On the Jointly Continuous Utility Representation Problem

Alessandro Caterino[a], Rita Ceppitelli[b,*], Ľubica Holá[c]

[a] *Via dei Narcisi 41A, 06100 Perugia, Italy*
[b] *Department of Mathematics and Computer Sciences, University of Perugia, Via L. Vanvitelli 1, 06123 Perugia, Italy*
[c] *Institute of Mathematics, Academy of Science, Štefánikova 49, 81473 Bratislava, Slovakia*

Abstract

Submetrizable k_ω-spaces seem to be interesting in the study of finite dimensional State Preference Models and in ordering of distributions of wellbeing. In these applications the existence of utilities representing families of preorders is important. In the present paper we are interested in the problem of the jointly continuous utility representations for submetrizable k_ω-spaces. We found a right and natural generalization of Back's Theorem. We improve the results in the paper A. Caterino, R. Ceppitelli, *Jointly continuous utility functions on submetrizable k_ω-spaces*, Topology and its Applications, **190**, (2015) 109-118.

Keywords: Jointly continuous utility functions, closed preorders, Back's Theorem, submetrizable k_ω-spaces, boundedly compact metric, Fell topology.
2010 MSC: 54F05, 91B16, 54B20

1. Introduction

In the present paper we are mainly interested in the problem of the jointly continuous utility representations.
Let (X, τ) be a Hausdorff topological space and let $CL(X)$ be the nonempty closed subsets of X. Let

$$\mathcal{P} = \{\preceq : \ \preceq \text{ is a preorder on } D(\preceq) \in CL(X)\} \subset CL(X \times X)$$

be the family of closed preorders defined on closed subsets $D \subset X$ and let $\mathcal{U} = \{(D, u) : D \in CL(X), u \in C(D, I\!\!R)\}$. The jointly continuous utility representation problem consists in finding suitable topologies on $\mathcal{P}$, X and $\mathcal{U}$ ensuring the existence of a continuous function $\nu : \mathcal{P} \to \mathcal{U}$ such that for every

*Corresponding author
Email addresses: `sandro.caterino@gmail.com` (Alessandro Caterino), `rita.ceppitelli@unipg.it` (Rita Ceppitelli), `hola@mat.savba.sk` (Ľubica Holá)

Preprint submitted to Elsevier *October 11, 2019*

Accepted 11 October 2019 https://doi.org/10.1016/j.topol.2019.106919

$\preceq\in\mathcal{P}\ \ \nu(\preceq)$ is a utility function for $\preceq$.
In this case the utility functions $\nu(\preceq)$ are partial maps with different choice sets.

A natural topology (convergence) on the set $\mathcal{P}$ of preorders should satisfy the following condition:

$$(*)\qquad x_\sigma\to x\,,\,y_\sigma\to y\,,\,\preceq_\sigma\to\preceq\,,\,x_\sigma\preceq_\sigma y_\sigma\Longrightarrow x\preceq y.$$

When X is second countable locally compact, a currently applied topology on $\mathcal{P}$ is the topology of closed convergence (the Fell topology) (see [17, 18, 20, 1]).

We particularly refer to the representation theorem of Back [1]. We recall that Back's result is based on Levin's Theorems [19].

Back in [1] endowed the space $\mathcal{P}$ with the Fell topology, the space $\mathcal{U}$ with a new topology τ_c and proved the existence of a continuous map from the space of total preorders $\mathcal{P}_{\mathcal{L}}$ to the space $\mathcal{U}$ of utility functions.

In our paper we will work with submetrizable k_ω-spaces.
Let $(X_n,\tau_n)_n$ be a closed tower, i.e. a countable increasing family of topological spaces such that each space X_n is closed in X_{n+1}. Let $(X,\tau)=lim_{\rightarrow}X_n$ be the inclusion inductive limit of $(X_n,\tau_n)_n$, where τ is the finest topology on X for which the inclusion maps $j_n:X_n\hookrightarrow X, n\in\mathbb{N}$ are continuous. It is well known that if each $X_n, n\in\mathbb{N}$ is a Hausdorff compact subspace, then X is a k_ω-space (i.e. X is a hemicompact k-space) and if every X_n is also metrizable it is submetrizable, too. We recall that a k_ω-space is submetrizable if and only if its compact subsets are metrizable [7]. Using this property, in [8] the authors proved that X is a submetrizable k_ω-space if and only if X is the inclusion inductive limit of a closed tower of Hausdorff second countable locally compact spaces.
Interesting examples of submetrizable k_ω-spaces that are not metrizable spaces are $lim_{\rightarrow}\mathbb{R}^n$ and the space S' of tempered distributions ([7], example 4.4.1). These spaces have applications in Mathematical Economics as shown in Section 5 and also in [8]. In these applications the existence of utilities representing families of preorders is important.

In [8] the authors introduced on $\mathcal{P}\subset CL(X\times X)$ a Fell-type topology $\bigcup_n F(\tau_n\times\tau_n)$ and generalized Back's Theorem for submetrizable k_ω-spaces. However, in Theorem 4.4 of this paper we proved that the topology $\bigcup_n F(\tau_n\times\tau_n)$ does not satisfy the property $(*)$ in non locally compact spaces. A necessary and sufficient condition for a topology defined on the set of preorders on an arbitrary topological space to satisfy property $(*)$ is given in Theorem 4.3.
In this paper using compatible uniform structures $\mathbf{U}$ on X, we introduced new topologies $\tau(\mathcal{G}),\mathcal{G}\in\mathbf{U}$ on $\mathcal{P}$ finer than $\bigcup_n F(\tau_n\times\tau_n)$, which satisfy the property $(*)$ and generalize Back's Theorem. Really, also the topology $\bigcap_{\mathcal{G}\in\mathbf{U}}\tau(\mathcal{G})$ is finer than $\bigcup_n F(\tau_n\times\tau_n)$ and generalizes Back's results, but we do not know whether satisfies the property $(*)$. For this reason we proposed a convergence on $\mathcal{P}$

which seems to be a right and natural generalization of Back's convergence for submetrizable k_ω-spaces and satisfies the property $(*)$.

2. Notation and preliminaries

Let $(X_n, \tau_n)_n$ be a countable increasing family of locally compact second countable Hausdorff spaces such that each space X_n is closed in X_{n+1}. Such a family $X_1 \subset X_2 \subset ... \subset X_n \subset ...$ is called *a closed tower*. The notion of a closed tower is currently used in the literature; see [23].

Let $\hat{X} = \bigsqcup_n (X_n \times \{n\})$ be the disjoint union of X_n. For every n, let $i_n : X_n \hookrightarrow \hat{X}$ be the canonical inclusion map defined by $i_n(x) = (x, n)$. The disjoint union topology on $\hat{X}$ is the largest topology on $\hat{X}$ for which the inclusion maps are continuous (i.e. the final topology for the family of functions $\{i_n\}_n$). Equivalently: $U \subset \hat{X}$ is open in $\hat{X}$ iff $i_n^{-1}(U)$ is open in X_n for every $n \in \mathbb{N}$.
The space $\hat{X}$ is locally compact, second countable and metrizable by the metric $\hat{d}((x,n),(y,m)) = d_n(x,y)$ if $m = n$, $\hat{d}((x,n),(y,m)) = 1$ elsewhere, where d_n is a compatible metric in X_n.

