Research Article

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Mathematical models for nonlocal elastic composite materials

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Abstract: In this paper we derive and solve nonlocal elasticity a model describing the elastic behavior of composite materials, involving the *fractional Laplacian* operator. In dimension one we consider in $(\mathcal{D})$ the case of a nonlocal elastic rod restrained at the ends, and we completely solve the problem showing the existence of a unique weak solution and providing natural sufficient conditions under which this solution is actually a classical solution of the problem. For the model $(\mathcal{D})$ we also perform numerical simulations and a parametric analysis, in order to highlight the response of the rod, in terms of displacements and strains, according to different values of the mechanical characteristics of the material. The main novelty of this approach is the extension of the central difference method by the numerical estimate of the fractional Laplacian operator through a finite-difference quadrature technique. For higher dimensions $N \geq 2$ we study more general problems for which the existence of weak solutions is proved via variational methods. The obtained results provide an original contribute in the knowledge of composite materials with properties of nonlocal elasticity.

Keywords: Nonlocal elasticity models, composite materials, fractional Laplacian operator, numerical simulations

MSC 2010: 35R11, 35A15, 35J60, 74B20, 74E30

1 Introduction

In the last years the nonlocal elasticity theory has been used in wider and wider engineering applications involving small-size devices and/or materials with marked microstructures. The driving force for developing advanced materials comes from society's call of large composite structures having solid mechanical features. This increasing need requires new engineering composite materials that work reliably and safely at the frontiers of cutting edge technologies, and pushes towards material systems in support of energy sustainability, [8].

The key tool in this theory is the analysis of the nonlocal interactions among different locations of the body, in terms of elastic central long-range body forces proportional to the volumes or masses for composite solids. In other words, in nonlocal phenomena, the displacement field u is not just reverting on its *infinitesimal average*, but instead it is influenced by its values at many scales, according to an *integral average* of the entire solid. As a feature of certain materials, nonlocality has been acknowledged in the literature since

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many years ago. In [7] it is shown that the elastic response of a material which presents distributed defects is necessarily nonlocal. Also in plasticity and in damage phenomena nonlocal behavior naturally arises, see for example [6, 24]. It is also worth noting that the homogenization of a composite with periodic microstructure produces a material with a nonlocal behavior, see [9].

The nonlocal effects are captured, in the equilibrium equations, by an integral term which is the resultant of all the long-range interactions. Hence, the corresponding equilibrium problem is ruled by one or more integro-differential equations in terms of u, which can be often tackled with certain tools of fractional calculus, cf. [12, 18, 19].

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real-powers of differentiation operators, generalizing the concept of integer-powers derivatives. An important point is that the fractional derivative of order $s \in \mathbb{R}^+$ at a point x has a local nature only when s is an integer. In the noninteger setting, it is not possible to view the fractional derivative at x of a function u depending only on the values of u very close to x, in the way that integer-power derivatives do. Therefore the theory should involve some sort of boundary conditions, concerning information on the function further out. Of course there are several ways of generalizing the differentiation operators to noninteger powers, leading to different results, see e.g. [13].

In this paper we derive and solve nonlocal elasticity models for composite materials involving the *fractional Laplacian* operator, since it is particularly useful in modeling materials exhibiting nonlocal behaviors caused by the effects of long range interactions among particles. We adopt the formalization of nonlocal elasticity proposed in [21, 22], which is particularly appealing because it allows us to use a continuum approach as in classical elasticity. More specifically, in the context of civil and mechanical engineering, it is of great interest to consider the nonlocal phenomena occurring in those elastic media which can be modeled as points connected each other by springs. Unlike what has been done up to now in the literature, concerning the 1-dimensional case, the use of the fractional Laplacian represents a new different approach to face these problems. One of the main advantages of this strategy is the possibility to generalize the analysis easily to higher dimensions $N \geq 2$. We should note though, in passing, that the fractional Laplacian operator occurs in several other engineering contexts, such as frequency-dependent attenuation, [13], wave transmission in nonlocal elasticity, [15], as well as in the so-called anomalous diffusion phenomena, [34, 42].

In the present paper, the fractional Laplacian operator $(-\Delta)^s$, with $s \in (0, 1)$, is defined pointwise by

$$(-\Delta)^s \varphi(x) = \mathscr{F}^{-1}(|\xi|^{2s} \mathscr{F}(\varphi(\xi))) \quad \text{for all } x, \xi \in \mathbb{R}^N, \tag{1.1}$$

along any function φ in $C_0^\infty(\mathbb{R}^N)$, $N \geq 1$. In (1.1)

$$\mathscr{F}\varphi(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} \varphi(x)\, dx \quad \text{for all } \xi \in \mathbb{R}^N,$$

is the Fourier transform of $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $\mathscr{F}^{-1}$ is the inverse Fourier transform, see Sections 2–3 for further details.

In the case $N = 1$, following somehow the approach used in [43], we give a detailed proof of the connection between the fractional Laplacian, the *Riemann–Liouville* and the *Caputo fractional derivatives* of order $2s$, taking $s \in (1/2, 1)$. Actually, inspired by [38], we consider the constitutive law for a 1-dimensional rod of finite length $2L$ having properties of nonlocal elasticity, and we derive the balance law for the rod in terms of $(-\Delta)^s$, that is

$$\begin{cases} -c\, u'' + \kappa\, (-\Delta)^s u + V(x) u = \dfrac{f(x)}{E} & \text{in } (-L, L), \\ u = 0 & \text{in } \mathbb{R} \setminus (-L, L). \end{cases} \tag{$\mathcal{D}$}$$

We assume *throughout the paper that* $c > 0$ *and* $\kappa \geq 0$. The cases $c = 0$ and $\kappa > 0$ can be treated in a similar way. The term $V(x)u$ represents an external spring, while $f(x)$ denotes an applied load. Moreover, *the potential* V *is a nonnegative and bounded measurable weight* and its stiffness is related to the position of the point along the rod. Finally, E is the Young modulus of the classical elasticity.

For $(\mathcal{D})$ we prove the existence of a unique *weak* solution and we provide natural sufficient conditions under which the weak solution is actually a classical solution of $(\mathcal{D})$, and vice versa we investigate when classical solutions of $(\mathcal{D})$ are weak solutions, cf. Section 2 for further details and comments.

In Section 3 we treat a nonlinear model for nonlocal elastic ideal infinite N-dimensional rods for any $N \geq 1$. Precisely, we consider

$$-c\,\Delta u + \kappa(-\Delta)^s u + V(x)u = \lambda\, w(x)|u|^{p-2}u + K(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{$\mathcal{P}_\lambda$}$$

with

$$2 < p < q < 2^*, \quad 2^* = \begin{cases} \infty, & \text{if } N \in \{1, 2\}, \\ \frac{2N}{N-2}, & \text{if } N \geq 3, \end{cases}$$

where $\lambda > 0$ is a physical constant and the *weight w satisfies*

$$w \in L^r(\mathbb{R}^N), \quad \textit{with } r = \frac{q}{q-p}. \tag{1.2}$$

In passing from a road of finite length to one of infinite length, we assume that *the potential V is a measurable positive weight, with the property that there exist V_1 and V_2, with $0 < V_1 \leq V_2$, such that*

$$V_1\|u\|_2^2 \leq \int_{\mathbb{R}^N} V(x)|u|^2\,dx \leq V_2\|u\|_2^2 \quad \textit{for all } u \in L^2(\mathbb{R}^N). \tag{1.3}$$

Here and in the following, for any exponent $\sigma \in [1, \infty)$ we denote by $\|\cdot\|_\sigma$ the usual norm of the Lebesgue space $L^\sigma(\mathbb{R}^N)$.

Moreover, *K is a positive measurable weight for which there exist real numbers K_1 and K_2, with* $0 < K_1 \leq K_2$, *such that*

$$0 < K_1 \leq K(x) \leq K_2 \quad \textit{for all } x \in \mathbb{R}^N. \tag{1.4}$$

Let $H^1(\mathbb{R}^N) = (H^1(\mathbb{R}^N), \|\cdot\|)$ be the standard Hilbert space, endowed with the norm

$$\|u\| = (\|u\|_2^2 + \|\nabla u\|_2^2)^{1/2}. \tag{1.5}$$

When $N = 1$ solutions of $(\mathcal{P}_\lambda)$ of class $H^1(\mathbb{R})$ are ground states in the sense that

$$\lim_{|x|\to\infty} u(x) = 0 \tag{1.6}$$

by virtue of [10, Corollary 8.9]. *From now on we assume that* (1.3)–(1.4) *hold*, unless otherwise specified.

Theorem 1.1. *There exists a threshold $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$ problem $(\mathcal{P}_\lambda)$ admits a nontrivial mountain pass solution u_λ in $H^1(\mathbb{R}^N)$. Moreover,*

$$\lim_{\lambda\to\infty} \|u_\lambda\| = 0. \tag{1.7}$$

For $N \geq 3$ a problem closely related to $(\mathcal{P}_\lambda)$, but in the critical case $q = 2^*$, is treated in the forthcoming paper [5]. While here, for $N \geq 2$, we deal with

$$-c\Delta u + \kappa(-\Delta)^s u + V(x)u = K(x)|u|^{q-2}u + f(x) \quad \text{in } \mathbb{R}^N, \quad 2 < q < 2^*, \tag{$\mathcal{P}$}$$

where f is a perturbation of class $L^2(\mathbb{R}^N)$. For $(\mathcal{P})$ we obtain the existence of nontrivial radial solutions *under* (1.3) *and* (1.4), *provided that V, K and f are radial.*

Theorem 1.2. *There exists $\delta > 0$ such that for all $f \in L^2(\mathbb{R}^N)$, with $\|f\|_2 \leq \delta$, problem $(\mathcal{P})$ admits a nontrivial radial mountain pass solution u in $H^1(\mathbb{R}^N)$.*

If $0 < \|f\|_2 \leq \delta$, then $(\mathcal{P})$ admits a second independent nontrivial radial solution v in $H^1(\mathbb{R}^N)$.

Theorem 1.2 is not contained in Theorem 1.1 even when $f \equiv 0$, since the case $\lambda = 0$ is not covered in Theorem 1.1. An interesting open problem is to extend Theorem 1.2 in the nonradial case assuming only (1.3) on V, and without changing the solution space $H^1(\mathbb{R}^N)$.

The paper is organized as follows. In Section 2 we first derive the balance law equation for a nonlocal elastic 1-dimensional finite rod in terms of a linear combination of the standard Laplacian operator and the fractional Laplacian operator and attain the model $(\mathcal{D})$. The existence of a unique *weak* solution of $(\mathcal{D})$ is proved in Theorem 2.1, while in Theorem 2.2 we provide sufficient conditions under which the weak solution of $(\mathcal{D})$ is also a classical solution of $(\mathcal{D})$. Finally, in Theorem 2.3 we show that every classical solution of $(\mathcal{D})$ is a weak solution of $(\mathcal{D})$.

Section 3 is devoted to the study of the existence of (weak) solutions for $(\mathcal{P}_\lambda)$ and contains the proof of Theorem 1.1, while in Section 4 we prove Theorem 1.2 for the existence of (weak) solutions for $(\mathcal{P})$.

In Section 5 we provide numerical simulations of the nonlocal elastic rod modeled in $(\mathcal{D})$. Moreover, a parametric analysis is performed with the aim of understanding the role of the mechanical characteristics of the material, with a special attention to their local and nonlocal effects.

Finally, in Section 6 we resume the obtained results and present an overview of comments and conclusions on the validity of the proposed models, both from a theoretical point of view and from an applicative perspective related to the numerical simulations developed in Section 5.

2 A linear model for a nonlocal elastic 1-dimensional rod

In the present section we introduce the problem of a 1-dimensional solid (a rod) with properties of nonlocal elasticity, according to the definition of [21]. In doing this, we use the approach proposed in [38].

Let us consider the rod centered at zero of length $2L$, so that $x = -L$ and $x = L$ are the coordinates of the extremes, with periodic boundary conditions. Put $I = (-L, L)$.

It is assumed that the *constitutive law* of the material is given by

$$\sigma = E\left(\beta_1\epsilon + \beta_2 k \int_{-L}^{L} \epsilon(\xi)g(x-\xi)\,d\xi - \int_{-L}^{x} V(\xi)u(\xi)\,d\xi\right),$$

where $\sigma = \sigma(x)$ is the tension, $\epsilon = \epsilon(x)$ is the strain, $V = V(x)$ is the stiffness associated to the restoring elastic forces and $u = u(x)$ is the displacement of the rod. The relation between the strain ϵ and the displacement u is assumed to be the same as in the classical elasticity theory, that is

$$\epsilon = u', \quad ' = \frac{d}{dx}. \tag{2.1}$$

Furthermore, E is the Young modulus of the material and k is a positive constant typical of the material. Moreover, β_1 and β_2 are physical coefficients such that $\beta_1 + \beta_2 = 1$, β_1 and $\beta_2 \in [0, 1]$. Therefore, σ depends on the convex combination between the local and nonlocal contributions. If $\beta_2 = 0$, the material is purely local, while if $\beta_2 = 1$, the material is purely nonlocal. Finally, g is an *attenuation function* which characterizes the nonlocal contribution to elasticity.

It is worth noting that the same expression for the constitutive law (except for the absence of the term containing $V(x)u$) has been obtained in [39]. In particular, in [39], using simple mechanical concepts, the author shows how nonlocal behavior is typical of a simple composite, constituted by alternating layers of stiff and compliant phases.

On the other hand, the *balance law* is

$$\sigma' + f = 0,$$

where $f = f(x)$ is the force (per unit volume) applied along the rod, and it is equivalent to

$$\begin{cases} -\beta_1\epsilon' - \beta_2 k \displaystyle\int_{-L}^{L} \epsilon'(\xi)g(x-\xi)\,d\xi + V(x)u = \dfrac{f(x)}{E}, \\ \epsilon(-L) = \epsilon(L) = 0, \end{cases} \tag{2.2}$$

where g is in $L^1_{\mathrm{loc}}(\mathbb{R})$, *the potential* V *is a measurable positive weight, with*

$$0 \le V(x) \le V_2 \quad \text{for all } x \in I, \tag{2.3}$$

and the strain $\epsilon \in C^1(\bar{I})$.

