_C(0,R),$$

$$\frac{R}{\psi(R)} \ge M^2 |BF| \, |\delta|_{L^1(0,a_F)} + M\, |\delta|_{L^1(0,a)}, \tag{2.22}$$

where M *is from* (1.5)*.*

Then (1.1) *has a mild solution in* $\overline{B}_C(0,R)$*.*

Proof. First of all, we show that under conditions (h1), (h2) and (h3*), the condition (h3) is satisfied in the case of time-independent radius. It is easy to see that (2.11) follows from (2.22) if $|\delta|_{L^1(0,a)} > 0$; otherwise (2.11) is trivially satisfied. Furthermore, since the function ψ is nondecreasing, we have

$$\int_r^R \frac{d\tau}{\psi(\tau)} \ge \frac{R-r}{\psi(R)}$$

and thus, by (2.22) and the definition of r (see (2.11)), condition (2.12) holds. According to Lemma 2.1 and Remark 2.2, in the case of time-independent radius, the condition (h3^0) is satisfied. Now Theorem 2.2 finishes the proof. □

Note that the condition (2.22) guarantees even more, namely that $N\left(\overline{U}\right) \subset \overline{U}$.

A much more applicable result can be derived from Theorem 2.2.

Theorem 2.3. *Assume that* (h1), (h2) *and* (h3) *hold. In addition assume that the following condition is satisfied:*

(h4*): $\Phi = \Psi + \Theta$, *where* $\Theta\left(\overline{U}\right)(t) \subset K$ *for a.a* $t \in [0,a]$, K *being a compact set in* X, *and there exists* $\gamma \in L^1_+(0,a)$ *such that for each countable set* $C \subset \overline{U}$,

$$\alpha(\Psi(C)(t)) \le \gamma(t)\,\alpha(C(t)), \quad \text{for a.a. } t \in [0,a] \tag{2.23}$$

and

$$\left(2M^2 |BF| + 2M\right) |\gamma|_{L^1(0,a_F)} < 1. \tag{2.24}$$

Then (1.1) *has a mild solution in* $\overline{U}$*.*

Proof. We shall check (h4). Since $\Theta\left(\overline{U}\right)(t) \subset K$ for a.a $t \in [0,a]$, K being a compact set in X, from (2.23) we see that (2.13) holds with $\omega(t,s) = \gamma(t)\,s$, $(t,s) \in V_R$. Now let $\varphi \in C[0,a]$ with graph$(\varphi) \subset V_R$, be any solution of (2.14), that is

$$\varphi(t) \le 2M^2 |BF| \, |\gamma\varphi|_{L^1(0,a_F)} + 2M\, |\gamma\varphi|_{L^1(0,t)}, \quad t \in [0,a]. \tag{2.25}$$

First we show that $\varphi(t) = 0$ for all $t \in [0,a_F]$. Indeed, from (2.25), since φ is nonnegative, we deduce

$$|\varphi|_{L^\infty(0,a_F)} \le |\varphi|_{L^\infty(0,a_F)} \left(2M^2 |BF| + 2M\right) |\gamma|_{L^1(0,a_F)}, \tag{2.26}$$

which in view of (2.24) gives $|\varphi|_{L^\infty(0,a_F)} = 0$. Then from the continuity of φ, we deduce $\varphi(t) = 0$ for all $t \in [0, a_F]$, as claimed. As a consequence, (2.25) reduces to

$$\varphi(t) \le 2M \int_{a_F}^{t} \gamma(s)\,\varphi(s)\,ds, \quad \text{for all} \ \ t \in [a_F, a],$$

and the remaining conclusion $\varphi(t) = 0$ for $t \in (a_F, a]$ follows from Gronwall's inequality. Then **(h4)** holds. By Theorem 2.2 the thesis is reached. □

Remark 2.4. In particular, the condition (2.23) holds if Ψ satisfies the Lipschitz inequality

$$|\Psi(u)(t) - \Psi(v)(t)| \le \gamma(t)\,|u(t) - v(t)|$$

for all $u, v \in \overline{U}$ and a.a. $t \in [0, a]$.