The family $\{(X_n, i_{m,n}) : m, n \in \mathbb{N}\}$ of spaces and inclusion maps $i_{m,n} : X_m \to X_n, m \leq n$ is an inductive spectrum with spaces X_n and connecting maps $i_{m,n}$. This inductive spectrum yields a limit space in the following way. Let $\sim$ be the equivalence relation on $\hat{X}$ defined by $(x,n) \sim (y,m) \Leftrightarrow x = y$. The quotient space $(X, \tau) = \hat{X}_{|\sim}$, where τ is the quotient topology, is, by the definition, the inclusion inductive limit of that spectrum and it is denoted by $lim_{\rightarrow} X_n$.
A model of the inclusion inductive limit is $X = \bigcup_n X_n$ with the topology τ so defined: a subset A is open in (X, τ) if and only if $A \cap X_n$ is open in (X_n, τ_n), for every $n \in \mathbb{N}$.
We will say that the tower $(X_n)_n$ determines the topology τ, that is τ is the finest topology on X for which the inclusion maps $j_n : X_n \hookrightarrow X, n \in \mathbb{N}$ are continuous.

Since each space X_n is closed in $X_{n+1}, n \in \mathbb{N}$, the space (X, τ) inherits some separation properties, in fact it is a T_4 space ([15]).
In [8] the authors studied the relationships between the submetrizable k_ω-spaces and the Hausdorff spaces which are the inductive limit of a countable increasing family of locally compact second countable subspaces.

Definition 2.1. *A topological space X is said to be a k_ω-space if X is the inclusion inductive limit of a countable increasing family $(K_n)_n$ of Hausdorff compact subspaces. The family $(K_n)_n$ is called a* k_ω-decomposition *of X.*

An interesting survey on k_ω-spaces can be found in [7, 15].
The following proposition is a restatement of the Proposition 2.8 of [8].

Proposition 2.2. *Let (X, τ) be a topological space. TFAE:*

(i) X is a submetrizable k_ω-space;
(ii) X is the inclusion inductive limit of a countable increasing family of metric compact spaces;
(iii) X is the inclusion inductive limit of a closed tower of Hausdorff second countable locally compact spaces.

If $(X_n)_n, (Y_n)_n$ are closed towers, the product topology on $lim_{\rightarrow}X_n \times lim_{\rightarrow}Y_n$ in general may not be the topology determined by the tower $(X_n \times Y_n)_n$ ([14]), but in the case of the k_ω-spaces this does not happen.

Proposition 2.3. ([15]). *Let $X = lim_{\rightarrow}X_n$ and $Y = lim_{\rightarrow}Y_n$ be k_ω-spaces. Then $lim_{\rightarrow}(X_n \times Y_n) = X \times Y$.*

Here are some examples of submetrizable k_ω-spaces that are not metrizable spaces. We recall that every first countable hemicompact space is locally compact.

Example 2.4. [12] Let $\mathbb{Z}, \mathbb{R}$ be spaces of integers and real numbers and $Y = \mathbb{R}/\mathbb{Z}$ be the quotient space obtained by identifying $\mathbb{Z}$ to the point $y_0 \in Y$. Clearly Y is a k-space but it is not metrizable since it is not first countable (y_0 fails to have a countable base of neighbourhoods). Moreover, Y is submetrizable and hemicompact.

Example 2.5. The space $lim_{\rightarrow}\mathbb{R}^n$ is a submetrizable k_ω-space of the *first category*, i.e., it is the union of a countable family of closed sets having empty interiors. Then $lim_{\rightarrow}\mathbb{R}^n$ is not locally compact, and hence not metrizable, because it is not a Baire space.

Example 2.6. The space S' of tempered distributions ([7], example 4.4.1) is another example of a submetrizable k_ω-space that is not metrizable. In fact S' is an infinite dimensional topological vector space, while every locally compact topological vector space has the finite dimension (see [22], Theorem 1.22).

3. The Fell topology on submetrizable k_ω-spaces

One of the most important topologies on the space of closed subsets of a topological space is the Fell topology.

Let (X, τ) be a Hausdorff topological space and let $CL(X)$ be the nonempty closed subsets of X. The Fell topology $F(\tau)$ on $CL(X)$ is the topology having as a subbase all sets of the form

$$U^- = \{B \in CL((X, \tau)) : B \cap U \neq \emptyset\},\ U \in \tau$$

$$(K^c)^+ = \{B \in CL((X, \tau)) : B \cap K = \emptyset\}, K \text{ compact in } (X, \tau).$$

A comprehensive reference on the Fell topology is [2].
We recall the following results:

Proposition 3.1. ([2], Proposition 5.1.2) *Let (X,τ) be a Hausdorff space. The following are equivalent:*

1. *$(CL(X), F(\tau))$ is Hausdorff;*
2. *$(CL(X), F(\tau))$ is regular;*
3. *$(CL(X), F(\tau))$ is completely regular;*
4. *(X,τ) is locally compact.*

Proposition 3.2. ([2], Corollary 5.1.4) *Let (X,τ) be a locally compact Hausdorff space. Then $(CL(X), F(\tau))$ is a locally compact Hausdorff space.*

Proposition 3.3. ([2], Theorem 5.1.5) *Let (X,τ) be a Hausdorff space. The following are equivalent:*

1. *(X,τ) is locally compact and second countable;*
2. *$(CL(X), F(\tau))$ is a Polish space (separable and metrizable with a complete metric);*
3. *$(CL(X), F(\tau))$ is metrizable.*

Theorem 3.4. *Let $(X_1,\tau_1) \subset (X_2,\tau_2) \subset ... \subset (X_n,\tau_n) \subset ...$ be a closed tower of locally compact second countable Hausdorff spaces.*
Then $(CL(X_1), F(\tau_1)) \subset (CL(X_2), F(\tau_2)) \subset ... \subset (CL(X_n), F(\tau_n)) \subset ...$ is also a closed tower of locally compact second countable Hausdorff spaces and $lim_{\rightarrow}(CL(X_n), F(\tau_n))$ is a submetrizable k_ω-space.

Proof. By Propositions 3.2 and 3.3 $(CL(X_n), F(\tau_n))$ is a locally compact second countable Hausdorff space for every n.
Moreover $F(\tau_n) = F(\tau_{n+1})_{|CL(X_n)}$ for every n (see [13], Lemma 3.1).
Finally $\underline{(CL(X_n), F(\tau_n))}$ is closed in $(CL(X_{n+1}), F(\tau_{n+1}))$. In fact, let $A \in CL(X_{n+1}), A \in \overline{CL(X_n)}$. Put $U = X_{n+1} \setminus X_n$. If $A \cap U \neq \emptyset$, then $A \in U^-$ but $U^- \cap CL(X_n) = \emptyset$, a contradiction.
By Proposition 2.2 $lim_{\rightarrow}(CL(X_n), F(\tau_n))$ is a submetrizable k_ω-space. $\square$

It seems interesting to investigate the relationship between the hyperspace $(CL(X), F(\tau))$ where $X = lim_{\rightarrow}X_n$ and the hyperspace $lim_{\rightarrow}(CL(X_n), F(\tau_n))$. By Proposition 3.1, the space $(CL(lim_{\rightarrow}X_n), F(\tau))$ is not Hausdorff unless $lim_{\rightarrow}X_n$ is locally compact.