Since the displacement u is also required to be $2L$-periodic, relation (2.1) is equivalent to the integral counterpart

$$u(x) = \int_{-L}^{x} \left[\epsilon(t) - \frac{1}{2L} \int_{-L}^{L} \epsilon(\tau)\, d\tau \right] dt.$$

Hence, (2.2) in terms of u becomes

$$\begin{cases} -\beta_1 u'' - \beta_2 k \displaystyle\int_{-L}^{L} u''(\xi) g(x-\xi)\, d\xi + V(x) u = \frac{f(x)}{E}, \\ u(-L) = u(L) = 0, \quad u'(-L) = u'(L) = 0, \end{cases} \tag{2.4}$$

with $u \in C^2(\bar{I})$. As in the model proposed in [38], the attenuation function g is of the form

$$g(\xi) = \frac{1}{\Gamma(2-2s)|\xi|^{2s-1}}, \qquad \frac{1}{2} < s < 1, \tag{2.5}$$

where Γ denotes the Euler function. *The restriction* $1/2 < s < 1$ *required for g justifies the physical validity of the model, that is the fact that g produces an attenuation effect.*

The first term on the left-hand side of (2.4) stands for the contribution coming from the classical elasticity theory. Indeed, taking $\beta_1 = 1$ and $\beta_2 = 0$ we get the canonical *elasticity law*

$$-u'' = \frac{f}{E}.$$

Before proceeding, let us recall some definitions and properties. Let s be a real number *such that* $0 < s < 1$ *and* $2 = [2s] + 1$. Note that this choice of s fits with the restriction assumed in (2.5).

Following [27] and [28, Theorem 2.1], we say that

$${}^{C}_{-L}D^{2s}_{x} u(x) = \frac{1}{\Gamma(2-2s)} \int_{-L}^{x} \frac{u''(\xi)}{(x-\xi)^{2s-1}}\, d\xi, \quad {}^{C}_{x}D^{2s}_{L} u(x) = \frac{1}{\Gamma(2-2s)} \int_{x}^{L} \frac{u''(\xi)}{(\xi-x)^{2s-1}}\, d\xi$$

are the *forward* and *backward Caputo fractional derivatives of order* $2s$ of u at $x \in I$.

Moreover, the *forward* and *backward Riemann–Liouville fractional derivatives of order* $2s$ of u at any point $x \in I$ are defined by

$${}_{-L}D^{2s}_{x} u(x) = \frac{1}{\Gamma(2-2s)} \cdot \frac{d^2}{dx^2} \int_{-L}^{x} \frac{u(\xi)}{(x-\xi)^{2s-1}}\, d\xi, \quad {}_{x}D^{2s}_{L} u(x) = \frac{1}{\Gamma(2-2s)} \cdot \frac{d^2}{dx^2} \int_{x}^{L} \frac{u(\xi)}{(\xi-x)^{2s-1}}\, d\xi,$$

see again [27, 28].

Now fix $x \in I$ as above. Then

$$\begin{aligned} \Gamma(2-2s)\, {}_{-L}D^{2s}_{x} u(x) &= \frac{d^2}{dx^2} \int_{-L}^{x} \frac{u(\xi)}{(x-\xi)^{2s-1}}\, d\xi = \frac{d^2}{dx^2} \int_{0}^{x+L} \frac{u(x-\xi)}{\xi^{2s-1}}\, d\xi \\ &= \frac{d}{dx} \left[\int_{0}^{x+L} \frac{u'(x-\xi)}{\xi^{2s-1}}\, d\xi + \frac{u(-L)}{(x+L)^{2s-1}} \right] \\ &= \int_{0}^{x+L} \frac{u''(x-\xi)}{\xi^{2s-1}}\, d\xi + \frac{u'(-L)}{(x+L)^{2s-1}} + \frac{(1-2s)u(-L)}{(x+L)^{2s}} \\ &= \int_{-L}^{x} \frac{u''(\xi)}{(x-\xi)^{2s-1}}\, d\xi + \frac{u'(-L)}{(x+L)^{2s-1}} + \frac{(1-2s)u(-L)}{(x+L)^{2s}}. \end{aligned}$$

Therefore,

$$ {}_{-L}D_x^{2s}u(x) = {}_{-L}^{\,C}D_x^{2s}u(x) + \frac{1}{\Gamma(2-2s)} \cdot \frac{u'(-L)}{(x+L)^{2s-1}} + \frac{1}{\Gamma(1-2s)} \cdot \frac{u(-L)}{(x+L)^{2s}}. $$

Similarly,

$$ \begin{aligned} \Gamma(2-2s)\,{}_xD_L^{2s}u(x) &= \frac{d^2}{dx^2}\int_x^L \frac{u(\xi)}{(\xi-x)^{2s-1}}\,d\xi = \frac{d^2}{dx^2}\int_0^{L-x} \frac{u(x+\xi)}{\xi^{2s-1}}\,d\xi \\ &= \int_x^L \frac{u''(\xi)}{(\xi-x)^{2s-1}}\,d\xi - \frac{u'(L)}{(L-x)^{2s-1}} + \frac{(1-2s)u(L)}{(L-x)^{2s}}, \end{aligned} $$

and so

$$ {}_xD_L^{2s}u(x) = {}_x^{C}D_L^{2s}u(x) - \frac{1}{\Gamma(2-2s)} \cdot \frac{u'(L)}{(L-x)^{2s-1}} + \frac{1}{\Gamma(1-2s)} \cdot \frac{u(L)}{(L-x)^{2s}}. $$

Thus, since u verifies the boundary conditions in (2.4), we get immediately in I

$$ \int_{-L}^{L} u''(\xi)g(x-\xi)\,d\xi = {}_{-L}^{\,C}D_x^{2s}u(x) + {}_x^{C}D_L^{2s}u(x) = {}_{-L}D_x^{2s}u(x) + {}_xD_L^{2s}u(x). $$

We note in passing that the above relation between the Riemann–Liouville and the Caputo fractional derivatives of order $2s$ have been somehow derived also in [2].

We next show that any solution u of (2.4) satisfies in I

$$ (-\Delta)^s u(x) = \frac{{}_{-L}D_x^{2s}u(x) + {}_xD_L^{2s}u(x)}{2\cos(s\pi)}, \qquad \frac{1}{2} < s < 1, $$

where $(-\Delta)^s$ is defined in (1.1). To this end we shall use an argument appeared in [43].

Since $(-\Delta)^s$ is a pseudo-differential operator well-defined in the Schwartz space and (2.4) is solved in the basic interval of periodicity, we have to extend u in the entire $\mathbb{R}$, putting $u = 0$ outside I. Clearly this extension is only of class $C^1(\mathbb{R})$ by the boundary conditions (2.4), with $u'' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\operatorname{supp} u \subset \bar{I}$. With abuse of notation *we indicate by u both the periodic displacement in I as well as the canonical extension to the entire* $\mathbb{R}$ *by putting* $u = 0$ *outside* I.

By (1.1) for $N = 1$ we have

$$ (-\Delta)^s u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}|\xi|^{2s}\,d\xi \int_{-\infty}^{\infty} e^{i\xi\eta}u(\eta)\,d\eta. \tag{2.6} $$

Integrating twice by parts, by the boundary conditions in (2.4) and the fact that $u'' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have

$$ \int_{-\infty}^{\infty} e^{i\xi\eta}u(\eta)\,d\eta = -\frac{1}{\xi^2}\int_{-\infty}^{\infty} e^{i\xi\eta}u''(\eta)\,d\eta. $$

In other words, by the Fubini theorem,

$$ (-\Delta)^s u(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} u''(\eta)\,d\eta \int_{-\infty}^{\infty} e^{i\xi(\eta-x)}|\xi|^{2s-2}\,d\xi. \tag{2.7} $$

Without loss of generality, let us assume that $\eta \neq x$, being $x \in \mathbb{R}$ fixed, and put for simplicity $\omega = x - \eta$. From now on the restriction that $2s > 1$ is crucial and gives

$$ \int_{-\infty}^{\infty} e^{i\xi(\eta-x)}|\xi|^{2s-2}\,d\xi = \mathscr{L}(i\omega) + \mathscr{L}(-i\omega), \tag{2.8} $$

where $\mathscr{L}$ is the Laplace transform of the function $\mathbb{R}^+ \ni t \mapsto t^{2s-2}$. Observe that $\mathscr{L}$ is holomorphic in the domain $\Omega = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$.

Clearly, for any $h \in \mathbb{R}^+$,

$$\mathscr{L}(h) = \frac{\Gamma(2s-1)}{h^{2s-1}}. \tag{2.9}$$

Consider

$$\Omega \ni z \mapsto z^{2s-1} = e^{(2s-1)Ln(z)},$$

where $Ln(z) = Ln(|z|) + iArg(z)$ is the principle value of the complex logarithmic function. It turns out that $z \mapsto z^{2s-1}$ is a single valued complex function which is holomorphic in Ω.

Since (2.9) holds in a set having one accumulation point in Ω, it follows that (2.9) is valid in the entire open connected set Ω by the identity principle of holomorphic functions. Thus, (2.8) becomes

$$\begin{aligned}\mathscr{L}(i\omega) + \mathscr{L}(-i\omega) &= \frac{\Gamma(2s-1)}{|\omega|^{2s-1}}\left[\frac{1}{i^{2s-1}} + \frac{1}{(-i)^{2s-1}}\right]\\ &= \frac{\Gamma(2s-1)}{|\omega|^{2s-1}}[i^{2s-1} + (-i)^{2s-1}]\\ &= \frac{\Gamma(2s-1)}{|\omega|^{2s-1}}\, 2\sin s\pi\\ &= -\frac{\pi}{|x-\eta|^{2s-1}\Gamma(2-2s)\cos\pi s}\end{aligned}$$

by the Euler identity and the fact that

$$\Gamma(2s-1)\Gamma(2-2s) = -\frac{\pi}{\sin 2\pi s}.$$

Therefore, from (2.7) we get

$$\begin{aligned}(-\Delta)^s u(x) &= \frac{1}{2\cos s\pi}\cdot\frac{1}{\Gamma(2-2s)}\int_{-\infty}^{\infty}\frac{u''(\eta)}{|x-\eta|^{2s-1}}\,d\eta\\ &= \frac{1}{2\cos s\pi}\int_{-L}^{L} u''(\eta)g(x-\eta)\,d\eta\\ &= \frac{{}_{-L}D_x^{2s}u(x) + {}_xD_L^{2s}u(x)}{2\cos s\pi},\end{aligned} \tag{2.10}$$

as claimed.

In conclusion, if $1/2 < s < 1$ and g is of the type (2.5), then problem (2.4) can be rewritten as

$$\begin{cases} -c\,u'' + \kappa\,(-\Delta)^s u + V(x)u = \dfrac{f(x)}{E} & \text{in } I,\\ u(x) = u'(x) = 0 & \text{in } \mathbb{R}\setminus I,\end{cases} \tag{2.11}$$

with $u \in C_0^1(\mathbb{R})$, $\operatorname{supp} u \subset \overline{I}$, $u' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $u'' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and the coefficients are $c = \beta_1 \in [0,1]$ and $\kappa = -2\beta_2 k\cos s\pi \geq 0$, since $k > 0$, $1/2 < s < 1$ and $\beta_2 \in [0,1]$, with $\beta_1 + \beta_2 = 1$.

It is interesting to note that, in the case $V \equiv 0$, the equation in (2.11) is obtained in [41], when studying a subgradient elasticity model of a lattice with weak spatial dispersion, by using a discrete approach and then letting the interparticle distance go to zero, expanding the lattice model to a continuous model. See also [29].

A somewhat related approach, which takes inspiration from [14] and [19], has been used also in the recent paper [38] to provide a physical interpretation of (2.11) when $V \equiv 0$. More precisely, if $V \equiv 0$, problem (2.11) characterizes the behavior of a point-spring model which has four kinds of springs. One of these springs has a local effect and derives from the classical elasticity Hooke law, while the remaining three springs model the nonlocal actions, and one of them connects directly the two extremes of the rod. As noted in the Introduction, the additional term $V(x)u$ represents the stiffness associated to further restoring elastic forces.

We recall that u indicates both the displacement in I as well as its canonical extension to the entire $\mathbb{R}$, by putting $u = 0$ outside $I = (-L, L)$. In particular, (2.11) can be weakly solved in the larger Sobolev space $H_0^1(I)$, using a standard density argument, provided that $f \in H^{-1}(I)$, where $H^{-1}(I)$ is the dual space of $H_0^1(I)$. For simplicity, *from now on in the section we assume the stronger requirement that* $f \in L^2(I)$.

By the Poincaré inequality $\|u\|_\infty \le \|u'\|_1$ for all $u \in H_0^1(I)$. Hence

$$\|u\|_2 \le |I|^{1/2}\|u\|_\infty \le |I|^{1/2}\|u'\|_1 \le |I| \cdot \|u'\|_2$$

for all $u \in H_0^1(I)$. We endow $H_0^1(I)$ with the equivalent Hilbertian norm

$$\|u\| = \|u'\|_2. \tag{2.12}$$

Similarly, $H_0^s(I) = (H_0^s(I), [\,\cdot\,]_s)$, where

$$[u]_s^2 = \int_{\mathbb{R}} |\xi|^{2s}\hat{u}^2\,d\xi = \int_{\mathbb{R}} |(-\Delta)^{s/2}u|^2\,dx, \quad \hat{u} = \mathscr{F}u,$$

by virtue of the Plancherel theorem. The embedding $H_0^1(I) \hookrightarrow H_0^s(I)$ is continuous, that is there exists $C_s > 0$ such that

$$[u]_s \le C_s\|u\| \quad \text{for all } u \in H_0^1(I). \tag{2.13}$$

Inequality (2.13) is a direct consequence of [17, Proposition 2.2 and remark after Theorem 2.4].