In the case of the superposition nonlinearity, namely if Φ is given by (1.2), from Theorem 2.3, we can deduce the following result.

Corollary 2.2 (case of superposition operator)**.** *Assume that the condition* (h2) *holds. Let $f : [0, a] \times \overline{B}(0, |R|_\infty) \to X$ be a mapping such that*

(h1$_f$): *$f(.,x)$ is measurable on $[0, a]$ for each $x \in \overline{B}(0, |R|_\infty)$;*

$f(t,.)$ is continuous on the ball $\overline{B}(0, R(t))$ for a.a. $t \in [0, a]$;

$|f(t,x)| \le \eta(t)$ for all $x \in \overline{B}(0, R(t))$ and a.a. $t \in [0, a]$, where $\eta \in L^1_+(0, a)$;

(h3$_f$): *there exist $\delta \in L^1_+(0, a)$ and a continuous nondecreasing function $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ with $\psi(s) > 0$ for all $s > 0$, such that*

$$|f(t,x)| \le \delta(t)\psi(|x|), \quad \text{for a.a.} \ \ t \in [0, a] \quad \text{and all} \ \ x \in \overline{B}(0, R(t)),$$

and (2.6), (2.7) *are satisfied;*

(h4*$_f$): *$f = g + h$, where $h(D)$ is relatively compact in X for $D := \{(t,x) : |x| \le R(t),\ t \in [0, a]\}$, and there exists $\gamma \in L^1_+(0, a)$ such that for each countable set $C \subset \overline{B}(0, R(t))$,*

$$\alpha(g(t, C)) \le \gamma(t)\,\alpha(C), \quad \text{for a.a.} \ \ t \in [0, a]$$

and (2.24) *holds.*

Then the problem

$$\begin{cases} u'(t) = A(t)u(t) + f(t, u(t)), & \text{for a.a. } t \in [0, a] \\ u(0) = F(u) \end{cases}$$

has a mild solution in $\overline{U}$.

3. Existence and localization of solutions for evolution systems

Consider n Banach spaces $(X_i, |.|_i)$, the product space $X = X_1 \times X_2 \times ... \times X_n$ and the Cauchy problem for an n-dimensional system, with nonlocal conditions

$$\text{(3.1)} \qquad \begin{cases} u_i'(t) = A_i(t)\,u_i(t) + \Phi_i(u_1, u_2, ..., u_n)(t), & \text{for a.a} \ \ t \in [0, a] \\ u_i(0) = F_i(u_1, u_2, ..., u_n), \end{cases}$$

$i=1,2,...,n$. Here, for each i, $\{A_i(t)\}_{t\in[0,a]}$ is a family of linear operators in the Banach space X_i generating an evolution operator T_i, Φ_i is a nonlinear mapping, and F_i is linear.

On the linear part of the i-equation we require the condition:

($\mathbf{A}_i$): $\{A_i(t)\}_{t\in[0,a]}$ is a family of linear not necessarily bounded operators ($A_i(t) : D(A_i) \subset X_i \to X_i$, $t \in [0,a]$, $D(A_i)$ is a dense subset of X_i not depending on t) generating a continuous evolution operator $T_i : \Delta \to \mathcal{L}(X_i, X_i)$.

Consider the vector-valued mappings, represented as column matrices, Φ and F acting from $C([0,a];X)$ into X,

$$\Phi = [\Phi_1, \Phi_2, ..., \Phi_n]^{\mathrm{tr}}, \quad F = [F_1, F_2, ..., F_n]^{\mathrm{tr}} \tag{3.2}$$

and the family $\{A(t)\}_{t\in[0,a]}$ of linear operators in X, where, for each $t \in [0,a]$, the opertator $A(t) : D(A) = \prod_{i=1}^{n} D(A_i) \to X$ is represented as diagonal matrix of operators,

$$A(t) = \begin{bmatrix} A_1(t) & ... & 0 \\ ... & A_2(t) & ... \\ 0 & ... & A_n(t) \end{bmatrix}.$$