Theorem 3.5. *Let $(X_1,\tau_1) \subset (X_2,\tau_2) \subset ... \subset (X_n,\tau_n) \subset ...$ be a closed tower of locally compact second countable Hausdorff spaces. There is a continuous inclusion map*

$$h : lim_{\rightarrow}(CL(X_n), F(\tau_n)) \hookrightarrow (CL(lim_{\rightarrow}X_n), F(\tau)).$$

Proof. First we note that $lim_{\rightarrow}CL(X_n) \subset CL(lim_{\rightarrow}X_n)$.
Up to homeomorphism we suppose $lim_{\rightarrow}CL(X_n) = \bigcup_n CL(X_n)$. So, if $E \in lim_{\rightarrow}CL(X_n)$ then $E \in CL(X_{n_0})$ for some $n_0 \in \mathbb{N}$. Therefore $E \cap X_n$ is closed,

for every $n \in \mathbb{N}$ and $E \in CL(lim_{\rightarrow} X_n)$.
To prove the continuity of the map h, it is sufficient to check only subbase elements of $F(\tau)$.
We denote by η the topology of the space $lim_{\rightarrow}(CL(X_n), F(\tau_n))$. Let U be an open set in $lim_{\rightarrow} X_n$. Then $U \cap X_n$ is open in X_n for every $n \in \mathbb{N}$.
Let $U^- = \{E \in CL(lim_{\rightarrow} X_n) : E \cap U \neq \emptyset\}$.
We will prove that $U^- \cap lim_{\rightarrow} CL(X_n)$ is η-open.
$U^- \cap lim_{\rightarrow} CL(X_n) = \{E \in lim_{\rightarrow} CL(X_n) : E \cap U \neq \emptyset\} =$
$= \bigcup_n \{E \in CL(X_n) : E \cap U \neq \emptyset\} = \bigcup_n \{E \in CL(X_n) : E \cap (U \cap X_n) \neq \emptyset\} =$
$= \bigcup_n (U \cap X_n)^-$ which is η-open.
Now, let K be a compact set in $lim_{\rightarrow} X_n$, then $K \subset X_n$ for some $n \in \mathbb{N}$.
If $(K^c)^+ = \{E \in CL(lim_{\rightarrow} X_n) : E \cap K = \emptyset\}$,
then $(K^c)^+ \cap lim_{\rightarrow} CL(X_n) = \{E \in lim_{\rightarrow} CL(X_n) : E \cap K = \emptyset\} =$
$\bigcup_n \{E \in CL(X_n) : E \cap K = \emptyset\} = \bigcup_n ((K \cap X_n)^c)^+$
that is η-open in $lim_{\rightarrow}(CL(X_n), F(\tau_n))$. □

Remark 3.6. In general, h is not a homeomorphism between $lim_{\rightarrow}(CL(X_n), F(\tau_n))$ and $h(lim_{\rightarrow}(CL(X_n), F(\tau_n)))$.
Put

$$X_n = \{0\} \cup [1/n, 1], n \in \mathbb{N}.$$

Then

$$h : lim_{\rightarrow}(CL(X_n), F(\tau_n)) \hookrightarrow (CL(lim_{\rightarrow} X_n), F(\tau))$$

is not an embedding.

For every n, $([1/n, 1]^c)^+ = \{0\}$. Thus $\{0\}$ is open in every $(CL(X_n), F(\tau_n))$ and so it is open in $lim_{\rightarrow}(CL(X_n), F(\tau_n))$.

Now, we show that $\{0\}$ is not open in $h(lim_{\rightarrow}(CL(X_n), F(\tau_n)))$.

Without loss of generality, let

$$\{0\} \in I = \{0\}^- \cap (K^c)^+,$$

where K is a compact subset of $lim_{\rightarrow} X_n$ ($\{0\}$ is open in $lim_{\rightarrow} X_n$).
Since $K \subset X_{m-1}$ for some $m \in \mathbb{N}$, then $1/m \notin K$, hence

$$\{0, 1/m\} \in I \cap CL(X_m) \subset I \cap h(lim_{\rightarrow}(CL(X_n), F(\tau_n))).$$

From the previous Remark 3.6 we can see that $lim_{\rightarrow} CL(X_n)$ can be a proper subset of $CL(lim_{\rightarrow} X_n)$. In fact, $[0, 1] \in CL(lim_{\rightarrow} X_n) \setminus lim_{\rightarrow} CL(X_n)$.

4. Jointly continuous utility functions on submetrizable k_ω-spaces

A *preference relation* $\preceq$ on a set (of alternatives) X is a *preorder*, that is a *reflexive* and *transitive* binary relation. The preference relation $\preceq$ is *complete* or

total if any two elements of X are comparable (i.e., for every $x, y \in X \times X$, either $x \preceq y$ or $y \preceq x$). The use of non-total preorders may be viewed as more realistic and adequate in order to explain the behavior of an individual. In Economics, *preference relations* are often described by means of *utility functions*. Mainly the literature is interested in representability of preorders by means of continuous utility functions.
A function $u : X \longrightarrow \mathbb{R}$ is a utility function representing a preference relation $\preceq$ if:

- $\forall x, y \in X\,, x \preceq y \Rightarrow f(x) \leq f(y)$;
- $\forall x, y \in X\,, x \prec y \Rightarrow f(x) < f(y)$.

Classical hypotheses for the existence of continuous representations of preorders are the *continuity* of $\preceq$ or the *closedness* of $\preceq$, stronger than continuity [5, 21].
A preference relation on a topological space X is *continuous (or semiclosed)* if for every $x \in X$ both the sets $(-\infty, x] = \{z \in X : z \preceq x\}$ and $[x, +\infty) = \{z \in X : x \preceq z\}$ are closed in X.
A preference relation $\preceq$ on a topological space X is said to be *closed* if its graph $\{(x, y) \in X \times X : x \preceq y\}$ is a closed subset of the topological product $X \times X$.
It is well known that if the preorder $\preceq$ is total, then $\preceq$ is closed iff it is continuous.
In the study of problems of Szpilrajn-type, Herden and Pallack [16] introduced the concept of a *weakly continuous* preorder; see also [4, 3].
A preference relation $\preceq$ on a topological space X is said to be *weakly continuous* if for every $x, y \in X$ such that $x \prec y$ there exists a continuous increasing function $u_{xy} : (X, \tau, \precsim) \longrightarrow (\mathbb{R}, \tau_{nat}, \leq)$ such that $u_{xy}(x) < u_{xy}(y)$.
If the preorder $\preceq$ is total, then $\preceq$ is weakly continuous iff it is continuous. Herden and Pallack ([16], Proposition 2.11) proved that every weakly continuous binary relation on a topological space X has a continuous refinement by a closed preorder.