We say that *u is a weak solution of* $(\mathcal{D})$ *if* $u \in H_0^1(I)$ *and u satisfies the identity*

$$c\langle u', \varphi'\rangle_2 + \kappa\langle u, \varphi\rangle_s + \int_I V(x)u\varphi\,dx = \frac{1}{E}\int_I f(x)\varphi\,dx \tag{2.14}$$

for all $\varphi \in H_0^1(I)$, where $\langle u', \varphi'\rangle_2 = \int_{\mathbb{R}} u'(x)\varphi'(x)\,dx$ and

$$\langle u, \varphi\rangle_s = \int_{\mathbb{R}} (-\Delta)^{s/2}u(x)(-\Delta)^{s/2}\varphi(x)\,dx. \tag{2.15}$$

A classical solution u of $(\mathcal{D})$ *is a function* $u \in C^2(\bar{I})$ *satisfying the equation in* $(\mathcal{D})$ *in the usual sense.* In this case clearly $u'' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

We are now able to prove

Theorem 2.1. *Assume that $c > 0$, $\kappa \ge 0$ and that V satisfies property* (2.3)*. Then for all $f \in L^2(I)$, problem* $(\mathcal{D})$ *admits a unique weak solution $u \in H_0^1(I)$, which is obtained by the Dirichlet principle, that is u is the unique global minimizer of*

$$\mathcal{J}(v) = \frac{1}{2}\left(c\|v'\|_2^2 + \kappa[v]_s^2 + \int_I V(x)|v|^2\,dx\right) - \frac{1}{E}\int_I f(x)v\,dx$$

in $H_0^1(I)$.

Proof. The form

$$a(v, \varphi) = c\langle v', \varphi'\rangle_2 + \kappa\langle v, \varphi\rangle_s + \int_I V(x)v\varphi\,dx$$

is bilinear and symmetric in $H_0^1(I) \times H_0^1(I)$. Moreover, a is continuous by the Cauchy–Schwarz inequality and (2.12)–(2.13), being for all $(v, \varphi) \in H_0^1(I) \times H_0^1(I)$

$$|a(v, \varphi)| \le c\|v'\|_2\|\varphi'\|_2 + \kappa C_s^2\|v'\|_2\|\varphi'\|_2 + V_2|I|^2\|v'\|_2\|\varphi'\|_2 = C\|v\| \cdot \|\varphi\|,$$

where $C = c + \kappa C_s^2 + V_2|I|^2$. Furthermore, a is coercive in $H_0^1(I)$, since for all $v \in H_0^1(I)$

$$a(v, v) \ge c\|v'\|_2^2 = c\|v\|^2,$$

where $c > 0$ by assumption. The functional

$$v \mapsto \frac{1}{E}\int_I f(x)v\,dx$$

is linear and continuous in $H_0^1(I)$, namely it is an element of the dual space $H^{-1}(I)$ of $H_0^1(I)$. Hence, the assertion follows as a direct application of the Lax–Milgram theorem, see [10, Corollary 5.8]. □

Theorem 2.2. *Suppose that the assumptions of Theorem* 2.1 *are satisfied. Assume furthermore that* V *and* f *are also continuous in* $\overline{I}$ *and that* $s < 3/4$. *If the weak solution* $u \in H_0^1(I)$ *determined in Theorem 2.1 admits second derivative* u'' *in* I, *with* $u'' \in L^2(I)$, *then* u *is a classical solution of* $(\mathcal{D})$, *that is* u *is of class* $C^2(\overline{I})$, *satisfies the equation in* $(\mathcal{D})$ *pointwise in* I *and verifies the boundary conditions in* $(\mathcal{D})$.

Proof. Let u be a weak solution of $(\mathcal{D})$, satisfying the assumptions of the theorem. Since $u'' \in L^2(\mathbb{R})$, it follows that $\xi \mapsto |\xi|^2\hat{u}(\xi) \in L^2(\mathbb{R})$ by [37, Theorem IX.27 (a)]. Clearly, $u \in H_0^s(I) \hookrightarrow H^s(\mathbb{R})$, being $H_0^1(I) \hookrightarrow H_0^s(I)$ continuously, so that $\xi \mapsto |\xi|^s\hat{u}(\xi) \in L^2(\mathbb{R})$.

Combining these facts, we get at once that $\xi \mapsto |\xi|^{2s}\hat{u}(\xi) \in L^2(\mathbb{R})$. Therefore, the Plancherel theorem gives that $(-\Delta)^s u$ is in $L^2(\mathbb{R})$.

We claim that actually $(-\Delta)^s u = \mathscr{F}^{-1}(|\xi|^{2s}\hat{u})$ is also continuous in $\mathbb{R}$, vanishes at infinity and is of class $L^1(\mathbb{R})$. To this end it is enough to show that $\xi \mapsto |\xi|^{2s}\hat{u}(\xi)$ is in $L^1(\mathbb{R})$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}} |\xi|^{2s}|\hat{u}|\,d\xi &\le \int_{\{|\xi|\le 1\}} |\xi|^{2s}|\hat{u}|\,d\xi + \int_{\{|\xi|\ge 1\}} |\xi|^{2(s-1)}|\xi|^2|\hat{u}|\,d\xi \\ &\le L_1 + \Big(\int_{\{|\xi|\ge 1\}} |\xi|^{4(s-1)}\,d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}} |\xi|^4|\hat{u}|^2\,d\xi\Big)^{1/2} < \infty,\end{aligned}$$

since $|\xi|^{2s}\hat{u} \in C(\mathbb{R})$, $\xi \mapsto |\xi|^2\hat{u}(\xi) \in L^2(\mathbb{R})$, as shown above, and $s < 3/4$ by assumption. This completes the proof of the claim.

For all $\varphi \in C_0^\infty(I)$ by (2.15), (2.6) and the Plancherel theorem we get

$$\begin{aligned}\langle u, \varphi\rangle_s &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{(-\Delta)^{s/2}u}\,\widehat{(-\Delta)^{s/2}\varphi}\,dx = \frac{1}{2\pi}\int_{\mathbb{R}} |\xi|^s\hat{u}\,|\xi|^s\hat{\varphi}\,d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}\,d\xi \int_{\mathbb{R}} e^{-i\xi x}(-\Delta)^s u\,dx = \frac{1}{2\pi}\int_{\mathbb{R}} (-\Delta)^s u\,dx \int_{\mathbb{R}} e^{i\xi x}\hat{\varphi}\,d\xi \\ &= \int_{\mathbb{R}} (-\Delta)^s u\varphi\,dx.\end{aligned} \tag{2.16}$$

Since u satisfies the identity (2.14) and $u'' \in L^2(I)$, integrating by parts the first term in (2.14), we have by (2.16) for all $\varphi \in C_0^\infty(I)$

$$\int_I \Big(-c\,u'' + \kappa\,(-\Delta)^s u + V(x)u - \frac{f(x)}{E}\Big)\varphi\,dx = 0,$$

where $-c\,u'' + \kappa\,(-\Delta)^s u + V(x)u - f(x)/E \in L^2(I)$. Therefore, $-c\,u'' + \kappa\,(-\Delta)^s u + V(x)u = f(x)/E$ a.e. in I, and since $\kappa\,(-\Delta)^s u + V(x)u - f(x)/E \in C(\overline{I})$, actually the equality holds everywhere in I, that is u verifies the equation in (2.11) pointwise in $I = (-L, L)$. Furthermore, $u'' \in C(\overline{I})$, since $u'' = (\kappa\,(-\Delta)^s u + V(x)u - f(x)/E)/c$ and $u \in C(\overline{I})$ since $u \in H_0^1(I)$ by assumption and $N = 1$. Hence, $u \in C^2(\overline{I})$. Therefore u is a classical solution of $(\mathcal{D})$. □

Theorem 2.3. *Suppose that the assumptions of Theorem 2.1 are satisfied. Then every classical solution* u *of* $(\mathcal{D})$ *is a weak solution of* $(\mathcal{D})$.

Proof. Let u be a classical solution of $(\mathcal{D})$, so that $u \in C^2(\overline{I})$ satisfies $(\mathcal{D})$ in the usual sense, $u'' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and the boundary conditions in $(\mathcal{D})$ are verified. Hence $u \in H^1(I)$ and so the classical solution u is in $H_0^1(I)$ by the boundary conditions in $(\mathcal{D})$ and [10, Theorem 8.12].

Since u is a classical solution of $(\mathcal{D})$, multiplying the equation in $(\mathcal{D})$ by any $\varphi \in C_0^\infty(I)$, by (2.16) and integration by parts we get

$$c\langle u', \varphi'\rangle_2 + \kappa\langle u, \varphi\rangle_s + \int_I V(x)u\varphi\,dx = \frac{1}{E}\int_I f(x)\varphi\,dx.$$

Since $u \in H_0^1(I)$, a standard density argument shows that the above identity indeed holds for all $\varphi \in H_0^1(I)$. Therefore u is a weak solution of $(\mathcal{D})$. □

3 A nonlinear model for infinite nonlocal elastic N-dimensional rods

In [43] (see in particular the details of the proof of Lemma 1) a precise expression for the nonlocal operator $(-\Delta)^s$ is derived in dimension $N = 1$ and for an ideal infinite rod, under the homogeneous boundary conditions $u(-\infty) = u(\infty) = u'(-\infty) = u'(\infty) = 0$ and under the assumptions that $u' \in L^1(\mathbb{R})$ and $u'' \in L^1(\mathbb{R})$. The request $u'' \in L^1(\mathbb{R})$ is fairly natural and already appears in [2], when u is defined in $\mathbb{R}$.

From an exact mathematical point of view, problem (2.11) for an ideal infinite rod becomes

$$\begin{cases} -c\,u'' + \kappa\,(-\Delta)^s u + V(x)u = \dfrac{f(x)}{E} \quad \text{in } \mathbb{R}, \\ u \in C_0^1(\mathbb{R}), \quad u'' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}). \end{cases} \tag{3.1}$$

In particular, (2.10) is converted into

$$\begin{aligned} (-\Delta)^s u(x) &= \frac{1}{2\cos s\pi} \cdot \frac{1}{\Gamma(2-2s)} \int_{-\infty}^{\infty} \frac{u''(\eta)}{|x-\eta|^{2s-1}}\,d\eta \\ &= \frac{1}{2\cos s\pi} \int_{-\infty}^{\infty} u''(\eta) g(x-\eta)\,d\eta \\ &= \frac{{}_{-\infty}D_x^{2s} u(x) + {}_xD_{\infty}^{2s} u(x)}{2\cos s\pi}. \end{aligned}$$

Therefore every solution of (3.1) satisfies

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} u'(x) = 0. \tag{3.2}$$

In this section, we extend the linear model (3.1) to the nonlinear version $(\mathcal{P}_\lambda)$ given in the Introduction.

First recall that the space $H^1(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the Hilbertian norm $\|\cdot\|$ defined in (1.5). Similarly, $H^s(\mathbb{R}^N)$, $s \in (0,1)$, is the Hilbert space, with norm

$$\|u\|_s = (\|u\|_2^2 + [u]_s^2)^{1/2}, \quad [u]_s^2 = \int_{\mathbb{R}^N} |\xi|^{2s}\hat{u}^2\,d\xi = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,dx, \quad \hat{u} = \mathscr{F}u,$$

by the Plancherel theorem.

The fractional Laplacian operator $(-\Delta)^s$ introduced in (1.1) can be equivalently defined by

$$(-\Delta)^s \varphi(x) = C_{N,s}\,\text{P.V.} \int_{\mathbb{R}^N} \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s}}\,dy, \tag{3.3}$$

see [17, Lemma 3.5], where

$$C_{N,s} = s\,2^{2s} \frac{\Gamma(s+N/2)}{\pi^{N/2}\Gamma(1-s)}. \tag{3.4}$$

Furthermore, for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^N)$ the already cited Plancherel theorem gives

$$\int_{\mathbb{R}^N} (-\Delta)^s \varphi\psi\,dx = \int_{\mathbb{R}^N} |\xi|^{2s}\hat{\varphi}\hat{\psi}\,d\xi = \int_{\mathbb{R}^N} (-\Delta)^{s/2}\varphi(-\Delta)^{s/2}\psi\,dx = \langle\varphi,\psi\rangle_s,$$

where for simplicity in notation $\hat{\varphi} = \mathscr{F}\varphi$, as before.

The embedding $H^1(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$ is continuous, that is there exists $C_s > 0$ such that

$$\|u\|_s \le C_s\|u\| \quad \text{for all } u \in H^1(\mathbb{R}^N). \tag{3.5}$$

Clearly C_s is here a different number than in (2.13). The proof of (3.5) is a combination of the arguments given in the proofs of [17, Propositions 2.1 and 2.2].

From here on we assume that *the potential* V *is a positive weight satisfying condition* (1.3) *given in the Introduction*. Assumption (1.3) guarantees that the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, V)$ is continuous, where

$$L^2(\mathbb{R}^N, V) = \Big\{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} V(x)|u|^2\,dx < \infty\Big\}$$

is the weighted Lebesgue space related to the positive potential V, endowed with the norm

$$\|u\|_{2,V} = \Big(\int_{\mathbb{R}^N} V(x)|u|^2\,dx\Big)^{1/2}.$$

Similarly, the weighted Lebesgue space

$$L^p(\mathbb{R}^N, w) = \Big\{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} w(x)|u|^p\,dx < \infty\Big\}$$

has norm

$$\|u\|_{p,w} = \Big(\int_{\mathbb{R}^N} w(x)|u|^p\,dx\Big)^{1/p}.$$

By (1.2) and [36, Lemma 2.6] the embedding $H^1(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^N, w)$ is compact. In particular, there exists a positive constant $C_w > 0$ depending also on p such that

$$\|u\|_{p,w} \le C_w\|u\| \tag{3.6}$$

for all $u \in H^1(\mathbb{R}^N)$.