Clearly, $A(t)x = [A_1(t)x_1, A_2(t)x_2, ..., A_n(t)x_n]^{\mathrm{tr}}$, $x \in D(A)$. Then looking at the elements of the product space X as column matrices, the system (3.1) can be written as

$$\begin{cases} u'(t) = A(t)u(t) + \Phi(u)(t), & \text{for a.a. } t \in [0,a] \\ u(0) = F(u), \end{cases}$$

which is exactly problem (1.1), this time, in a vectorial form, in the product space $X = X_1 \times X_2 \times ... \times X_n$. Thus, all previous results are applicable and yield existence theorems for the system (3.1). However, like in [5], we can take advantage from the splitting of this vectorial equation into n equations and obtain more refined results under conditions allowing the operators F_i and Φ_i to behave independently as much as possible. This will be possible by exploiting the vectorial nature of the system and by using matrix conditions instead of scalar ones. For instance, instead of speaking globally about the support of the operator F, as shown by (1.4), we shall consider the *support of F with respect to each variable u_i*, $i = 1,2,...,n$, as being the minimal closed subinterval $[0,a_i]$ of $[0,a]$ with the property

$$\begin{aligned} F(u_1,...,u_{i-1},u_i,u_{i+1},...,u_n) &= F(u_1,...,u_{i-1},v_i,u_{i+1},...,u_n) \\ \text{whenever } u_i &= v_i \text{ on } [0,a_i]. \end{aligned}$$

Also, we are interested not only on the existence of a mild solution $u = (u_1, u_2, ..., u_n)$ of the problem (3.1), but also on the localization of each component u_i individually. Thus, the solutions are sought in a bounded closed subset $\overline{U}$ of $C([0,a];X)$, of the form $\overline{U} = \overline{U}_1 \times \overline{U}_2 \times ... \times \overline{U}_n$ with

$$\overline{U}_i := \{v \in C([0,a];X_i) : |v(t)|_i \le R_i(t) \text{ for all } t \in [0,a]\},$$

where $R_i \in C[0,a]$ are given functions with $R_i(t) > 0$ for all $t \in [0,a]$, $i = 1,2,...,n$.

Let us define the family $\{T(t,s)\}_{(t,s)\in\Delta}$ of linear operators from X to X, where, for each $(t,s)\in\Delta$, $T(t,s)$ is represented as diagonal matrix

$$T(t,s)=\left[\begin{array}{ccc} T_1(t,s) & ... & 0 \\ ... & T_2(t,s) & ... \\ 0 & ... & T_n(t,s) \end{array}\right],$$

and so

$$T(t,s)\,x=[T_1(t,s)\,x_1,T_2(t,s)\,x_2,...,T_n(t,s)\,x_n]^{\mathrm{tr}}\ ,\quad x\in X.$$

We shall assume the analogue conditions to (h1) and (h2):

(H1): $\Phi_i:\overline{U}\to L^1(0,a;X_i)$ is continuous, $i=1,2,...,n$;
(H2): $F_i:C([0,a];X)\to X_i$ is a linear and continuous mapping, $i=1,2,...,n$, and the operator from X to X, $x\mapsto x-F(T(.,0)\,x)$ has an inverse B.

Note that, using the vectorial notations Φ and F given in (3.2), the conditions (H1), (H2) appear identical to (h1), (h2), respectively.