In the present paper we are mainly interested in the problem of the jointly continuous utility representations on submetrizable k_ω-spaces. We refer in particularly to the representation theorem of Back [1] whose result is based on Levin's Theorems [19]. Let

$$\mathcal{P} = \{\preceq :\ \preceq \text{ is a preorder on } D(\preceq) \in CL(X)\} \subset CL(X \times X)$$

be the family of closed preorders defined on closed subsets $D \subset X$ and let

$$\mathcal{U} = \{(D, u) : D \in CL(X), u \in C(D, I\!R)\}$$

be the family of continuous real functions defined on closed subsets of X (partial maps).
Our aim is to find suitable topologies on $\mathcal{P}$ and $\mathcal{U}$ ensuring the existence of a continuous function $\nu : \mathcal{P} \to \mathcal{U}$ such that for every $\preceq \in \mathcal{P}$ $\nu(\preceq)$ is a utility

function for $\preceq$.
A natural topology on the set $\mathcal{P}$ of preorders should satisfy the following condition:

$$(*) \qquad x_\sigma \to x\,,\, y_\sigma \to y\,,\, \preceq_\sigma \to \preceq\,,\, x_\sigma \preceq_\sigma y_\sigma \Longrightarrow x \preceq y.$$

When X is a Hausdorff second countable locally compact space, a currently applied topology on $\mathcal{P}$ is the topology of closed convergence (the Fell topology) (see [17, 18, 20, 1]). In this setting Back in [1] endowed the space $\mathcal{U}$ with a new topology τ_c and proved the existence of a continuous map from the space of total preorders $\mathcal{P}_{\mathcal{L}}$ to the space $\mathcal{U}$ of utility functions.

Definition 4.1. *Given a topological space (X, τ), the τ_c-topology on $\mathcal{U}$ is the topology that has as a subbase the sets*

$$[G] = \{(D, u) \in \mathcal{U} : \ D \cap G \neq \emptyset\}$$

$$[K : I] = \{(D, u) \in \mathcal{U} : \ u(D \cap K) \subset I\}$$

where G is a τ-open subset of X, $K \subset X$ is a τ-compact and $I \subset \mathbb{R}$ is open (possibly empty).

Back considered the spaces $(\mathcal{P}_{\mathcal{L}}, F(\tau \times \tau)_{|\mathcal{P}_{\mathcal{L}}})$ and $(\mathcal{U}, \tau_c)$ and proved the following:

Theorem 4.2. (Back, [1]) *Let X be a Hausdorff locally compact and second countable space. There exists a continuous map $\nu : \mathcal{P}_{\mathcal{L}} \to \mathcal{U}$ such that $\nu(\preceq)$ is a utility function for every $\preceq \in \mathcal{P}_{\mathcal{L}}$. Any such map ν is actually a homeomorphism of $\mathcal{P}_{lns}$ onto $\nu(\mathcal{P}_{lns})$, where $\mathcal{P}_{lns}$ is the family of total locally non-satiated preorders.*

We recall that a preorder $\preceq$ is called *locally non-satiated* if for each $x \in D(\preceq)$ and each neighbourhood U of x there is $y \in U$ such that $x \prec y$.

In [11] it was pointed out that Back's proof holds also in the case of closed preorders not necessarily total. Then using the same technique it is possible to prove the existence of a continuous map $\nu : (\mathcal{P}, F(\tau \times \tau)_{|\mathcal{P}}) \to (\mathcal{U}, \tau_c)$.
Of course, the hypothesis according to which the preorders are total cannot be dropped to prove that ν is a homeomorphism of $\mathcal{P}_{\mathrm{lns}}$ onto $\nu(\mathcal{P}_{\mathrm{lns}})$.

The submetrizable $k_\omega -$ *spaces* have an important role in the study of the utility representation problem. $lim_{\to}\mathbb{R}^n$ and the space S' of tempered distributions are submetrizable k_ω-spaces which are not metrizable. These spaces have applications in Mathematical Economics as it is shown in Section 5 and in [8]. Levin's Theorems have been generalized in the case of the consumption space X is a submetrizable k_ω-space and $\mathcal{P}$ is metrizable or both spaces are hemicompact, submetrizable and their product is a k-space [12].

Our aim is to generalize Back's result for a submetrizable k_ω-space X.

The next theorem gives a necessary and sufficient condition for a topology defined on the set of preorders of an arbitrary topological space to satisfy the property $(*)$.

Theorem 4.3. *Let X be a topological space. Let η be a topology on $\mathcal{P}$. The following conditions are equivalent:*

1. *η satisfies the property $(*)$;*
2. *whenever $\{\preceq_\sigma : \sigma \in \Sigma\}$ converges to $\preceq$ in $(\mathcal{P}, \eta)$, then $Ls \preceq_\sigma \subseteq \preceq$.*

Proof. (2) $\Rightarrow$ (1) is trivial.
(1) $\Rightarrow$ (2) Let $\{\preceq_\sigma : \sigma \in \Sigma\}$ be a net in $\mathcal{P}$ which converges to $\preceq$ in $(\mathcal{P}, \eta)$.
Let $(x, y) \in Ls \preceq_\sigma$. We will show that there is a net $\{(x_i, y_i) : i \in I\}$ such that $\{(x_i, y_i) : i \in I\}$ converges to (x, y), $(x_i, y_i) \in \ll_i$ for every $i \in I$ and $\{\ll_i : i \in I\}$ is a subnet of $\{\preceq_\sigma : \sigma \in \Sigma\}$. Then by (1) we are done. ($\{\ll_i : i \in I\}$ converges to $\preceq$ in $(\mathcal{P}, \eta), \{(x_i, y_i) : i \in I\}$ converges to (x, y), thus $(x, y) \in \preceq$.)
Let $\mathcal{U}(x, y)$ be a base of open neighbourhoods of (x, y). Consider $\mathcal{U}(x, y) \times \Sigma$ with the natural direction; i.e.

$$(V, \zeta) \trianglelefteq (U, \sigma) \Leftrightarrow U \subseteq V \quad \text{and} \quad \zeta \leq \sigma.$$