In the same way also the weighted Lebesgue space

$$L^q(\mathbb{R}^N, K) = \Big\{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} K(x)|u|^q\,dx < \infty\Big\}$$

is equipped with the norm

$$\|u\|_{q,K} = \Big(\int_{\mathbb{R}^N} K(x)|u|^q\,dx\Big)^{1/q}.$$

We note in passing that the embedding $L^q(\mathbb{R}^N, K) \hookrightarrow L^p(\mathbb{R}^N, w)$ is continuous by (1.2), (1.4) and the fact that $p < q$.

By (1.4) the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, K)$ is continuous. This is a trivial consequence of the continuity of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ and (1.4).

In the special case $N = 1$, then [10, Theorem 8.8] assures that $\|u\|_\infty \le \sqrt{2}\,\|u\|$ for all $u \in H^1(\mathbb{R})$, so that, by the interpolation theorem and the fact that $2 < q < \infty$, it results

$$\|u\|_q \le \|u\|_\infty^{1-2/q}\|u\|_2^{2/q} \le 2^{1/2-1/q}\|u\|.$$

Consequently, $\|u\|_{q,K} \le C_q\|u\|$ for all $u \in H^1(\mathbb{R})$ by (1.4), where $C_q = K_2^{1/q}2^{1/2-1/q}$ and $C_q \to 2^{-1/2}$ as $q \to \infty$. In any case,

$$S_q = \inf_{\substack{v\in H^1(\mathbb{R})\\ v\neq 0}} \frac{\|v\|}{\|v\|_{q,K}} \ge \frac{2^{1/q-1/2}}{K_2^{1/q}} = \frac{1}{C_q} > 0.$$

Similarly, when $N \ge 2$, the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous by [10, Corollaries 9.10 and 9.11] and the fact that $2 < q < 2^*$. By (1.4) there exists a constant C_q such that $\|u\|_{q,K} \le C_q\|u\|$ for all $u \in H^1(\mathbb{R}^N)$. For $N = 2$, the results of [31, Section 1.4.8] and a short calculation give

$$C_q = 2^{-1+2/q}(K_2\pi)^{1/q}\sqrt{\frac{q}{e}}$$

and $C_q \to \infty$ as $q \to \infty$. However,

$$S_q = \inf_{\substack{v\in H^1(\mathbb{R}^2)\\ v\neq 0}} \frac{\|v\|}{\|v\|_{q,K}} \ge \frac{1}{C_q} > 0.$$

In conclusion,

$$S_q = \inf_{\substack{v \in H^1(\mathbb{R}^N) \\ v \neq 0}} \frac{\|v\|}{\|v\|_{q,K}} > 0 \tag{3.7}$$

for all $N \geq 1$.

Problems $(\mathcal{P}_\lambda)$ and $(\mathcal{P})$ can be weakly solved in $H^1(\mathbb{R}^N)$. *A (weak) solution u of $(\mathcal{P}_\lambda)$ is a function of class $H^1(\mathbb{R}^N)$ satisfying the identity*

$$c\langle \nabla u, \nabla \varphi \rangle_2 + \kappa \langle u, \varphi \rangle_s + \langle u, \varphi \rangle_{2,V} = \lambda \langle u, \varphi \rangle_{p,w} + \langle u, \varphi \rangle_{q,K} \tag{3.8}$$

for all $\varphi \in H^1(\mathbb{R}^N)$, where $\langle \nabla u, \nabla \varphi \rangle_2 = \int_{\mathbb{R}^N} \nabla u \nabla \varphi \, dx$ and

$$\langle u, \varphi \rangle_s = \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi \, dx, \quad \langle u, \varphi \rangle_{2,V} = \int_{\mathbb{R}^N} V(x) u \varphi \, dx,$$

$$\langle u, \varphi \rangle_{p,w} = \int_{\mathbb{R}^N} w(x) |u|^{p-2} u \varphi \, dx, \qquad \langle u, \varphi \rangle_{q,K} = \int_{\mathbb{R}^N} K(x) |u|^{q-2} u \varphi \, dx.$$

Clearly $\nabla u = u'$ when $N = 1$.

If $N = 1$, then [10, Corollary 8.9] guarantees that any solution $u \in H^1(\mathbb{R})$ satisfies the limit condition (1.6). Furthermore, in this case if u'' exists and u'' is in $L^2(\mathbb{R})$, then $u \in H^2(\mathbb{R})$ and also (3.2) holds.

Problem $(\mathcal{P}_\lambda)$ has a variational structure and the underlying Euler–Lagrange functional $J_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated to $(\mathcal{P}_\lambda)$ is defined by

$$J_\lambda(u) = \frac{1}{2}(c\|\nabla u\|_2^2 + \kappa [u]_s^2 + \|u\|_{2,V}^2) - \frac{\lambda}{p} \|u\|_{p,w}^p - \frac{1}{q} \|u\|_{q,K}^q \tag{3.9}$$

for all $u \in H^1(\mathbb{R}^N)$. Indeed, J_λ is of class $C^1(H^1(\mathbb{R}^N))$ and solutions of $(\mathcal{P}_\lambda)$ correspond to critical points of J_λ in $H^1(\mathbb{R}^N)$.

In order to find a solution of $(\mathcal{P}_\lambda)$, we intend to apply the mountain pass theorem of Ambrosetti and Rabinowitz [1] to the functional J_λ at a special level $c_{\lambda,\cdot}$. We put for simplicity

$$m = \min\{c, V_1\}, \quad M = \max\{c, \kappa C_s^2, V_2\}, \tag{3.10}$$

where C_s is defined in (3.5). Of course, $m > 0$ by (1.3) and the fact that $c > 0$ by assumption. *Throughout the section we assume the validity of* (1.2)–(1.4) *and* (3.10).

Lemma 3.1. *For any $\lambda > 0$ there exist two positive constants α and ρ such that $J_\lambda(u) \geq \alpha > 0$ for any $u \in H^1(\mathbb{R}^N)$, with $\|u\| = \rho$, and there exists a radial function $e \in C_0^\infty(\mathbb{R}^N)$, with $J_\lambda(e) < 0$ and $\|e\| > \rho$.*

Proof. Fix $\lambda > 0$. By (3.9), (3.10), (3.6) and (3.7), for all $u \in H^1(\mathbb{R}^N)$

$$J_\lambda(u) \geq \frac{m}{2}\|u\|^2 - \frac{\lambda}{p}\|u\|_{p,w}^p - \frac{1}{q}\|u\|_{q,K}^q \geq \left(\frac{m}{2} - \frac{\lambda C_w^p}{p}\|u\|^{p-2} - \frac{C_q^q}{q}\|u\|^{q-2}\right)\|u\|^2.$$

Taking $\rho \in (0, 1]$ so small that $m/2 - \lambda C_w^p \rho^{p-2}/p - C_q^q \rho^{q-2}/q > 0$, we get that

$$J_\lambda(u) \geq \alpha = \rho^2 \left(\frac{m}{2} - \frac{\lambda C_w^p \rho^{p-2}}{p} - \frac{C_q^q \rho^{q-2}}{q}\right)$$

for all $u \in H^1(\mathbb{R}^N)$, with $\|u\| = \rho$.

Let $u_0 \in C_0^\infty(\mathbb{R}^N)$ be a radial function such that $\|u_0\| = 1$. For all $t > 0$

$$J_\lambda(tu_0) \leq \frac{M}{2} t^2 - \frac{\lambda}{p} \|u_0\|_{p,w}^p t^p.$$

Thus $J_\lambda(tu_0) \to -\infty$ as $t \to \infty$, since $2 < p$ and $\lambda > 0$. Take $\tau_\lambda > 0$ so large that $e = \tau_\lambda u_0$ has the property that $\|e\| \geq 2$ and $J_\lambda(e) < 0$. In particular, $\|e\| > \rho$, being $\rho \in (0, 1]$. □

From the proof of Lemma 3.1 it is evident that if $e = \tau_{\lambda_0} u_0$ is selected at some $\lambda_0 > 0$, then $J_\lambda(e) < 0$ for all $\lambda \geq \lambda_0$. Moreover, $\|e\| \geq 2 > \rho$ for all $\rho = \rho(\lambda) \in (0, 1]$ and for all $\lambda \geq \lambda_0$.

Fix $\lambda > 0$ and put

$$c_\lambda = \inf_{g\in\Gamma}\max_{t\in[0,1]} J_\lambda(g(t)), \quad \Gamma = \{g \in C([0,1], H^1(\mathbb{R}^N)) : g(0) = 0,\ g(e) < 0\}.$$

Clearly, $c_\lambda > 0$ by Lemma 3.1.

We recall that $(u_k)_k \subset H^1(\mathbb{R}^N)$ is a *Palais–Smale sequence for J_λ at level c_λ* if

$$J_\lambda(u_k) \to c_\lambda \quad \text{and} \quad J'_\lambda(u_k) \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N) \tag{3.11}$$

as $k \to \infty$. Furthermore, J_λ is said to satisfy *the Palais–Smale condition in $H^1(\mathbb{R}^N)$ at level c_λ* if any Palais–Smale sequence $(u_k)_k \subset H^1(\mathbb{R}^N)$ at level c_λ admits a convergent subsequence in $H^1(\mathbb{R}^N)$.

Before proving the relatively compactness of the Palais–Smale sequences for J_λ, we introduce an asymptotic behavior of the levels c_λ, as proved in the study of Kirchhoff fractional Dirichlet problems in [3, 23]. This fact will be essential not only to get (1.7), but above all to overcome the lack of compactness.

Lemma 3.2. *It results*

$$\lim_{\lambda\to\infty} c_\lambda = 0.$$

Proof. Fix $\lambda_0 > 0$. Let $e \in C_0^\infty(\mathbb{R}^N)$ be the radial function obtained in Lemma 3.1 for J_{λ_0}. Hence J_λ satisfies the mountain pass geometry at 0 and e for all $\lambda \ge \lambda_0$. Fix $\lambda \ge \lambda_0$. Then there exists $t_\lambda > 0$ verifying $J_\lambda(t_\lambda e) = \max_{t\ge0} J_\lambda(te)$. Hence, $\langle J'_\lambda(t_\lambda e), e\rangle = 0$ and so

$$Mt_\lambda\|e\|^2 \ge t_\lambda(c\|\nabla e\|_2^2 + \kappa[e]_s^2 + \|e\|_{2,V}^2) = \lambda\, t_\lambda^{p-1}\|e\|_{p,w}^p + t_\lambda^{q-1}\|e\|_{q,K}^q \ge \lambda\, t_\lambda^{p-1}\|e\|_{p,w}^p. \tag{3.12}$$

Therefore, $\{t_\lambda\}_{\lambda\ge\lambda_0}$ is bounded, since $p > 2$, $\lambda \ge \lambda_0 > 0$ and e depends only on λ_0.

Therefore there exists $\tau \in \mathbb{R}_0^+$ such that

$$\limsup_{\lambda\to\infty} t_\lambda = \tau.$$

Clearly $\tau = 0$. Otherwise,

$$\limsup_{\lambda\to\infty} (\lambda t_\lambda^{p-2}) = \infty,$$

and this would contradict (3.12). In conclusion, $\tau = 0$, so that

$$\lim_{\lambda\to\infty} t_\lambda = 0. \tag{3.13}$$

Consider now the path $g(t) = te$, $t \in [0,1]$, belonging to Γ. By Lemma 3.1

$$0 < c_\lambda \le \max_{t\in[0,1]} J_\lambda(g(t)) \le J_\lambda(t_\lambda e) \le \frac{M}{2}\|e\|^2 t_\lambda^2,$$

and letting $\lambda \to \infty$ we get the assertion by (3.13), since e depends only on λ_0. □

Now, we are ready to prove the Palais–Smale condition at level c_λ, adapting in a suitable way the main ideas already used in Lemma 3.4 of [3] for a degenerate Kirchhoff fractional Dirichlet problem.

Lemma 3.3. *There exists $\lambda^* > 0$ such that for all $\lambda \ge \lambda^*$ the functional J_λ satisfies the Palais–Smale condition in $H^1(\mathbb{R}^N)$ at level c_λ.*

Proof. Take $\lambda > 0$. Let $(u_k^\lambda)_k \subset H^1(\mathbb{R}^N)$ be a Palais–Smale sequence for J_λ at level c_λ. In particular,

$$\begin{aligned} J_\lambda(u_k^\lambda) - \frac{1}{p}\langle J'_\lambda(u_k^\lambda), u_k^\lambda\rangle &\ge m\Big(\frac12 - \frac1p\Big)\|u_k^\lambda\|^2 + \Big(\frac1p - \frac1q\Big)\|u_k^\lambda\|_{q,K}^q \\ &\ge m\Big(\frac12 - \frac1p\Big)\|u_k^\lambda\|^2. \end{aligned}$$

Hence, (3.11) yields at once that as $k \to \infty$

$$c_\lambda + d_\lambda\|u_k^\lambda\| + o(1) \ge m\Big(\frac12 - \frac1p\Big)\|u_k^\lambda\|^2.$$

Therefore, $(u_k^\lambda)_k$ is bounded in $H^1(\mathbb{R}^N)$, being $p > 2$, and there exists u_λ in $H^1(\mathbb{R}^N)$ such that, up to a subsequence, it follows that

$$\begin{aligned} &u_k^\lambda \rightharpoonup u_\lambda \text{ in } H^1(\mathbb{R}^N), \text{ in } H^s(\mathbb{R}^N) \text{ and in } L^q(\mathbb{R}^N, K),\\ &\nabla u_k^\lambda \rightharpoonup \nabla u_\lambda \text{ in } [L^2(\mathbb{R}^N)]^N, \quad \text{and } u_k^\lambda \to u_\lambda \text{ in } L^p(\mathbb{R}^N, w),\\ &u_k^\lambda \to u_\lambda \text{ a.e. in } \mathbb{R}^N, \quad \|u_k^\lambda\| \to \alpha_\lambda, \quad \|u_k^\lambda - u_\lambda\|_{q,K} \to \ell_\lambda\\ &\|\nabla u_k^\lambda\|_2 \to g_\lambda, \quad [u_k^\lambda]_s \to s_\lambda, \quad \|u_k^\lambda\|_{2,V} \to V_\lambda. \end{aligned} \tag{3.14}$$

Furthermore, by (3.11)

$$c_\lambda + o(1) \ge m\Big(\frac{1}{2} - \frac{1}{p}\Big)\|u_k^\lambda\|^2 + \Big(\frac{1}{p} - \frac{1}{q}\Big)\|u_k^\lambda\|_{q,K}^q. \tag{3.15}$$

Thus, by (3.14) and (3.15) we have

$$c_\lambda \ge m\Big(\frac{1}{2} - \frac{1}{p}\Big)\alpha_\lambda^2. \tag{3.16}$$

We first assert that

$$\lim_{\lambda\to\infty} \alpha_\lambda = 0. \tag{3.17}$$

Otherwise $\limsup_{\lambda\to\infty} \alpha_\lambda = \alpha > 0$. Hence, by (3.16)

$$\limsup_{\lambda\to\infty} c_\lambda \ge m\Big(\frac{1}{2} - \frac{1}{p}\Big)\alpha^2 > 0,$$

which is impossible by Lemma 3.2 and proves assertion (3.17).