Like F, the linear operator B from X to X can be naturaly looked as a column matrix

$$B=[B_1,B_2,...,B_n]^{\mathrm{tr}},$$

where $B_i\in\mathcal{L}(X,X_i)$. Moreover, thanks to the linearity of the operators B_i and F_i, B and F can be identified to a matrix

$$B=[B_{ij}]_{1\le i,j\le n}\,,\quad F=[F_{ij}]_{1\le i,j\le n}\,,$$

whose entries $B_{ij}\in\mathcal{L}(X_j,X_i)$, $F_{ij}\in\mathcal{L}(C([0,a];X_j),X_i)$ are given by

$$\begin{array}{rcl} B_{ij}(x_j) & = & B_i(0,0,...,x_j,0,...,0) \\ F_{ij}(u_j) & = & F_i(0,0,...,u_j,0,...,0)\,, \end{array}$$

with $x_j\in X_j$, $u_j\in C([0,a];X_j)$ on the j-th position. Then

$$B_i(x)=\sum_{j=1}^n B_{ij}(x_j)\,,\quad \text{for every}\ \ x\in X,$$

$$F_i(u)=\sum_{j=1}^n F_{ij}(u_j)\,,\quad \text{for every}\ \ u\in C([0,a];X)\,.$$

Let G denote the linear mapping BF from $C([0,a];X)$ to X. According to the above explanations,

$$G(u)=[G_1(u),G_2(u),...,G_n(u)]^{\mathrm{tr}},\quad G=[G_{ij}]_{1\le i,j\le n}\,,$$

where $G_i\in\mathcal{L}(C([0,a];X),X_i)$, $G_{ij}\in\mathcal{L}(C([0,a];X_j),X_i)$ and

$$G_{ij}(u_j)=G_i(0,0,..,u_j,0,...,0)$$

with u_j on the j-th position. Thanks again to the linearity of the operators, we have

$$\begin{aligned} G_i(u) &= B_i(F(u)) = \sum_{k=1}^{n} B_{ik}(F_k(u)) = \sum_{k=1}^{n} B_{ik}\left(\sum_{j=1}^{n} F_{kj}(u_j)\right) \\ &= \sum_{k,j=1}^{n} B_{ik}(F_{kj}(u_j)) \end{aligned}$$

and

$$G_{ij}(u_j) = \sum_{k=1}^{n} B_{ik} F_{kj}(u_j).$$

Using the above notations, letting M_i be such that $|T_i(t,s)|_{\mathcal{L}(X_i,X_i)} \leq M_i$ for all $(t,s) \in \Delta$, and denoting for simplicity

$$|G_{ij}| = |G_{ij}|_{\mathcal{L}(C([0,a];X_j),X_i)},$$

we can state our next assumption:

(H3): for each $i = 1,2,...,n$, there exist $\delta_i \in L^1_+(0,a)$ and a continuous nondecreasing function $\psi_i : \mathbf{R}_+ \to \mathbf{R}_+$ with $\psi_i(s) > 0$ for all $s > 0$, such that

$$|\Phi_i(u)(t)| \leq \delta_i(t)\, \psi_i(|u_i(t)|_i) \quad \text{for a.a. } t \in [0,a] \text{ and all } u \in \overline{U}, \tag{3.3}$$

$$r_i := M_i \sum_{j=1}^{n} |G_{ij}|\, M_j\, |\delta_j(.)\, \psi_j(R_j(.))|_{L^1(0,a_j)} < \min_{t\in[0,a]} R_i(t), \tag{3.4}$$

where $[0,a_j]$ is the support of F with respect the variable u_j, and

$$\int_{r_i}^{R_i(t)} \frac{d\tau}{\psi_i(\tau)} \geq M_i\, |\delta_i|_{L^1(0,t)} \quad \text{for all } t \in [0,a]. \tag{3.5}$$

Note that the support of F in this case is given by $a_F = \max_{1\leq i\leq n} a_i$.

Finally, if we denote by α_i the Kuratowski measure of noncompactness on X_i, then we can state the vectorial analogue of the condition (h4*):

(H4): for each $i = 1,2,...,n$, $\Phi_i = \Psi_i + \Theta_i$, where $\Theta_i\left(\overline{U}\right)(t) \subset K_i$ for a.a $t \in [0,a]$, K_i being a compact set in X_i, and there exist $\gamma_{ij} \in L^1_+(0,a)$ $(1 \leq j \leq n)$, such that for each countable set $C \subset \overline{U}$,

$$\alpha_i(\Psi_i(C)(t)) \leq \sum_{j=1}^{n} \gamma_{ij}(t)\, \alpha_j(C_j(t)), \quad \text{for a.a. } t \in [0,a],$$

and

$$\rho(H) < 1 \tag{3.6}$$

for the matrix

$$H = 2\left(|\mathcal{G}|\, |\widetilde{\gamma}|_{L^1(0,a_F)} + |\gamma|_{L^1(0,a_F)}\right).$$