For every $(U, \sigma) \in \mathcal{U}(x, y) \times \Sigma$ put $\ll_{U,\sigma} = \preceq_\sigma$. It is easy to verify that $\{\ll_{U,\sigma} : (U, \sigma) \in \mathcal{U}(x, y) \times \Sigma\}$ is a subnet of $\{\preceq_\sigma : \sigma \in \Sigma\}$.
For every $U \in \mathcal{U}(x, y)$ put

$$H_U = \{\zeta \in \Sigma : \ll_{U,\zeta} \cap U \neq \emptyset\}.$$

Put $I = \{(U, \zeta) \in \mathcal{U}(x, y) \times \Sigma : \zeta \in H_U\}$. Then I is cofinal in $\mathcal{U}(x, y) \times \Sigma$.
Let $(U, \sigma) \in \mathcal{U}(x, y) \times \Sigma$. Then there is $\zeta \in \Sigma$ such that $\sigma \leq \zeta$ and $\ll_{U,\zeta} \cap U \neq \emptyset$; i.e. $\zeta \in H_U, (U, \zeta) \in I$ and $(U, \sigma) \trianglelefteq (U, \zeta)$. Thus the net $\{\ll_{U,\zeta} : (U, \zeta) \in I\}$ is a subnet of $\{\preceq_\sigma : \sigma \in \Sigma\}$.
For every $(U, \zeta) \in I$ choose $(x_{U,\zeta}, y_{U,\zeta}) \in \ll_{U,\zeta} \cap U$.
It is easy to verify that $\{(x_{U,\zeta}, y_{U,\zeta}) : (U, \zeta) \in I\}$ converges to (x, y). $\square$

In the following (X, τ) will denote a submetrizable k_ω-space that is the inclusion inductive limit of a closed tower $(X_n, \tau_n)_n$ of Hausdorff second countable locally compact spaces.

In [8] the authors introduced on $\mathcal{P} \subset CL(X \times X)$ a Fell-type topology $\bigcup_n F(\tau_n \times \tau_n)$ generated by all sets of the form

$$U^- = \{\preceq \in \mathcal{P} : \preceq \cap\, U \neq \emptyset\},\ U \text{ open in } (X_n, \tau_n) \times (X_n, \tau_n) \quad \text{for some} \quad n \in \mathbb{N}$$

$$(K^c)^+ = \{\preceq \in \mathcal{P} : \preceq \cap\, K = \emptyset\},\ K \text{ compact in } (X, \tau) \times (X, \tau).$$

Note that when (X, τ) is a submetrizable k_ω-space which is not locally compact, neither the Fell topology nor the topology $\bigcup_n F(\tau_n \times \tau_n)$ satisfies the condition $(*)$ as it is proved in the following:

Theorem 4.4. *Let $\mathcal{P}$ be equipped with the topology $\bigcup_n F(\tau_n \times \tau_n)$. Then $\bigcup_n F(\tau_n \times \tau_n)$ satisfies the property $(*)$ if and only if X is locally compact.*

Proof. Suppose X is not locally compact. There must exist $x \in X$ such that for every open neighborhood U of x and every compact $K \subset X$ there is $x_{U,K} \in U \setminus K$. Let $\mathcal{U}(x)$ be a base of open neighborhoods of x and $K(X)$ the family of all compact sets in X.
We will order elements of $\mathcal{U}(x) \times K(X)$ as follows:

$$(U,K) \trianglelefteq (V,C) \Longleftrightarrow V \subset U \quad \text{and} \quad C \supset K.$$

$(\mathcal{U}(x) \times K(X), \trianglelefteq)$ is a directed set. Let $y \in X$ be such that $y \neq x$. For every $(U,K) \in \mathcal{U}(x) \times K(X)$ put $\preceq_{U,K} = \{(x_{U,K}, x_{U,K}), (y,y)\}$ and $\preceq = \{(y,y)\}$.
It is easy to verify that $\{\preceq_{U,K} : (U,K) \in \mathcal{U}(x) \times K(X)\}$ converges to $\preceq$ in $(\mathcal{P}, \bigcup_n F(\tau_n \times \tau_n))$ and that $(x,x) \in Ls \preceq_{U,K}$, however $(x,x) \notin \preceq$. $\square$

Now, let $\mathcal{G}$ be a compatible uniformity on X and $\mathcal{B}$ be a base of closed symmetric elements from $\mathcal{G}$. We will define a new topology $\tau(\mathcal{G})$ on $\mathcal{P}$. $\tau(\mathcal{G})$ is generated by all sets of the form

$$U^- = \{\preceq \in \mathcal{P} : \preceq \cap\, U \neq \emptyset\},\ U \text{ open in } (X_n,\tau_n) \times (X_n,\tau_n) \quad \text{for some} \quad n \in \mathbb{N}$$

$$[(B[x] \times B[y])^c]^+ = \{\preceq \in \mathcal{P} : \preceq \cap\, (B[x] \times B[y]) = \emptyset\},\ B \in \mathcal{B},\ x,y \in X.$$

Of course two different compatible uniformities on X can give two different topologies.

Example 4.5. Let $X = \{1/n : n \in \mathbb{N}\}$ be equipped with the usual Euclidean topology. Let d be the usual Euclidean metric induced on X. Let $\rho : X \times X \to I\!R$ be the metric defined as follows: $\rho(1/n, 1/m) = |n - m|$. Of course both d and ρ generate the topology on X. Denote by $\mathcal{G}_d$ the uniformity generated by the metric d and by $\mathcal{G}_\rho$ the uniformity generated by the metric ρ. Of course $\mathcal{G}_d$ and $\mathcal{G}_\rho$ are different. We will show that also $\tau(\mathcal{G}_d)$ and $\tau(\mathcal{G}_\rho)$ are different.

For every $n \in \mathbb{N}$ put $\preceq_n = \{(1/n, 1/n), (1,1)\}$ and $\preceq = \{(1,1)\}$. It is easy to verify that $\{\preceq_n : n \in \mathbb{N}\}$ $\tau(\mathcal{G}_\rho)$-converges to $\preceq$ (realize that every ρ-closed ball contains only finitely many elements of X). However $\{\preceq_n : n \in \mathbb{N}\}$ fails to $\tau(\mathcal{G}_d)$-converge to $\preceq$. $\preceq \in [(B_{1/3}[1/3] \times B_{1/3}[1/3])^c]^+$, where $B_{1/3}[1/3] = \{z \in X : d(z, 1/3) \leq 1/3\}$. For every $n \geq 2$ we have $\preceq_n \cap (B_{1/3}[1/3] \times B_{1/3}[1/3]) \neq \emptyset$.

Theorem 4.6. *Let $\mathcal{G}$ be a compatible uniformity on X. The topology $\tau(\mathcal{G})$ on $\mathcal{P}$ satisfies the condition $(*)$ and it is finer than the topology $\bigcup_n F(\tau_n \times \tau_n)$.*

Proof. It is easy to verify that $\tau(\mathcal{G})$ satisfies $(*)$. In fact, let $(x_\sigma, y_\sigma, \preceq_\sigma)$ be a net converging to $(x, y, \preceq)$ with $(x_\sigma, y_\sigma) \in \preceq_\sigma$ and suppose that $(x, y) \notin \preceq$. Then, there exist $B[x]$, $B[y]$, $B \in \mathcal{B}$, such that $(B[x] \times B[y]) \cap \preceq = \emptyset$, that is, $\preceq \in [(B[x] \times B[y])^c]^+$. It follows that, there is σ_0 such that, for every $\sigma \geq \sigma_0$, $\preceq_\sigma \in [(B[x] \times B[y])^c]^+$, that is, $\preceq_\sigma \cap (B[x] \times B[y]) = \emptyset$. This is a contradiction, since (x_σ, y_σ) converges to (x, y).