Moreover, $\|u_\lambda\| \le \lim_k \|u_k^\lambda\| = \alpha_\lambda$ since $u_k^\lambda \rightharpoonup u_\lambda$, so that (3.7) and (3.17) imply at once

$$\lim_{\lambda\to\infty} \|u_\lambda\|_{q,K} = \lim_{\lambda\to\infty} \|u_\lambda\| = 0. \tag{3.18}$$

By (3.14) and the fact that $|u_k^\lambda|^{q-2}u_k^\lambda \rightharpoonup |u_\lambda|^{q-2}u_\lambda$ in $L^{q'}(\mathbb{R}^N, K)$ by [4, Proposition A.8], where q' is the Hölder conjugate of q, we have that u satisfies (3.8) for all $\varphi \in H^1(\mathbb{R}^N)$. Hence, u_λ is a critical point of J_λ in $H^1(\mathbb{R}^N)$, that is u_λ is a solution of $(\mathcal{P}_\lambda)$. In particular, (3.11) and (3.14) imply that as $k \to \infty$

$$\begin{aligned} o(1) &= \langle J_\lambda'(u_k^\lambda) - J_\lambda'(u_\lambda), u_k^\lambda - u_\lambda\rangle\\ &= c\|\nabla u_k^\lambda - \nabla u_\lambda\|_2^2 + \kappa[u_k^\lambda - u_\lambda]_s^2 + \|u_k^\lambda - u_\lambda\|_{2,V}^2 - \lambda \int_{\mathbb{R}^N} w(x)(|u_k^\lambda|^{p-2}u_k^\lambda - |u_\lambda|^{p-2}u_\lambda)(u_k^\lambda - u_\lambda)\,dx\\ &\quad - \int_{\mathbb{R}^N} K(x)(|u_k^\lambda|^{q-2}u_k^\lambda - |u_\lambda|^{q-2}u_\lambda)(u_k^\lambda - u_\lambda)\,dx\\ &= c(g_\lambda^2 - \|\nabla u_\lambda\|_2^2) + \kappa(s_\lambda^2 - [u_\lambda]_s^2) + V_\lambda^2 - \|u_\lambda\|_{2,V}^2 - \|u_k^\lambda\|_{q,K}^q + \|u_\lambda\|_{q,K}^q + o(1)\\ &= c(g_\lambda^2 - \|\nabla u_\lambda\|_2^2) + \kappa(s_\lambda^2 - [u_\lambda]_s^2) + V_\lambda^2 - \|u_\lambda\|_{2,V}^2 - \|u_k^\lambda - u_\lambda\|_{q,K}^q + o(1), \end{aligned} \tag{3.19}$$

since (3.14) gives that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} w(x)(|u_k^\lambda|^{p-2}u_k^\lambda - |u_\lambda|^{p-2}u_\lambda)(u_k^\lambda - u_\lambda)\,dx = 0$$

and by (3.14) and the celebrated Brézis–Lieb lemma, see [11],

$$\begin{aligned} \|\nabla u_k^\lambda - \nabla u_\lambda\|_2^2 &= \|\nabla u_k^\lambda\|_2^2 - \|\nabla u_\lambda\|_2^2 + o(1) = g_\lambda^2 - \|\nabla u_\lambda\|_2^2 + o(1),\\ [u_k^\lambda - u_\lambda]_s^2 &= [u_k^\lambda]_s^2 - [u_\lambda]_s^2 + o(1) = s_\lambda^2 - [u_\lambda]_s^2 + o(1),\\ \|u_k^\lambda - u_\lambda\|_{2,V}^2 &= \|u_k^\lambda\|_{2,V}^2 - \|u_\lambda\|_{2,V}^2 + o(1) = V_\lambda^2 - \|u_\lambda\|_{2,V}^2 + o(1),\\ \|u_k^\lambda - u_\lambda\|_{q,K}^q &= \|u_k^\lambda\|_{q,K}^q - \|u_\lambda\|_{q,K}^q + o(1) \end{aligned} \tag{3.20}$$

as $k \to \infty$.

Therefore, by (3.19) we have the main formula

$$\ell_\lambda^q = c(g_\lambda^2 - \|\nabla u_\lambda\|_2^2) + \kappa(s_\lambda^2 - [u_\lambda]_s^2) + V_\lambda^2 - \|u_\lambda\|_{2,V}^2,$$

which, in particular, yields by (3.20) that

$$\ell_\lambda^q \ge c \lim_{k\to\infty} \|\nabla u_k^\lambda - \nabla u_\lambda\|_2^2 + V_1 \lim_{k\to\infty} \|u_k^\lambda - u_\lambda\|_2^2 \ge m \lim_{k\to\infty} \|u_k^\lambda - u_\lambda\|^2, \tag{3.21}$$

by (3.10) and the facts that $s_\lambda \geq [u_\lambda]_s$ and $\kappa \geq 0$. Using (3.14), (3.15) and (3.20) and letting $k \to \infty$, Lemma 3.2 and (3.18) give

$$0 = \lim_{\lambda \to \infty} c_\lambda \geq \left(\frac{1}{p} - \frac{1}{q}\right) \lim_{\lambda \to \infty} (\ell_\lambda^q + \|u_\lambda\|_{q,K}^q).$$

In particular,

$$\lim_{\lambda \to \infty} \ell_\lambda = 0. \tag{3.22}$$

By (3.21) and (3.7), for all $\lambda \in \mathbb{R}^+$

$$\ell_\lambda^q \geq m\, S_q^2 \ell_\lambda^2.$$

We claim that there exists $\lambda^* > 0$ such that $\ell_\lambda = 0$ for all $\lambda \geq \lambda^*$. Otherwise there would exist infinitely many λ_n, with $\lambda_n \to \infty$ as $n \to \infty$, such that for all n

$$\ell_{\lambda_n}^{q-2} \geq m\, S_q^2 > 0,$$

which is impossible by (3.22). This proves the claim.

In conclusion, there exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$

$$\lim_{k \to \infty} \|u_k^\lambda - u_\lambda\|_{q,K}^q = \ell_\lambda^q = 0$$

and so by (3.21)

$$\lim_{k \to \infty} \|u_k^\lambda - u_\lambda\| = 0,$$

as required. □

Proof of Theorem 1.1. Lemmas 3.1 and 3.3 guarantee that there exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$ the functional J_λ satisfies all the assumptions of the mountain pass theorem in $H^1(\mathbb{R}^N)$. Hence, there exists a critical point $u_\lambda \in H^1(\mathbb{R}^N)$ for J_λ at level c_λ for all $\lambda \geq \lambda^*$. Since $J_\lambda(u_\lambda) = c_\lambda > 0 = J_\lambda(0)$, we have that $u_\lambda \not\equiv 0$. Finally, the asymptotic behavior (1.7) holds thanks to (3.18). □

4 The model $(\mathcal{P})$

In this section $N \geq 2$ and as usually $s \in (0, 1)$. We denote by $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ the closed subspace of $H^1(\mathbb{R}^N)$, consisting of all the radial functions of $H^1(\mathbb{R}^N)$. Clearly $H^1_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ continuously.

From now on *we assume that the positive weights V and K satisfying* (1.3)–(1.4) *are also radial* and we continue to use the notation (3.10). Hence all the properties (3.5)–(3.7) are still valid. The perturbation $f \in L^2(\mathbb{R}^N)$ is also assumed radial, without further mentioning.

Problem $(\mathcal{P})$ has a variational structure and solutions of $(\mathcal{P})$ correspond to the critical points of the underlying Euler–Lagrange functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}(c\|\nabla u\|_2^2 + \kappa[u]_s^2 + \|u\|_{2,V}^2) - \frac{1}{q}\|u\|_{q,K}^q - \int_{\mathbb{R}^N} f(x)u\,dx \tag{4.1}$$

for all $u \in H^1(\mathbb{R}^N)$. Indeed, J is of class $C^1(H^1(\mathbb{R}^N))$ and $v \in H^1(\mathbb{R}^N)$ is a (weak) *solution* of $(\mathcal{P})$ if

$$c\langle \nabla v, \nabla \varphi\rangle_2 + \kappa\langle v, \varphi\rangle_s + \langle u, \varphi\rangle_{2,V} = \langle u, \varphi\rangle_{q,K} - \int_{\mathbb{R}^N} f(x)\varphi\,dx \tag{4.2}$$

for all $\varphi \in H^1(\mathbb{R}^N)$. While $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ is a (weak) *radial solution* of $(\mathcal{P})$ in the sense of $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ if (4.2) holds for all $\varphi \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$.

In order to find the critical points of J, we intend to apply both the mountain pass theorem of [1] and the *Ekeland* variational principle given in [20]. For a wide selection of applications of critical point theory to fractional elliptic differential problems we refer to the recent monograph [32].

Lemma 4.1. *There exist three positive numbers α, δ and ρ such that $J(u) \geq \alpha > 0$ for any $u \in H^1(\mathbb{R}^N)$, with $\|u\| = \rho$, and a function $e \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)$, with $J(e) < 0$ and $\|e\| > \rho$, provided that $\|f\|_2 \leq \delta$.*

In particular, $J(u) \geq \alpha > 0$ for any $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$, with $\|u\| = \rho$, provided that $\|f\|_2 \leq \delta$.

Proof. The proof is similar to that of Lemma 3.1. By the continuity of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, K)$ and the Hölder inequality,

$$\begin{aligned} J(u) &\geq \frac{m}{2}\|u\|^2 - \frac{1}{q}\|u\|^q_{q,K} - \int_{\mathbb{R}^N} f(x)u\,dx \\ &\geq \frac{m}{2}\|u\|^2 - \frac{1}{q}\|u\|^q_{q,K} - \|f\|_2\|u\| \\ &\geq \Big\{\Big(\frac{m}{2} - \frac{1}{q}C_q^q\|u\|^{q-2}\Big)\|u\| - \|f\|_2\Big\}\|u\| \end{aligned}$$

for all $u \in H^1(\mathbb{R}^N)$. Hence $J(u) \geq \alpha$ for all $u \in H^1(\mathbb{R}^N)$ with $\|u\| = \rho$, provided that $\rho \in (0, 1]$ and δ are so small that $a = m/2 - C_q^q\rho^{q-2}/q > 0$, $0 < \delta < a\rho$ and $\|f\|_2 \leq \delta$.

Take $u_0 \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)$ such that $\|u_0\| = 1$. By (4.1) for all $t > 0$

$$J(tu_0) \leq \frac{M}{2}t^2 - \frac{1}{q}\|u\|^q_{q,K}t^q - t\int_{\mathbb{R}^N} f(x)u_0\,dx \leq \frac{M}{2}t^2 - \frac{1}{q}\|u\|^q_{q,K}t^q + \|f\|_2\|u_0\|_2 t.$$

Thus $J(tu_0) \to -\infty$ as $t \to \infty$, being $q > 2$. Take τ sufficiently large so that $e = \tau u_0$ has the property that $\|e\| \geq 2$ and $J(e) < 0$. In particular, $\|e\| > \rho$, since $\rho \in (0, 1]$.

The last part of the lemma is trivial. □

We say that $(u_k)_k \subset H^1_{\mathrm{rad}}(\mathbb{R}^N)$ is a *Palais–Smale sequence for J* if

$$(J(u_k))_k \text{ is bounded in } \mathbb{R} \text{ and } J'(u_k) \to 0 \quad \text{in } H^{-1}_{\mathrm{rad}}(\mathbb{R}^N) \tag{4.3}$$

as $k \to \infty$. Furthermore, *J is said to satisfy the Palais–Smale condition in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$* if any Palais–Smale sequence $(u_k)_k \subset H^1_{\mathrm{rad}}(\mathbb{R}^N)$ admits a convergent subsequence in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$.

Lemma 4.2. *The functional J satisfies the Palais–Smale condition in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$.*

Proof. Let $(u_k)_k \subset H^1_{\mathrm{rad}}(\mathbb{R}^N)$ be a Palais–Smale sequence for J. Then, there exists $C > 0$ such that $|J(u_k)| \leq C$ and $|\langle J'(u_k), u_k\rangle| \leq C\|u_k\|$ for all k. In particular,

$$\begin{aligned} C + C\|u_k\| &\geq J(u_k) - \frac{1}{q}\langle J'(u_k), u_k\rangle \\ &\geq m\Big(\frac{1}{2} - \frac{1}{q}\Big)\|u_k\|^2 - \frac{1}{q'}\|f\|_2\|u_k\|_2 \\ &\geq m\Big(\frac{1}{2} - \frac{1}{q}\Big)\|u_k\|^2 - \frac{1}{q'}\|f\|_2\|u_k\| \end{aligned}$$

for all k. Hence, the sequence $(u_k)_k$ is bounded in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Therefore there exist a function $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and a subsequence of $(u_k)_k$, still called $(u_k)_k$, such that

$$\begin{aligned} &u_k \rightharpoonup u \text{ in } H^1_{\mathrm{rad}}(\mathbb{R}^N) \text{ and in } H^s_{\mathrm{rad}}(\mathbb{R}^N), \\ &u_k \rightharpoonup u \text{ in } L^2(\mathbb{R}^N), \quad \nabla u_k \rightharpoonup \nabla u \text{ in } [L^2(\mathbb{R}^N)]^N, \\ &u_k \to u \text{ in } L^q(\mathbb{R}^N, K), \quad u_k \to u \text{ a.e. in } \mathbb{R}^N, \\ &|u_k| \leq \psi \text{ a.e. in } \mathbb{R}^N, \quad \text{with } \psi \in L^q(\mathbb{R}^N, K), \end{aligned} \tag{4.4}$$

by [30, Proposition I.1], being $N \geq 2$.