Here $\rho(H)$ is the spectral radius of H and $|\mathcal{G}|$, $|\gamma|_{L^1(0,a_F)}$, $|\widetilde{\gamma}|_{L^1(0,a_F)}$ are the matrices

$$
\begin{aligned}
|\mathcal{G}| &= [M_i\,|G_{ij}|]_{1\le i,j\le n}\,,\\
|\gamma|_{L^1(0,a_F)} &= \left[M_i\,|\gamma_{ij}|_{L^1(0,a_F)}\right]_{1\le i,j\le n}, \quad |\widetilde{\gamma}|_{L^1(0,a_F)} = \left[M_i\,|\widetilde{\gamma}_{ij}|_{L^1(0,a_F)}\right]_{1\le i,j\le n}\,,
\end{aligned}
$$

where $\widetilde{\gamma}_{ij}(t) = \gamma_{ij}(t)$ for $t \in [0,a_i]$, $\widetilde{\gamma}_{ij}(t) = 0$ for $t \in (a_i, a]$.

Theorem 3.1. *Under the conditions* **(H1)-(H4)**, *the problem* (3.1) *has a mild solution in* $\overline{U}$.

Proof. The problem (3.1) is equivalent to the fixed point equation for the nonlinear operator (2.2) in $C([0,a];X)$, $N = N_1 + N_2$, where for each $i = 1,2,...,n$,

$$
N_{2i}(u)(t) = \int_0^t T_i(t,s)\Phi_i(u)(s)\,ds \tag{3.7}
$$

and

$$
\begin{aligned}
N_{1i}(u)(t) &= T_i(t,0)\,G_i(N_2(u))\\
&= T_i(t,0)\sum_{j=1}^{n} G_{ij}(N_{2j}(u))\\
&= T_i(t,0)\sum_{j=1}^{n} G_{ij}\left(\chi_{a_j}(N_{2j}(u))\right).
\end{aligned} \tag{3.8}
$$

Here $\chi_{a_j} : C([0,a];X_j) \to C([0,a];X_j)$ is given by

$$
\chi_{a_j}(v)(t) = \begin{cases} v(t) & \text{if } t \in [0,a_j] \\ v(a_j) & \text{if } t \in (a_j, a], \end{cases}
$$

for all $v \in C([0,a];X_j)$.

We shall apply Mönch's continuation theorem in the Banach space $C([0,a];X)$, to the open bounded set $U = U_1 \times U_2 \times ... \times U_n$, where

$$
U_i = \{v \in C([0,a];X_i) : |u_i(t)|_i < R_i(t) \text{ for } t \in [0,a]\} \quad (1 \le i \le n)
$$

and to the element $u_0 = 0$. Let $u = \lambda N(u)$ for some $u \in \overline{U}$ and $\lambda \in (0,1)$. From (3.3), (3.7), we have

$$
|N_{2i}(u)(t)|_i \le M_i \int_0^t |\Phi_i(u)(s)|\,ds \le M_i\,|\delta_i(.)\,\psi_i(|u_i(.)|_i)|_{L^1(0,t)} \tag{3.9}
$$

Also, from (3.8) and (3.9),

$$
\begin{aligned}
|N_{1i}(u)(t)|_i &\le M_i \sum_{j=1}^{n} |G_{ij}|\,\left|\chi_{a_j}(N_{2j}(u))\right|_{C([0,a];X_j)}\\
&\le M_i \sum_{j=1}^{n} |G_{ij}|\,M_j \left|\delta_j(.)\,\psi_j\left(|u_j(.)|_j\right)\right|_{L^1(0,a_j)}\\
&\le r_i.
\end{aligned} \tag{3.10}
$$