Now, let K be a compact set in $(X, \tau) \times (X, \tau)$ and $\preceq \in \mathcal{P}$ be such that $\preceq \in (K^c)^+$. There must exist $B \in \mathcal{B}$ such that

$$(B \times B)[K] \cap \preceq = \emptyset.$$

Let $(x_i, y_i) \in K$, $i = 1, 2, \ldots, n$ be such that

$$K \subset \bigcup_{i=1}^{n} B[x_i] \times B[y_i].$$

Then

$$\bigcap_{i=1}^{n} [(B[x_i] \times B[y_i])^c]^+ \in \tau(\mathcal{G})$$

and

$$\preceq \in \bigcap_{i=1}^{n} [(B[x_i] \times B[y_i])^c]^+ \subset (K^c)^+.$$

There is σ_0 such that, for every $\sigma \geq \sigma_0$, $\preceq_\sigma \in \bigcap_{i=1}^{n} [(B[x_i] \times B[y_i])^c]^+ \subset (K^c)^+$.
□

Now we will define a new topology $\tau_c(\mathcal{G})$ on the space $\mathcal{U}$ of utility functions which is finer than the generalized compact open topology τ_c of Back. $\tau_c(\mathcal{G})$ is generated by all sets of the form

$$[K : I] = \{(D, u) \in \mathcal{U} : \ u(D \cap K) \subset I\}$$
$$[G] = \{(D, u) \in \mathcal{U} : \ G \cap D \neq \emptyset\}$$
$$(B[x]^c)^+ = \{(D, u) \in \mathcal{U} : \ B[x] \cap D = \emptyset\}$$

where G is open in (X_n, τ_n) for some $n \in \mathbb{N}$, $K \subset X$ is compact, $I \subset \mathbb{R}$ is open (possibly empty), $B \in \mathcal{B}$ and $x \in X$.

Theorem 4.7. *Let (X, τ) be a submetrizable k_ω-space, the inclusion inductive limit of a closed tower $(X_n, \tau_n)_n$ of Hausdorff second countable locally compact spaces. Let $\mathcal{G}$ be a compatible uniformity on X. There exists a continuous map*

$$\nu : (\mathcal{P}, \tau(\mathcal{G})) \to (\mathcal{U}, \tau_c(\mathcal{G}))$$

such that $\nu(\preceq)$ is a utility function for every $\preceq \in \mathcal{P}$.

Proof. By Theorem 3.6 in [8] there exists a continuous map

$$\nu : (\mathcal{P}, \bigcup_n F(\tau_n \times \tau_n)) \to (\mathcal{U}_\tau, \tau_c)$$

such that $\nu(\preceq)$ is a utility function for every $\preceq \in \mathcal{P}$.
By Theorem 4.6 $\tau(\mathcal{G})$ is finer than the topology $\bigcup_n F(\tau_n \times \tau_n)$.
Thus $\nu : (\mathcal{P}, \tau(\mathcal{G})) \to (\mathcal{U}_\tau, \tau_c)$ is continuous.
Now we will show that $\nu : (\mathcal{P}, \tau(\mathcal{G})) \to (\mathcal{U}_\tau, \tau_c(\mathcal{G}))$ is continuous.
Let $\{\preceq_\sigma : \sigma \in \Sigma\}$ converge to $\preceq$ in $(\mathcal{P}, \tau(\mathcal{G})$. Let $B \in \mathcal{B}$ and $x \in X$ be such that $\nu(\preceq) \in (B[x]^c)^+$. Thus $D(\preceq) \cap B[x] = \emptyset$. That means that $\preceq \in [(B[x] \times B[x])^c]^+$. There is $\sigma_o \in \Sigma$ such that $\preceq_\sigma \in [(B[x] \times B[x])^c]^+$ for every $\sigma \geq \sigma_o$; i.e. $D(\preceq_\sigma) \in (B[x]^c)^+$ for every $\sigma \geq \sigma_o$.
Now let G be an open set in (X_n, τ_n), for some $n \in \mathbb{N}$ such that $D(\preceq) \in [G]$. Then $\preceq \in (G \times G)^-$ and $G \times G$ is open in $(X_n, \tau_n) \times (X_n, \tau_n)$. There must exist $\sigma_0 \in \Sigma$ such that for every $\sigma \geq \sigma_0, \preceq_\sigma \cap (G \times G) \neq \emptyset$; i.e. $D(\preceq_\sigma) \cap G \neq \emptyset$ for every $\sigma \geq \sigma_0$. □

Our Theorem 4.7 can be considered a generalization of the first part of Back's Theorem (Theorem 4.2). In fact any locally compact, separable metric space has a compatible metric d such that (Y, d) is a boundedly compact space ([24]). We say that a metric space (Y, d) is boundedly compact ([2]) if every closed bounded subset is compact. Denote by $\mathcal{G}_d$ the uniformity generated by the metric d. It is easy to verify that $\tau(\mathcal{G}_d)$ coincides with the Fell topology.

Notice that also the following theorem holds.

Theorem 4.8. *Let (X, τ) be a submetrizable k_ω-space, the inclusion inductive limit of a closed tower $(X_n, \tau_n)_n$ of Hausdorff second countable locally compact spaces. Let $\boldsymbol{U}$ be the family of all compatible uniformities on X. There exists a continuous map $\nu : (\mathcal{P}, \bigcap_{\mathcal{G} \in \boldsymbol{U}} \tau(\mathcal{G})) \to (\mathcal{U}, \tau_c)$ such that $\nu(\preceq)$ is a utility function for every $\preceq \in \mathcal{P}$.*

Since we do not know whether the topology $\bigcap_{\mathcal{G} \in \mathbf{U}} \tau(\mathcal{G})$ satisfies the property $(*)$, we propose a convergence on $\mathcal{P}$ which seems to be a right and natural generalization of Back's convergence for submetrizable k_ω-spaces and satisfies the property $(*)$.