For all $\varphi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)$ by (4.4) and the continuity of the embeddings $H^1_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, V)$ we have

$$\begin{aligned} \langle J'(u), \varphi\rangle &= \langle J'(u_k), \varphi\rangle - c\langle\nabla u_k - \nabla u, \nabla\varphi\rangle_2 - \kappa\langle u_k - u, \varphi\rangle_s \\ &\quad - \int_{\mathbb{R}^N} V(x)(u_k - u)\varphi\,dx - \int_{\mathbb{R}^N} K(x)(|u_k|^{q-2}u_k - |u|^{q-2}u)\varphi\,dx = o(1) \quad \text{as } k \to \infty. \end{aligned}$$

Therefore, $\langle J'(u), \varphi\rangle = 0$ for all $\varphi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)$ and by a standard density argument $J'(u) = 0$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. In other words, u is a critical point of $J|_{H^1_{\mathrm{rad}}(\mathbb{R}^N)}$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$.

Now, by (4.2) it results that

$$0 = \langle J'(u), u\rangle = c\|\nabla u\|_2^2 + \kappa[u]_s^2 + \|u\|_{2,V}^2 - \|u\|_{q,K}^q - \int_{\mathbb{R}^N} f(x)u\,dx \tag{4.5}$$

and by (4.4) it follows that

$$\int_{\mathbb{R}^N} f(x)u\,dx = \lim_{k\to\infty}\int_{\mathbb{R}^N} f(x)u_k\,dx$$

being $f \in L^2(\mathbb{R}^N)$. Thus, from (4.5) we get

$$\begin{aligned}\limsup_{k\to\infty}(c\|\nabla u_k\|_2^2 + \kappa[u_k]_s^2) &= \limsup_{k\to\infty}\Big(\|u_k\|_{q,K}^q - \|u_k\|_{2,V}^2 + \int_{\mathbb{R}^N} f(x)u_k\,dx\Big)\\ &\le \|u\|_{q,K}^q - \|u\|_{2,V}^2 + \int_{\mathbb{R}^N} f(x)u\,dx\\ &= c\|\nabla u\|_2^2 + \kappa[u]_s^2\\ &\le \liminf_{k\to\infty} c\|\nabla u_k\|_2^2 + \liminf_{k\to\infty}\kappa[u_k]_s^2\\ &\le \liminf_{k\to\infty}(c\|\nabla u_k\|_2^2 + \kappa[u_k]_s^2),\end{aligned}$$

since $\|u_k\|_{q,K}^q \to \|u\|_{q,K}^q$ by (4.4), and

$$\|u\|_{2,V}^2 \le \liminf_{k\to\infty}\|u_k\|_{2,V}^2 \tag{4.6}$$

by (1.3) and (4.4). Hence

$$\lim_{k\to\infty}(c\|\nabla u_k\|_2^2 + \kappa[u_k]_s^2) = c\|\nabla u\|_2^2 + \kappa[u]_s^2.$$

Therefore, using again (4.5) and (4.6), we have

$$\begin{aligned}\limsup_{k\to\infty}\|u_k\|_{2,V}^2 &= \limsup_{k\to\infty}\Big(\|u_k\|_{q,K}^q - c\|\nabla u_k\|_2^2 - \kappa[u_k]_s^2 + \int_{\mathbb{R}^N} f(x)u_k\,dx\Big)\\ &= \|u\|_{q,K}^q - c\|\nabla u\|_2^2 - \kappa[u]_s^2 + \int_{\mathbb{R}^N} f(x)u\,dx\\ &= \|u\|_{2,V}^2 \le \liminf_{k\to\infty}\|u_k\|_{2,V}^2,\end{aligned}$$

and so

$$\lim_{k\to\infty}\|u_k\|_{2,V}^2 = \|u\|_{2,V}^2. \tag{4.7}$$

Thus, (4.4) and (4.7) lead to

$$\|u_k - u\|_{2,V}^2 = \|u_k\|_{2,V}^2 + \|u\|_{2,V}^2 - 2\int_{\mathbb{R}^N} V(x)u_k u\,dx \to 0$$

as $k \to \infty$. Finally, (1.3) implies

$$\lim_{k\to\infty}\|u_k - u\|_2 = 0. \tag{4.8}$$

Similarly, as $k \to \infty$

$$c\|\nabla u_k - \nabla u\|_2^2 + \kappa[u_k - u]_s^2 \to 0.$$

In conclusion, from the fact that $c > 0$ and $\kappa \ge 0$, the above limit and (4.8) give

$$\lim_{k\to\infty}\|u_k - u\| = 0, \tag{4.9}$$

as required. □

Fix $f \in L^2(\mathbb{R}^N)$, with $\|f\|_2 \le \delta$, where $\delta > 0$ is determined in Lemma 4.1, and put

$$c_e = \inf_{g\in\Gamma} \max_{t\in[0,1]} J(g(t)), \quad \Gamma = \{g \in C([0,1], H^1_{\mathrm{rad}}(\mathbb{R}^N)) : g(0) = 0,\ g(1) = e\}.$$

Clearly $c_e > 0$ by Lemma 4.1.

Lemma 4.2 yields that J satisfies the Palais–Smale condition in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ in particular at the level c_e. In other words, any Palais–Smale sequence $(u_k)_k$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, with the property

$$J(u_k) \to c_e \quad \text{and} \quad J'(u_k) \to 0 \quad \text{as } j \to \infty,$$

admits a convergent subsequence in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$.

Proof of Theorem 1.2. First we prove that $(\mathcal{P})$ admits a nontrivial radial mountain pass solution $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ at level c_e. Lemmas 4.1 and 4.2 guarantee that J satisfies all the assumptions of the mountain pass theorem in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Hence there exists a Palais–Smale sequence $(u_k)_k$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ for J at the critical value c_e. Now, Lemma 4.2 implies that, up to a subsequence, $(u_k)_k$ converges to some u in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Moreover, $J(u) = c_e > 0 = J(0)$. Therefore, u is a nontrivial critical point of $J|_{H^1_{\mathrm{rad}}(\mathbb{R}^N)}$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$.

Let us now assume that f *is nontrivial*, that is $0 < \|f\|_2 \le \delta$. In constructing a second independent nontrivial radial solution of $(\mathcal{P})$ we somehow follow the main ideas contained in [35]. We first claim that it is possible to find $\varphi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} f(x)\varphi\, dx > 0,$$

being f nontrivial. Indeed, by density, there exists $(f_k)_k \subset C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)$ such that $f_k \to f$ in $L^2(\mathbb{R}^N)$ as $k \to \infty$. Hence, for a fixed k sufficiently large we have $\|f_k - f\|_2 \le \|f\|_2/2$, and so

$$\int_{\mathbb{R}^N} f(x) f_k\, dx \ge -\|f_k - f\|_2\|f\|_2 + \|f\|_2^2 \ge \frac{\|f\|_2^2}{2} > 0.$$

The claim is proved taking $\varphi = f_k \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)$.

Now, for any $t > 0$ it results that

$$J(t\varphi) \le \frac{M}{2}\|\varphi\|^2 t^2 - \frac{1}{q}\|\varphi\|^q_{q,K} t^q - t\int_{\mathbb{R}^N} f(x)\varphi\, dx.$$

This implies that for a fixed $t_0 \in (0,1)$ sufficiently small $J(t_0\varphi) < 0$ and $\|t_0\varphi\| < \rho$, with ρ given in Lemma 4.1. Clearly $t_0\varphi$ is in $C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)$. Therefore

$$c_0 = \inf\{J(\omega) : \omega \in \overline{B_\rho}\} < 0, \quad B_\rho = \{\omega \in H^1_{\mathrm{rad}}(\mathbb{R}^N) : \|\omega\| < \rho\}.$$

By the *Ekeland* variational principle and Lemma 4.1 there exists a sequence $(u_k)_k \in B_\rho$ such that

$$c_0 \le J(u_k) \le c_0 + \frac{1}{k} \quad \text{and} \quad J(\omega) \ge J(u_k) - \frac{1}{k}\|\omega - u_k\|$$

for all $\omega \in \overline{B_\rho}$. In particular, for all $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$, with $\|v\| = 1$, and for all $k \in \mathbb{N}$, we have for $t > 0$ sufficiently small

$$\frac{J(u_k + tv) - J(u_k)}{t} \ge -\frac{1}{k}.$$

Since $J \in C^1(H^1_{\mathrm{rad}}(\mathbb{R}^N))$, the last inequality gives at once that

$$\|J'(u_k)\|_{X'} \le \frac{1}{k} \quad \text{for all } k \in \mathbb{N},$$

where X' denotes the dual space of $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Hence $(u_k)_k$ is a bounded Palais–Smale sequence for J in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Thus Lemma 4.2 assures the existence of some $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ such that $J'(v) = 0$ and $J(v) = c_0 < 0$. Moreover, v is actually in B_ρ by Lemma 4.1 and so v is the second radial solution of $(\mathcal{P})$ in the sense of $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and v is independent of the mountain pass solution u constructed in the first part.

Therefore, u and v are two nontrivial critical points of $J|_{H^1_{\mathrm{rad}}(\mathbb{R}^N)}$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ when f is nontrivial.

We claim that u and v are two critical points of J in the entire $H^1(\mathbb{R}^N)$, that is solutions of $(\mathcal{P})$ in the sense of definition (4.2).

The Hilbert space $H^1(\mathbb{R}^N)$ is a uniformly convex and reflexive Banach space, so that to show the claim it is enough to apply the principle of symmetric criticality due to R.S. Palais as stated in [16, Proposition 3.1]. Indeed, here $E = H^1(\mathbb{R}^N)$, $\Sigma = H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and

$$G = \{g : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N) : g(u) = u \circ A,\ A \in \mathrm{SO}(N)\},$$

where $\mathrm{SO}(N)$ is the special orthogonal group of the $N \times N$ orthogonal matrices A such that

$$AA^T = I, \quad A^* = A, \quad \det A = 1.$$

The claim is a consequence of [16, Proposition 3.1], since $J(u \circ A) = J(u)$ for all $u \in H^1(\mathbb{R}^N)$ and for all $A \in \mathrm{SO}(N)$.

Indeed, fixed $u \in H^1(\mathbb{R}^N)$, for all $A \in \mathrm{SO}(N)$, recalling that V, K, f and u are radial, we have

$$\begin{aligned}
\|\nabla(u \circ A)\|_2^2 &= \int_{\mathbb{R}^N} |\nabla(u(Ax))|^2\, dx = \int_{\mathbb{R}^N} |\nabla(u(x'))|^2\, dx' = \|\nabla u\|_2^2,\\
\|u \circ A\|_{2,V}^2 &= \int_{\mathbb{R}^N} V(x)|u(Ax)|^2\, dx = \int_{\mathbb{R}^N} V(x')|u(x')|^2\, dx' = \|u\|_{2,V}^2,\\
\|u \circ A\|_{q,K}^q &= \int_{\mathbb{R}^N} K(x)|u(Ax)|^q\, dx = \int_{\mathbb{R}^N} K(x')|u(x')|^q\, dx' = \|u\|_{q,K}^q,\\
\int_{\mathbb{R}^N} f(x)u(Ax)\, dx &= \int_{\mathbb{R}^N} f(x')u(x')\, dx',
\end{aligned}$$

since $|x| = |Ax| = |x'|$ and $\det A = 1$. Moreover, by [40, Chapter IV, Theorem 1.1], we get

$$\begin{aligned}
[u \circ A]_s^2 &= \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(u \circ A)|^2(x)\, dx = \int_{\mathbb{R}^N} |\xi|^{2s}|\widehat{u \circ A}|^2(\xi)\, d\xi\\
&= \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}|^2(A\xi)\, d\xi = \int_{\mathbb{R}^N} |x'|^{2s}|\hat{u}|^2(x')\, dx'\\
&= \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2(x')\, dx' = [u]_s^2.
\end{aligned}$$

Therefore, for all $u \in H^1(\mathbb{R}^N)$

$$J(u \circ A) = \frac{1}{2}(c\|\nabla(u \circ A)\|_2^2 + \kappa[u \circ A]_s^2 + \|u \circ A\|_{2,V}^2) - \frac{1}{q}\|u \circ A\|_{q,K}^q - \int_{\mathbb{R}^N} f(x)(u \circ A)(x)\, dx = J(u)$$

for all $A \in \mathrm{SO}(N)$, as stated. This completes the proof. □

As a final comment to the proof of Theorem 1.2 we remark that even if Theorem 1.1 and its Corollary 1.2 in [40, Chapter IV] are stated for functions of class $L^1(\mathbb{R}^N)$, a standard density argument extends these results in $L^2(\mathbb{R}^N)$, as actually used above.

5 Numerical simulations for $(\mathcal{D})$

In this section we provide numerical computations for problem $(\mathcal{D})$ which has completely been solved from a pure analytical point of view in Section 2.

Before starting, we recall that model $(\mathcal{D})$ describes the displacement $u = u(x)$ of a finite rod of length $2L$, restrained at the boundary, under the effects of external forces acting along the axis. More precisely, $f = f(x)$ represents an external force per unit volume, while $V(x)u$ stands for a restoring elastic force whose stiffness could possibly depend on x. The quantity E is the Young modulus of the material; the constants $c > 0$ and $\kappa \geq 0$ satisfy

$$c = \beta_1 \quad \text{and} \quad \kappa = -2\beta_2 k \cos s\pi,$$

where β_1 and $\beta_2 = 1 - \beta_1$ are the weights described in Section 2, and k is a further constant related to the nonlocal nature of the material.