Then, since $u = \lambda N(u)$, for each $t \in [0,a]$, one has

$$|u_i(t)|_i \leq \lambda \left(r_i + M_i \left| \delta_i(.) \psi_i(|u_i(.)|_i) \right|_{L^1(0,t)} \right) =: c_i(t).$$

Next we follow the same argument as in the proof of Lemma 2.1 in order to show that

$$c_i(t) < R_i(t) \quad \text{for every} \quad t \in [0,a].$$

To check condition (b) of Theorem 1.1, let $C \subset \overline{U}$ be countable and $C \subset \overline{\text{conv}}(\{0\} \cup N(C))$. Then, for each i,

$$\begin{aligned} \varphi_i(t) := {} & \alpha_i(C_i(t)) = \alpha_i(N_i(C)(t)) \\ \leq {} & \alpha_i(N_{1i}(C)(t)) + \alpha_i(N_{2i}(C)(t)), \quad t \in [0,a]. \end{aligned} \tag{3.11}$$

Using (1.7), **(H4)** and (3.11) we obtain for a.e. $t \in [0,a]$

$$\begin{aligned} \alpha_i(N_{2i}(C)(t)) & \leq 2 \int_0^t M_i \alpha_i(\Phi_i(C)(s))\, ds \\ & \leq 2M_i \int_0^t \sum_{j=1}^n \gamma_{ij}(s)\, \alpha_j(C_j(s))\, ds \\ & = 2M_i \int_0^t \sum_{j=1}^n \gamma_{ij}(s)\, \varphi_j(s)\, ds. \end{aligned} \tag{3.12}$$

This, in view of (3.8), yields

$$\begin{aligned} \alpha_i(N_{1i}(C)(t)) & \leq M_i \alpha_i \left(\sum_{j=1}^n G_{ij} \left(\chi_{a_j}(N_{2j}(C)) \right) \right) \\ & \leq M_i \sum_{j=1}^n |G_{ij}|\, \alpha_{C_j} \left(\chi_{a_j}(N_{2j}(C)) \right), \end{aligned}$$

where α_{C_j} is the Kuratowski measure of noncompactness on $C([0,a];X_j)$. Furthermore, by (3.12) and **(H4)**, we get

$$\begin{aligned} \alpha_{C_j} \left(\chi_{a_j}(N_{2j}(C)) \right) & = \max_{t \in [0,a]} \alpha_j \left(\chi_{a_j}(N_{2j}(C))(t) \right) = \max_{t \in [0,a_j]} \alpha_j((N_{2j}(C))(t)) \\ & \leq 2M_j \int_0^{a_j} \sum_{k=1}^n \gamma_{jk}(s)\, \varphi_k(s)\, ds = 2M_j \int_0^{a_F} \sum_{k=1}^n \widetilde{\gamma}_{jk}(s)\, \varphi_k(s)\, ds. \end{aligned}$$

Then

$$\alpha_i(N_{1i}(C)(t)) \leq M_i \sum_{j=1}^n |G_{ij}|\, 2M_j \int_0^{a_F} \sum_{k=1}^n \widetilde{\gamma}_{jk}(s)\, \varphi_k(s)\, ds. \tag{3.13}$$

Now from (3.11)-(3.13) we find

$$\varphi_i(t) \leq 2 \sum_{j=1}^n M_i |G_{ij}| \int_0^{a_F} \sum_{k=1}^n M_j \widetilde{\gamma}_{jk}(s)\, \varphi_k(s)\, ds + 2 \int_0^t \sum_{k=1}^n M_i \gamma_{ik}(s)\, \varphi_k(s)\, ds.$$