Definition 4.9. *Let (X, τ) be a submetrizable k_ω-space, that is the inclusion inductive limit of a closed tower $(X_n, \tau_n)_n$ of Hausdorff second countable locally compact spaces. We say that a net $\{\preceq_\sigma : \sigma \in \Sigma\} \subset \mathcal{P}$ $\mathcal{R}$- converges to $\preceq$ if $L_s \preceq_\sigma \subseteq \preceq$ and if $\preceq \in U^-$, U open in $(X_n, \tau_n) \times (X_n, \tau_n)$, $n \in \mathbb{N}$ there is $\sigma_0 \in \Sigma$ such that for every $\sigma \geq \sigma_0$, $\preceq_\sigma \in U^-$.*

Theorem 4.10. *Let (X, τ) be a submetrizable k_ω-space, the inclusion inductive limit of a closed tower $(X_n, \tau_n)_n$ of Hausdorff second countable locally compact spaces. There exists a map $\nu : \mathcal{P} \to \mathcal{U}$ such that*

1. *$\nu(\preceq)$ is a utility function for every $\preceq\in\mathcal{P}$,*
2. *if $\{\preceq_\sigma : \sigma\in\Sigma\}$ $\mathcal{R}$- converges to $\preceq$, then $\{\nu(\preceq_\sigma) : \sigma\in\Sigma\}$ τ_c- converges to $\nu(\preceq)$.*

Proof. By Theorem 3.6 in [8] there exists a continuous map

$$\nu : (\mathcal{P}, \bigcup_n F(\tau_n\times\tau_n)) \to (\mathcal{U}_\tau, \tau_c)$$

such that $\nu(\preceq)$ is a utility function for every $\preceq\in\mathcal{P}$.
Let $\{\preceq_\sigma : \sigma\in\Sigma\}$ $\mathcal{R}$- converge to $\preceq$ and K be a compact set in $X\times X$ such that $\preceq\in(K^c)^+$. Suppose that for every $\sigma\in\Sigma$ there is $\eta_\sigma\geq\sigma$ such that $\preceq_{\eta_\sigma}\cap K\neq\emptyset$. Let $(x_{\eta_\sigma}, y_{\eta_\sigma})\in\preceq_{\eta_\sigma}\cap K$. The net $\{(x_{\eta_\sigma}, y_{\eta_\sigma}) : \sigma\in\Sigma\}\subset K$ has a cluster point $(x,y)\in K$. It is easy to verify that $(x,y)\in L_s \preceq_\sigma\subset\preceq\cap K$, a contradiction. Thus $\{\preceq_\sigma : \sigma\in\Sigma\}$ converges to $\preceq$ also in the topology $\bigcup_n F(\tau_n\times\tau_n)$. By the above $\{\nu(\preceq_\sigma) : \sigma\in\Sigma\}$ τ_c- converges to $\nu(\preceq)$. □

Corollary 4.11. *Let (X,τ) be a submetrizable k_ω-space, the inclusion inductive limit of a closed tower $(X_n,\tau_n)_n$ of Hausdorff second countable locally compact spaces. There exists a map $\nu:\mathcal{P}\to\mathcal{U}$ such that*

1. *$\nu(\preceq)$ is a utility function for every $\preceq\in\mathcal{P}$,*
2. *if a sequence $\{\preceq_n : n\in\mathbb{N}\}$ $\mathcal{R}-$converges to $\preceq$ and $\{x_n : n\in\mathbb{N}\}$ $\tau-$ converges to x, then $\nu(\preceq_n)(x_n)\ \to\ \nu(\preceq)(x)$ for every $x\in D(\preceq)$, $x_n\in D(\preceq_n), n\in\mathbb{N}$.*

Proof. By Theorem 4.10 there exists a map $\nu:\mathcal{P}\to\mathcal{U}$ such that

1. $\nu(\preceq)$ is a utility function for every $\preceq\in\mathcal{P}$ and
2. if $\{\preceq_n : n\in\mathbb{N}\}$ $\mathcal{R}$- converges to $\preceq$, then $\{\nu(\preceq_n) : n\in\mathbb{N}\}$ τ_c- converges to $\nu(\preceq)$.

Let $K=\{x_n : n\in\mathbb{N}\}\cup\{x\}$. Let I be an open set in $I\!R$ such that $\nu(\preceq)(x)\in I$, that is $(D(\preceq),\nu(\preceq))\in[K:I]$. There is $n_0\in\mathbb{N}$ such that $(D(\preceq_n),\nu(\preceq_n))\in[K:I]$, thus $\nu(\preceq_n)(x_n)\in I$ for every $n\geq n_0$. □

When X is a Hausdorff second countable locally compact space, Theorem 4.10 generalizes the first part of Back's Theorem,

5. Applications

Theorem 4.7 allows us to represent families of closed preorders defined on closed subsets of $lim_\to\mathbb{R}^n=\bigcup_n\mathbb{R}^n$ in real and varied applications.

5.1. An ordering of distributions of wellbeing

In a countable set of possible people, the British philosopher and economist J.Broome in [6] considers distributions (of wellbeing, income or something else) related to all finite populations and assumes that the size of the population may change. So, a distribution of wellbeing of a given finite population with n members (n - population) is a vector of $\mathbb{R}^n$ and the set of the distributions corresponding to all n - populations can be identified to a topological subspace of the n - dimensional Euclidean space $\mathbb{R}^n$. Broome considers the set $\mathcal{F}$ of all distributions when the population-size varies as a subspace of the disjoint union topological space $\bigsqcup_n (R^n \times \{n\})$. He defines a total preorder $\preceq$ on $\mathcal{F}$ and applies the representation Debreu's Theorem to prove the existence of a continuous utility function representing the ordering of distributions.
We think that a natural setting is to use $lim_{\rightarrow}\mathbb{R}^n = \bigcup_n \mathbb{R}^n$ which is closer to the reality.

How to define a preference relation in a set of distributions?
Classical tools are, for example, the Lorenz curves.
Let $X = (x_1, x_2, ..., x_N)$ be a distribution such that $x_1 \leq x_2 ... \leq x_N$.

Definition 5.1. *The Lorenz curve L_X is the continuous piecewise linear function connecting the points* $(\frac{i}{N}, \frac{\sum_{k=1}^{i} x_k}{\sum_{k=1}^{N} x_k}), i = 0, ..., N,\ (L_X(0) = 0)$.

The Lorenz curves L_X of the distribution X indicate, for each percentage cumulative poorest people, the percentage of total income from these owned. They are compared with the bisector "the perfect equality line" and are an effective way to showing inequality of income within and between countries. The Lorenz curves define a preorder on the set $\mathcal{F}$ of all distributions:

Definition 5.2. *For every pair X, Y of distributions,*
$X \preceq Y$ ***in the Lorenz ordering*** $\Longleftrightarrow L_X(w) \leq L_Y(w)$ *for every* $w \in [0, 1]$.

Note that the Lorenz ordering is a partial preorder, in fact if two curves are crossing, they cannot be compared.
Moreover, rather than only one preorder defined on the whole space of distributions, can be more appropriate to consider the family of closed partial preorders defined on closed sets of distributions, for example distributions of the same size or related to populations of the same size.
In [9] the authors had generalized Broome's result in this setting by applying Back's Theorem. As in Broome, $\mathcal{F}$ is considered a subspace of the disjoint union topological space $\bigsqcup_n (R^n \times \{n\})$.
The results of the present paper allow us in a natural way to consider $\mathcal{F}$ as a subspace of $lim_{\rightarrow}\mathbb{R}^n = \bigcup_n \mathbb{R}^n$ (the quotient space of $\bigsqcup_n(\mathbb{R}^n \times \{n\})$ by the equivalence relation $\sim$: $(x, n) \sim (y, m) \Leftrightarrow x = y$ for every $m, n \in \mathbb{N}$).
Put $\mathcal{P}_{\mathcal{F}} \subset \mathcal{P}$ the space of the closed preorders defined on closed sets of distributions, Theorem 4.7 proves the existence of jointly continuous utilities that represent $\mathcal{P}_{\mathcal{F}}$.