Due to the extreme difficulty in defining an explicit analytical expression for u, numerical methods could help in finding approximated solutions of the problem. In particular, we look for a discrete form of the problem seeking the values $u_i = u(x_i)$ reached by u in a finite number of points $x_i \in [-L, L]$ which are equidistant with step $h = 2L/(n-1)$, namely

$$x_i = -L + ih, \quad i = 0, 1, \dots, n-1.$$

The terms in the equation of $(\mathcal{D})$ are evaluated as follows. The second derivative of the solution u'' is expressed by means of the central finite difference formula, that is

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

To estimate $(-\Delta)^s u$, thanks to the main formulas (2.10) and (3.3), we are able to follow the discretization procedure proposed in [26]. More precisely, using the notation of [26], we compute

$$(-\Delta_h)^s u_i = \sum_{j=1}^{\infty}(2u_i - u_{i+j} - u_{i-j})w_j = \sum_{j=-\infty}^{\infty}(u_i - u_{i-j})w_j, \tag{5.1}$$

where the weights w_j are determined explicitly by the semi-exact quadrature rules related to the main weight function given in (3.3), that is $\nu(\xi) = C_{1,2s}|\xi|^{-1-2s}$, $\xi \in \mathbb{R} \setminus \{0\}$, $s \in (1/2, 1)$, where $C_{1,2s}$ is defined in (3.4). We refer to [26] for further details.

A simple truncation of the sum in (5.1) at a finite value $M \geq n-1$ of the index j gives as in [26]

$$\begin{aligned}(-\Delta_h)^s_M u_i &= \int_{-Mh}^{Mh} [u(x_i) - u(x_i - y)]\nu(y)\,dy + u(x_i)\int_{|y|>Mh} \nu(y)\,dy \\ &= \sum_{j=-M}^{M}(u_i - u_{i-j})w_j + \frac{C_{1,2s}}{s(Mh)^{2s}}\,u(x_i),\end{aligned}$$

since $Mh \geq 2L$. As remarked in [26], the procedure converges to the finite differences method when $s \to 1$.

Hence, the discrete version of the equation in $(\mathcal{D})$ is transformed into the problem of finding the values u_i such that

$$\begin{cases} -cu_i'' + \kappa(-\Delta_h)^s_M u_i + V(x_i)u_i = \dfrac{f(x_i)}{E} & \text{for } i = 1, \dots, n-2, \\ u_0 = u_{n-1} = 0. \end{cases} \tag{5.2}$$

The goal is to find the zeros of the above system of n functions in n variables. To this end we use the Python programming language. In particular, we employ the function "fsolve", based on Powell's hybrid method as implemented in MINPACK (see [33]).

The main ideas of the Powell hybrid method, as reported in [25], are briefly resumed in the following. Problem (5.2) can be rewritten as

$$\mathbf{P}(\mathbf{U}) = \mathbf{0}, \quad \mathbf{U} = (u_0, u_1, \dots, u_{n-1}), \quad \mathbf{P} : \mathbb{R}^n \to \mathbb{R}^n.$$

Start from an estimate $\mathbf{U}_0$, and find the next estimate to the solution

$$\mathbf{U}_0 + \Delta\mathbf{U},$$

where $\Delta\mathbf{U}$ is derived from

$$(\gamma\,\mathbb{I} + J^T(\mathbf{U}_0)J(\mathbf{U}_0))\Delta\mathbf{U} = -J(\mathbf{U}_0)\mathbf{P}(\mathbf{U}_0),$$

where γ is a suitable positive parameter, and $\mathbb{I}$ is the unit $n \times n$ matrix. Moreover, in order to increase efficiency, the Jacobian matrix J is approximated by means of a Broyden update with the formula

$$J(\mathbf{U}_0 + \Delta\mathbf{U}) = J(\mathbf{U}_0) + \frac{(\mathbf{V} - J(\mathbf{U}_0)\Delta\mathbf{U})\Delta\mathbf{U}^T}{\Delta\mathbf{U}^T\Delta\mathbf{U}}, \quad \mathbf{V} = \mathbf{P}(\mathbf{U}_0 + \Delta\mathbf{U}) - \mathbf{P}(\mathbf{U}_0),$$

instead of recalculating J.

We underline that the choice of the Powell hybrid method is motivated by the fact that it could be employed to perform numerical simulations also for more general problems than ($\mathcal{D}$), as ($\mathcal{P}_\lambda$) and ($\mathcal{P}$), since it is particularly suited also for nonlinear systems.

5.1 The case of a discontinuous force *f*

In this section we apply the numerical procedure described before to two cases, taking somehow inspiration from [14, 19].

Case study #1. We consider the rod loaded by two forces having equal magnitude and opposite direction (and therefore the resultant force is zero) applied close to the mid-span, as shown in Figure 1.

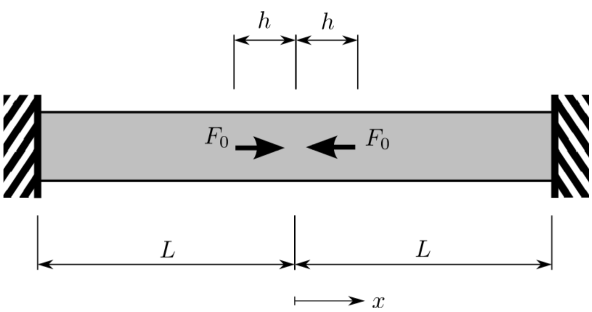

Figure 1. Rod loaded by two forces having equal magnitude and opposite direction.

Let A be the cross section area of the rod. The applied forces F_0 have been modeled assuming a uniform load on a rod portion spanning a length equal to $h/2$ before and after the application point of the force

$$f(x) = \begin{cases} F_0/Ah, & x \in [-2h, -h], \\ -F_0/Ah, & x \in [h, 2h], \\ 0, & \text{elsewhere}. \end{cases}$$

The values adopted for the parameters are

$$\begin{aligned} &\beta_1 = 0.5, &&\beta_2 = 0.5, && s = 3/4, \\ &E = 14400\,\text{MPa}, && k = 0.015\,\text{mm}^{2s-2}, && V = 0, \\ &F_0 = 1000\,\text{N}, && A = 100\,\text{mm}^2, && h = 2. \end{aligned}$$

The values obtained for the displacements u and the strain ϵ are shown in Figure 2.

The obtained results are in good agreement with those reported in [19].

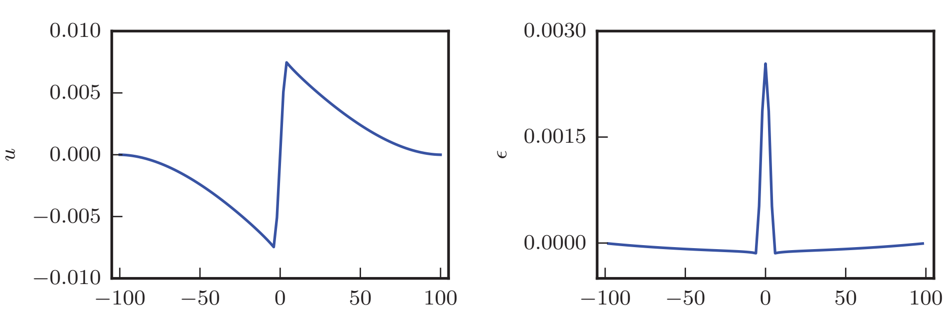

Figure 2. Displacements u and strain ϵ for the Case study #1.

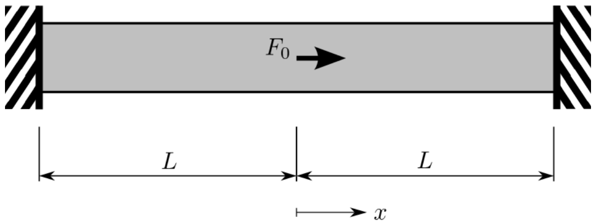

Figure 3. Rod loaded with a force at mid-span.

Case study #2. As before, let us consider the rod of length $2L$, centered at zero and with cross section area A. Here we consider an applied force F_0 at the mid-span, as shown in Figure 3.

The force F_0 has been modeled assuming a uniform load on a portion spanning a length h before and after the application point of the force

$$f(x) = \begin{cases} F_0/Ah, & x \in [-h, h], \\ 0, & \text{elsewhere}. \end{cases}$$

The values adopted for the parameters are

$$\beta_1 = 0.5, \qquad \beta_2 = 0.5, \qquad s = 3/4,$$
$$E = 14400\,\text{MPa}, \qquad k = 0.071\,\text{mm}^{2s-2}, \qquad V = 0,$$
$$F_0 = 360\,\text{N}, \qquad A = 100\,\text{mm}^2, \qquad h = 2.$$

The values obtained for the displacements u and the strain ϵ are shown in Figure 4.

The obtained results are in good agreement with those reported in [14].

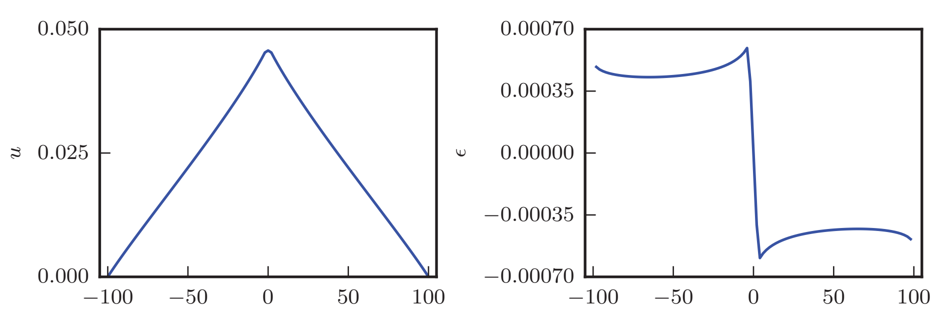

Figure 4. Displacements u and strain ϵ for the Case study #2.

5.2 The case of a continuous force *f*

Let us consider the distributed continuous external force

$$f(x) = f_0\, e^{-x^2/2(L/16)^2}$$

schematized in Figure 5. The following values have been used

$$\beta_1 = 0.5, \qquad \beta_2 = 0.5, \qquad s = 3/4,$$
$$E = 72000\,\text{MPa}, \qquad k = 0.5\,\text{mm}^{2s-2}, \qquad h = 2,$$
$$f_0 = 36\,\text{N}\,\text{mm}^{-3},$$

Two constant levels for the function V have been considered, namely $V = 0$ and $V = 0.0005$. The values obtained for the displacements u and the strains ϵ are shown in Figure 6.

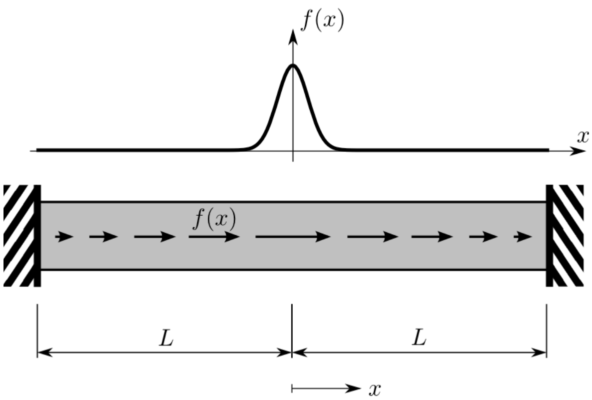

Figure 5. Rod loaded with a distributed continuous force f.

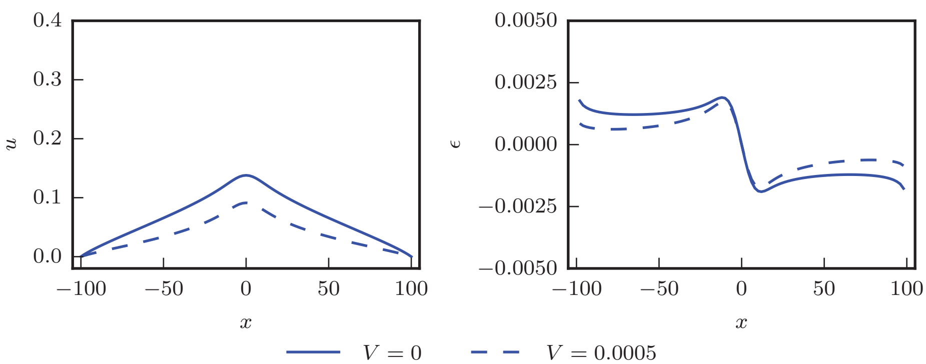

Figure 6. Displacements u and strains ϵ for the case of a continuous force f.

We remark that when $V = 0$ the results obtained for the displacement u and the strain ϵ are very similar to the analogous one got for the Case study #2 treated in the Section 5.1 for a discontinuous force f. In fact, the external force F_0 considered in the Case study #2 of Section 5.1 can be seen as the limit of the distributed force f used here when the interval of x in which f is significantly different from zero becomes very small.