If we denote

$$\begin{aligned}\gamma(t) &= [M_i\gamma_{ij}(t)]_{1\le i,j\le n}, \quad \widetilde{\gamma}(t)=[M_i\widetilde{\gamma}_{ij}(t)]_{1\le i,j\le n},\\ \varphi(t) &= [\varphi_1(t),\varphi_2(t),...,\varphi_n(t)]^{\mathrm{tr}},\end{aligned} \tag{3.14}$$

then the above inequalities for $i=1,2,...,n$, can be put under the vectorial form as

$$\varphi(t)\le 2|\mathcal{G}|\int_0^{a_F}\widetilde{\gamma}(s)\varphi(s)\,ds+2\int_0^t\gamma(s)\varphi(s)\,ds,\qquad t\in[0,a]. \tag{3.15}$$

Finally we follow the same argument as in the proof of Theorem 2.3, in order to show that $\varphi\equiv 0$ on $[0,a]$. The only one difference is that for $t\in[0,a_F]$, from (3.15), we have

$$\varphi(t)\le 2\left(|\mathcal{G}|\,|\widetilde{\gamma}|_{L^1(0,a_F)}+|\gamma|_{L^1(0,a_F)}\right)|\varphi|_{L^\infty(0,a_F)}=H\,|\varphi|_{L^\infty(0,a_F)},$$

whence

$$|\varphi|_{L^\infty(0,a_F)}\le H\,|\varphi|_{L^\infty(0,a_F)}, \tag{3.16}$$

where by $|\varphi|_{L^\infty(0,a_F)}$ we mean the column matrix of entries $|\varphi_i|_{L^\infty(0,a_F)}$. Then (3.16) is equivalent to the matrix inequality

$$(I-H)\,|\varphi|_{L^\infty(0,a_F)}\le 0. \tag{3.17}$$

By (3.6), the entries of the matrix $(I-H)^{-1}$ are nonnegative, so in (3.17) we can multiply to the left by $(I-H)^{-1}$ without changing the inequality, to obtain $|\varphi|_{L^\infty(0,a_F)}$ ≤ 0. Hence $\varphi(t)=0$ for all $t\in[0,a_F]$. The Gronwall's inequality implies that $\varphi(t)=0$ for all $t\in[a_F,a]$. Taking into account of (3.16) and (3.14) we can say that for each $i=1,2,...,n$ and for all $t\in[0,a]$,

$$\varphi_i(t)=\alpha_i(C_i(t))=0,$$

so $C_i(t)$ is relatively compact in X_i and $C(t)=\prod_{i=1}^n C_i(t)$ is relatively compact in X. Following the same argument of the proof of Lemma 2.2 we have that C is equicontinuous, so we can say that the condition **(h4^0)** is satisfied. On the other hand, by using **(H1)-(H3)**, as in the proof of Lemma 2.1 we can deduce **(h3^0)**. Therefore Theorem 2.1 provides the existence of at least one mild solution $u=(u_1,u_2,...,u_n)$ where $u_i\in\overline{U_i}$, $i=1,2,...,n$. □

Remark 3.1. In general, we have the matrix inequality $|\widetilde{\gamma}|_{L^1(0,a_F)}\le|\gamma|_{L^1(0,a_F)}$. In particular, if $a_1=a_2=...=a_n\ (=a_F)$, i.e. $[0,a_F]$ is the support of F with respect to all variables, one has $\gamma(t)=\widetilde{\gamma}(t)$ for all $t\in[0,a_F]$, which gives $|\widetilde{\gamma}|_{L^1(0,a_F)}=|\gamma|_{L^1(0,a_F)}$ and $H=2(|\mathcal{G}|+I)\,|\gamma|_{L^1(0,a_F)}$.

To conclude, let us underline the combined contribution of the functions δ_i, ψ_i, R_i, γ_{ij} and numbers M_i and a_i to the conditions of Theorem 3.1. In particular, note the different contribution of the support intervals $[0,a_i]$, $i=1,2,...,n$, in realizing the assumptions (3.4) and (3.6). As smaller a_i are, more chance for (3.4), (3.6) exists. In the limit case, where $a_i=0$ for all i, that is for the classical Cauchy problem, the conditions (3.4) and (3.6) are trivially satisfied.

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