5.2. A family of finite dimensional State Preference Models

Theorem 4.7 can be applied in the theory of finite dimensional State Preference Models. A state preference Model is a model in Mathematical Finance concerned with financial markets where the preorders are defined by linear operators.
Let $\mathcal{M}$ be a family of markets where each market $M \in \mathcal{M}$ has a finite number of goods and will be observed only two times, the initial time and the final time. A portfolio of the market M with n goods, $n \in \mathbb{N}$, is a vector $x \in \mathbb{R}^n$. The set of the portfolios of n - goods can be identified with a topological subspace of the n - dimensional Euclidean space $\mathbb{R}^n$
The set $\mathcal{F}$ of all portfolios when the number of the goods varies, can be considered as a subspace of $lim_{\rightarrow}\mathbb{R}^n = \bigcup_n \mathbb{R}^n$.
We suppose that every portfolio $x \in \mathcal{F}$ can assume a finite number of possible values at the final time which depend on the states of the world.
Let $M \in \mathcal{M}$ be a market with n goods, $n \in \mathbb{N}$ and let m be the possible states of the world, $m \in \mathbb{N}$. The values that the portfolio $x \in \mathbb{R}^n$ takes on the m states of the world can be represented by a $m \times n$ matrix A:

$$A = \begin{pmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... & ... & ... & ... & ... \\ a_{m1} & ... & a_{mj} & ... & a_{mn} \end{pmatrix}$$

where a_{ij} is the value of a unit of the $j-th$ good in the $i-th$ state of the world. So the vector Ax represents the final value of portfolio x on the m-states of the world.
The matrix A generates a natural preorder on the set $\mathcal{F}$ of all portfolios:

$$x \preceq_A x' \text{ iff } Ax \leq Ax' \text{ iff } A(x'-x) \text{ is a vector with nonnegative components.}$$

Let $\mathcal{A}$ be the set of all $m \times n$ real matrix, for every $m, n \in I\!N$. For every $A \in \mathcal{A}, \preceq_A$ is a closed partial preorder defined on closed subsets of $\mathcal{F}$. Put $\mathcal{P}_{\mathcal{F}} = \{\preceq_A : A \in \mathcal{A}\} \subset \mathcal{P}$. Theorem 4.7 proves the existence of jointly continuous utilities that represent $\mathcal{P}_{\mathcal{F}}$.
In [10] the authors had considered $\mathcal{F}$ as a subspace of the disjoint union topological space $\bigsqcup_n (R^n \times \{n\})$ and applied Back's Theorem 4.2 to represent $\{\preceq_A : A \in \mathcal{A}\}$.

`Acknowledgements.` *This research was carried out within the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni, Istituto Nazionale di Alta Matematica (Italy). The paper was supported by grant "Metodi di Teoria dell'Approssimazione, Analisi Reale, Analisi Nonlineare e loro applicazioni", Fondo di Ricerca di Base, 2018, dell'Università degli Studi di Perugia, Italia. Ľ. Holá would like to thank grant Vega 2/0006/16.*

References

[1] K. Back, *Concepts of similarity for utility functions*, Journ. of Math. Econ., **15** (1986), 129-142.

[2] G. Beer, *Topologies on Closed and Closed Convex Sets*, Kluwer Academic Publishers, 1993.

[3] G. Bosi, A. Caterino, R. Ceppitelli, *Existence of continuous utility functions for arbitrary binary relations: some sufficient conditions*, Tatra Mountains Math. Publ., **46** (2010), 15-27, DOI:10.2478/v10127-010-0015-8.

[4] G. Bosi, G. Herden, *On a possible continuous analogue of the Szpilrajn Theorem and its strengthening by Dushnik and Miller*, Order **23** (2006), 271-296.

[5] D. S. Bridges and G. B. Mehta, Representations of Preference Orderings, Springer-Verlag, Berlin, 1995.

[6] J. Broome, *Representing an ordering when the population varies.*, Social Choice and Welfaire, **20** (2003), 243-246.

[7] C. Castaing, P. Raynaud de Fitte, M. Valadier, *Young Measures on Topological Spaces with Applications in control Theory and Probability Theory*, Kluwer Academic Publishers Dordrecht 571, 2004.

[8] A. Caterino, R. Ceppitelli, *Jointly continuous utility functions on submetrizable k_ω-spaces*, Topology and its Applications, **190** (2015), 109-118, DOI: 10.1016/j.topol.2015.04.012.

[9] A. Caterino, R. Ceppitelli, An application of Back's Theorem to an ordering of distributions of wellbeing, Proceedings of 14th Conference on Applied Mathemathics Aplimat 2015, Bratislava, Slovak Republic, (2015) 191-198.

[10] D. Carfì, A. Caterino, R. Ceppitelli, State preference models and jointly continuous utilities, Proceedings of 15th Conference on Applied Mathemathics Aplimat 2016, Bratislava, Slovak Republic, (2016) 163-176.

[11] A. Caterino, R. Ceppitelli, Ľ. Holá, *Some generalizations of Back's Theorem*, Topology and its Applications, **160** (2013), 2386-2395, DOI:10.1016/j.topol.2013.07.033.

[12] A. Caterino, R. Ceppitelli, F. Maccarino, *Continuous utility functions on submetrizable hemicompact k-spaces*, Applied General Topology, **10** (2009), 187-195, DOI:10.4995/agt.2009.1732.

[13] G. Di Maio, Ľ. Holá, E. Meccariello, *Notes on hit-and-miss topologies*, Rostock Math. Kolloq. **52** (1998), 19-32.

[14] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[15] S. P. Franklin, B. V. Smith Thomas, *A survey of k_ω-spaces*, Topology Proceedings **2** (1977), 111-124.

[16] Herden, G., Pallack, A.: On the continuous analogue of the Szpilrajn Theorem I, Mathematical Social Sciences **43**, 115-134 (2002).

[17] W. Hildebrand, *On economies with many agents*, Journal of Economic Theory, **2** (1970), 1-188.

[18] W. Hildebrand, *Core and equilibria of a large economy*, Princeton University Press, Princeton, 1974.

[19] V. L. Levin, *A continuous utility theorem for closed preorders on a σ-compact metrizable space*, Soviet Math. Dokl. **28** (1983), 715-718.

[20] A. Mas-Colell, *On the continuous representation of preorders*, International Economic Review, **18** (1977), 509-513.

[21] L. Nachbin, *Topology and order*, Van Nostrand, Princeton, 1965.

[22] W. Rudin, *Functional Analysis*, McGraw-Hill Inc., 1973.

[23] K. Sakai, *Geometric Aspects of General Topology*, Springer Monographs in Mathematics, Springer Japan 2013, DOI:10.1007/978-4-431-54397-8

[24] H. Vaughan, *On locally compact metrizable spaces*, Bull. Amer. Math. Soc. **43** (1937), 532-535.