5.3 Parametric analysis

In this subsection we perform a parametric investigation on the problem $(\mathcal{D})$ in order to highlight the different response of the rod under the action of the continuous external force f introduced in Section 5.2, both in the nonlocal and in the *purely local* setting. The mechanical characteristics are

$$E = 72000\,\mathrm{MPa}, \quad k = 0.5\,\mathrm{mm}^{2s-2}, \quad V = 0,$$
$$f_0 = 36\,\mathrm{N\,mm}^{-3}, \quad h = 2.$$

Moreover, we recall that

$$c = \beta_1, \quad \kappa = -2\beta_2 k \cos s\pi.$$

Changing β_1 and β_2. First, we have considered the behavior of the rod according to different values of the parameter $\beta_1 \in [0, 1]$. In the *purely local* case $\beta_1 = 1$ (and therefore $\beta_2 = 0$) problem $(\mathcal{D})$ becomes

$$\begin{cases} -u'' + V(x)u = \dfrac{f(x)}{E} & \text{in } (-L, L), \\ u = 0 & \text{in } \mathbb{R} \setminus (-L, L). \end{cases} \tag{5.3}$$

If $V \equiv 0$, then the solution of (5.3) is

$$u(x) = \hat{u}(x) + Ax + B,$$

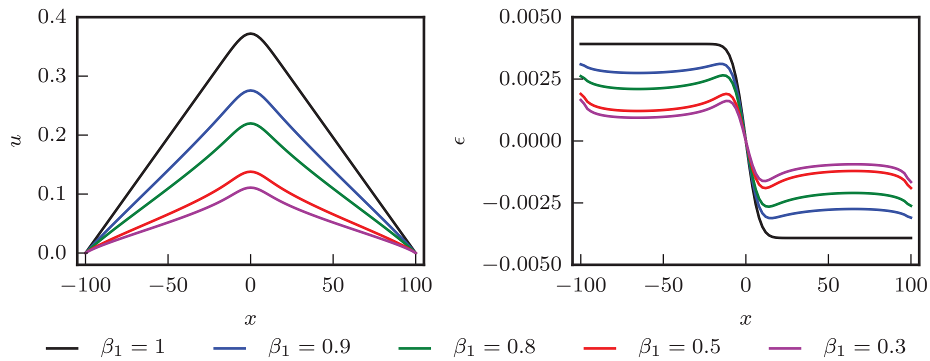

Figure 7. Parametric analysis on varying β_1, with $V = 0$.

where

$$\hat{u} = \frac{\sqrt{\pi}}{256} f_0 L^2 \Big(\frac{e^{-128x^2/L^2}}{\sqrt{\pi}} + \frac{2^{7/2} x \operatorname{erf}(2^{7/2} x)}{L} \Big), \quad A = -\frac{1}{2L}[\hat{u}(L) - \hat{u}(-L)], \quad B = -\frac{1}{2}[\hat{u}(L) + \hat{u}(-L)],$$

and the error function is defined as

$$\operatorname{erf}(\mathrm{x}) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\, dt.$$

The obtained displacements and strains are shown in Figure 7. Note that the graphics corresponding to the case $\beta_1 = 0.5$ coincide exactly with the ones plotted in Figure 6.

As can be observed, for a fixed value of k, for example $k = 0.5$, in the nonlocal setting ($\beta_1 < 1$) the displacement u is smaller than in purely local case $\beta_1 = 1$. The interesting point is that the values of u decrease as far as β_1 does, that is as far as the nonlocal effect increases.

Moreover, the different behavior of the strains, induced by nonlocality, can be clearly observed. Indeed, if $\beta_1 = 1$, the strain of the rod is constant in the region where f reaches values very close to zero, that is in the set where f is negligible from an engineering point of view. On the contrary, the rod with effective nonlocal properties ($\beta_1 < 1$) is such that the strain ϵ reaches its greatest values (in magnitude) at the end points and at the mid-span of the rod.

Under the action of a continuous external force f, it is particularly interesting to consider also the contribute of an effective positive constant potential V. The solution of problem (5.3) is now given by

$$\begin{aligned} u(x) = \frac{f_0\sqrt{\pi}}{4\sqrt{k\rho}} \Bigg\{ & \frac{e^{\sqrt{k}x}[\operatorname{erf}(\frac{\sqrt{k}-2\rho L}{2\sqrt{\rho}}) + e^{2\sqrt{k}L}\operatorname{erf}(\frac{\sqrt{k}+2\rho L}{2\sqrt{\rho}})]}{e^{2\sqrt{k}L-k/4\rho} + e^{-k/4\rho}} \\ & + \frac{e^{-\sqrt{k}x}[\operatorname{erf}(\frac{\sqrt{k}-2\rho L}{2\sqrt{\rho}}) + e^{2\sqrt{k}L}\operatorname{erf}(\frac{\sqrt{k}+2\rho L}{2\sqrt{\rho}})]}{e^{2\sqrt{k}L-k/4\rho} + e^{-k/4\rho}} \\ & - e^{k/4\rho-\sqrt{k}x}\Big[\operatorname{erf}\Big(\frac{\sqrt{k}-2\rho L}{2\sqrt{\rho}}\Big) + e^{2\sqrt{k}x}\operatorname{erf}\Big(\frac{\sqrt{k}+2\rho x}{2\sqrt{\rho}}\Big)\Big]\Bigg\}, \end{aligned}$$

where

$$\rho = \Big(\frac{8\sqrt{2}}{L}\Big)^2.$$

The results, obtained for $V = 0.0005$, are shown in Figure 8.

Considerations similar to those of the case $V = 0$ can be done.

Changing *s*. The analysis has been enlarged with numerical simulations concerning the variation of the fractional exponent s. In this context, the values $\beta_1 = 0.5$ and $\beta_2 = 0.5$ have been fixed, while the other parameters are set as in the previous case. In particular, note that the chosen value of k implies that if $s \to 1$, then $-cu'' + \kappa(-\Delta)^s u \to -u''$ and therefore the purely local behavior is obtained.

The results are shown, in terms of displacements u and strains ϵ, in Figure 9 for the case $V = 0$ and in Figure 10 for the case $V = 0.0005$.

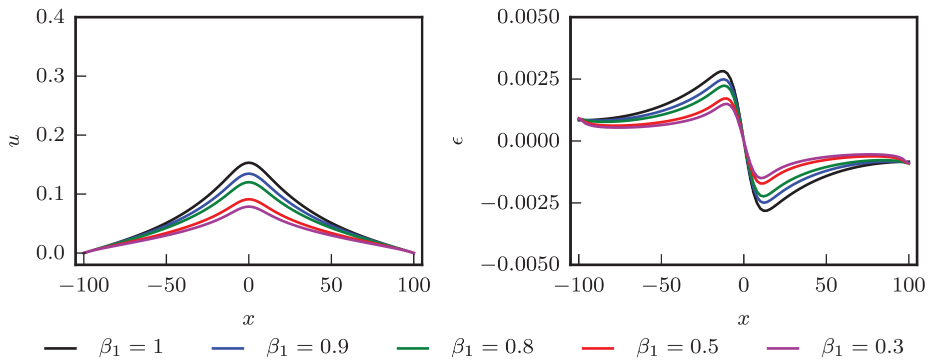

Figure 8. Parametric analysis on varying β_1, with $V > 0$.

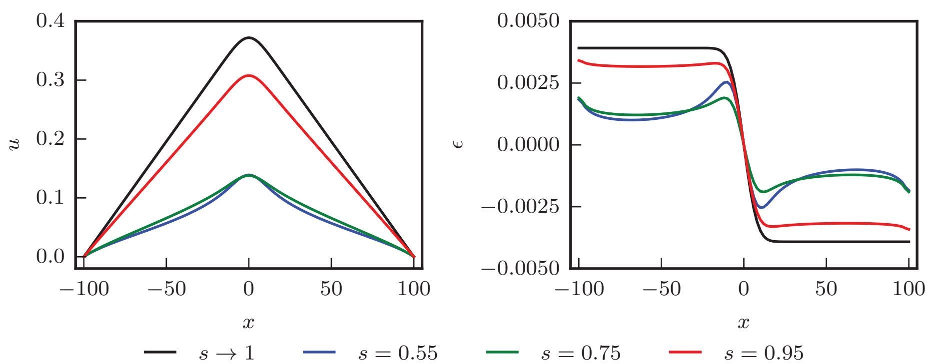

Figure 9. Parametric analysis on varying s, with $V = 0$.

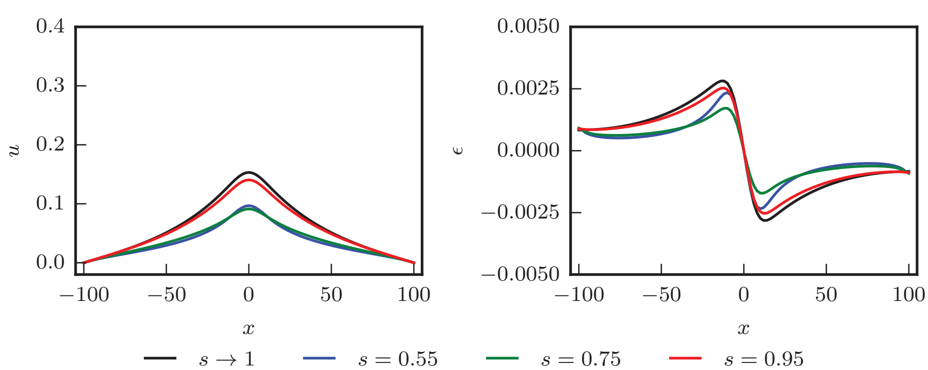

Figure 10. Parametric analysis on varying s, with $V > 0$.

It is interesting to note an evident increase of the strain ϵ near the end points and at the mid-span of the rod, when s is close to $1/2$.

Changing k. A further investigation we present here concerns the effects on u and ϵ deriving from different values of k. Such effects are schematized in Figure 11. In particular, we note that the value $k = 0.353$ has been chosen according to

$$c = \kappa \implies \beta_1 = -2\beta_2 k \cos s\pi \implies k = -\frac{1}{2}\cos\frac{3}{4}\pi.$$

In this case, we obtain a solution which does not coincide with the one obtained in the local case, and this difference may be used to assess the influence of the fractional Laplacian in problem $(\mathcal{D})$. As it is shown in Figure 11, the displacements and the corresponding strains increase as the value of k decrease.

Concerning Figure 11, we recall that the purely local case refers to problem (5.3).

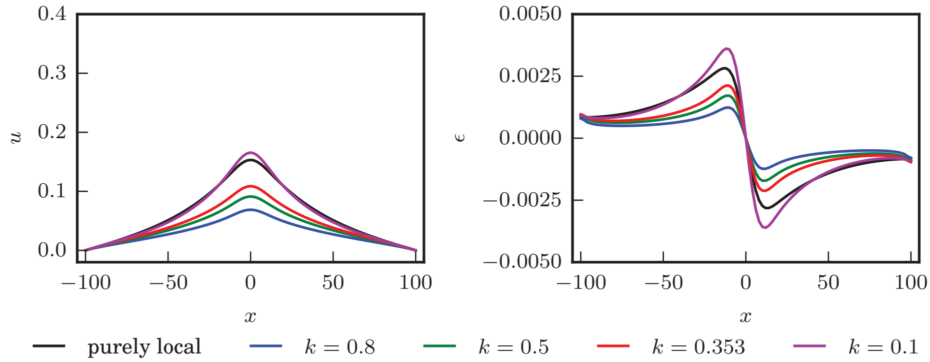

Figure 11. Parametric analysis on varying k, with $V > 0$.

Changing V. Finally, the action of the restoring force V has been studied and the corresponding results are presented in Figure 12. As it was expected, since the term $V(x)u$ represents an elastic restoring force, the displacements and the strains decrease as the value of V increase.

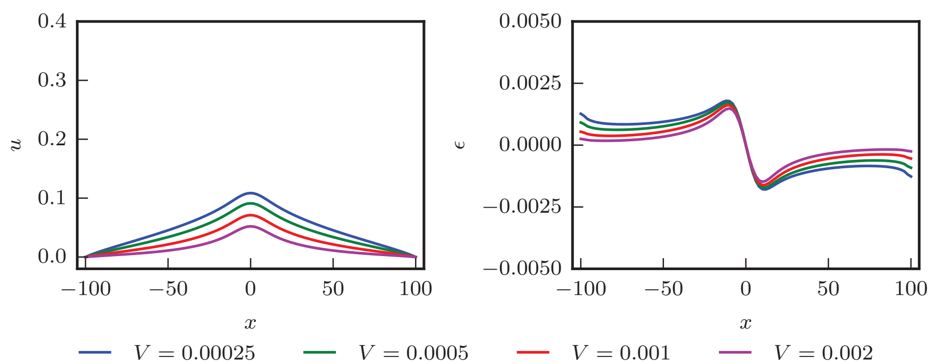

Figure 12. Parametric analysis on varying V.

6 Conclusions and perspectives

Composite materials are acquiring an important role in the development of innovative solutions which are environmentally sustainable and energetically efficient. The possibility to design the characteristics of these composites, using the properties of the single phases and their mutual arrangement, is precious and leads to significant outcomes. However, the resulting material, when considered at the macro-scale, is often characterized by nonlocal properties, in the sense that the stress at a point depends not only on the strain at the point but also on the strain at distant points. For materials exhibiting nonlocal behavior, the models of the classical mechanics are not sufficient and a new investigative approach turns to be necessary.

In this paper, following the approach proposed by Eringen, a nonlocal constitutive model for a rod has been obtained. The equation governing the problem contains a term involving the fractional Laplacian operator which accounts for the nonlocal part of the response. It has been shown that the problem of the rod subjected to external forces has a unique weak solution, and under reasonable conditions the weak solution is actually a classical solution of the problem. Moreover, the model has been extended to a nonlinear multidimensional problem depending on a real parameter λ, for which nontrivial solutions have been determined for all λ behind a threshold $\lambda^* > 0$.

In order to estimate the solution of problem $(\mathcal{D})$, numerical simulations have been produced. In particular, the problem has been discretized by mean of finite differences. The fractional Laplacian term has been evaluated following the approach due to Huang and Oberman. The obtained results have been validated by comparison with the exact solution known in the purely local case and with previous results in the literature. The numerical investigations have highlighted the importance of the role of the parameters in the nonlocal model, especially for what concerns the order s of the fractional Laplacian and the interactions between the local and nonlocal nature of the material.

The analysis carried out allows us to enlarge the present knowledge about always more efficient and energetically sustainable structural systems, employed in technical fields, trough the of linear and nonlinear models describing the elastic behavior of innovative composite materials. The obtained results could be useful in the identification of the mechanical characteristics of a composite material exhibiting a nonlocal behavior and in developing models to fit experimental data. The proposed approach seems to open new perspectives for the investigation and the analysis of other interesting engineering problems, concerning effective applications. The procedure appears to be particularly fruitful in the field of the mechanics of nanomaterials, plane problems with eventual radial symmetry (circular plates, pipes) and nonlinear systems.

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