TITLE: Convergence in variation and a characterization of the absolute continuity

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Convergence in variation and a characterization of the absolute continuity

Abstract

We study approximation results for a family of Mellin integral operators of the form

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}^N_+} K_w(\mathtt{t}, f(\mathtt{st}))\frac{d\mathtt{t}}{\langle \mathtt{t}\rangle},\ \mathtt{s} \in \mathbb{R}^N_+,\ w > 0,$$

where $\{K_w\}_{w>0}$ is a family of kernels, $\langle \mathtt{t}\rangle := \prod_{i=1}^N t_i$, $\mathtt{t} = (t_1, \ldots, t_N) \in \mathbb{R}^N_+$, and f is a function of bounded variation on $\mathbb{R}^N_+$. The starting point of this study is motivated by the important applications that approximation properties of certain families of integral operators have in image reconstruction and in other fields. In order to treat such problems, to work in $BV-$spaces in the multidimensional setting of $\mathbb{R}^N_+$ becomes crucial: for this reason we use a multidimensional concept of variation in the sense of Tonelli, adapted from the classical definition to the present setting of $\mathbb{R}^N_+$ equipped with the Haar measure. Using such definition of variation, we obtain a convergence result proving that $V[T_w f - f] \to 0$, as $w \to +\infty$, whenever f is an absolutely continuous function; moreover we also study the problem of the rate of approximation. In case of regular kernels, we finally prove a characterization of the absolute continuity in terms of the convergence in variation by means of the Mellin-type operators $\{T_w f\}_{w>0}$.

Key-words: Mellin integral operators, convergence in variation, absolutely continuous functions, multidimensional variation
AMS subject classification: 41A35, 41A25, 26B30, 26B99, 26D10

1 Introduction

In this paper we investigate the approximation properties of the following nonlinear family of Mellin-type integral operators:

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}^N_+} K_w(\mathtt{t}, f(\mathtt{st}))\frac{d\mathtt{t}}{\langle \mathtt{t}\rangle}, \tag{I}$$

$w > 0,\ \mathtt{s} = (s_1, \ldots, s_N) \in \mathbb{R}^N_+ :=]0, +\infty)^N$, where $\mathtt{st} := (s_1t_1, \ldots, s_Nt_N)$ and $\langle \mathtt{t}\rangle := \prod_{i=1}^N t_i$, for f belonging to the space of functions of bounded variation on $\mathbb{R}^N_+$. In particular we will prove that, under suitable assumptions on the kernel functions, if f is absolutely continuous, then

$$V[T_w f - f] \to 0,\ w \to +\infty, \tag{II}$$

and furthermore

$$V[T_w f - f] = O(w^{-\alpha}),\ w \to +\infty,$$

$0 < \alpha \leq 1$, for functions belonging to a Lipschitz class of Zygmund type. Moreover, in the case of absolutely continuous kernel functions, it is possible to prove that the convergence result (II) can be reversed, so that we obtain a characterization of the absolute continuity on $\mathbb{R}^N_+$ in terms of the convergence in variation of the operators

(I). In order to do this, a crucial step is to use a notion of absolute continuity (the *log-absolute continuity*), introduced in [1], which is proved to be equivalent to the classical one (see Section 5). These results contain, as particular case, the case of linear integral operators (see [2]). We point out that the present nonlinear setting is not only important from a mathematical point of view, allowing to develop a more general treatment of the theory, but it is also useful in several applications to describe nonlinear processes that cannot be approached by means of linear integral operators.

The set of bounded variation functions plays an important role, apart from a mathematical point of view, for the applications implied by several families of integral operators acting on this space. For example, when these operators are used in order to reconstruct images, the setting of $BV-$spaces is suitable to describe jumps of grey-levels of the image since, from a mathematical point of view, they are represented by discontinuities (see also, e.g. [3, 4]). Among the families of integral operators above mentioned, the Mellin type operators are particularly important because of their applications in optical physics and engineering (see, e.g., [5, 6, 7, 8]). Indeed they revealed to be useful for signal reconstruction when the samples are not uniformly spaced, as in the classical Shannon Sampling Theorem, but exponentially spaced: in fact such model is useful to represent situations in which information accumulates near the time $t = 0$ (see, e.g., [9, 10]). On the other hand, the importance of Mellin analysis is well-known, not only in approximation theory (we refer to [11, 12] for an extensive theory about Mellin operators, while, for other results about homothetic-type and discrete operators in various setting, one can see, e.g., [13, 14, 15, 16, 17, 18, 2, 19, 20, 21, 22]), but also in several other fields, because of its wide applications (see, e.g., [23, 24, 25, 26]).

In order to treat the case of Mellin integral operators, the most natural way to set up our study is to work in $\mathbb{R}^N_+$ endowed with the Haar measure $\mu(A) := \int_A \langle \mathtt{t} \rangle^{-1} \, d\mathtt{t}$, where A is a Borel subset of $\mathbb{R}^N_+$.

Another important role is played by the multidimensional setting and, in view of the frame in which we work, we will use the multidimensional concept of variation introduced in [2], that is inspired by the Tonelli approach (see [27], and also [28, 29]).

Finally we recall that approximation problems in $BV-$spaces were studied in the literature from several points of view, by means of different notions of variation, such as, e.g., the classical variation ([30]), the distributional variation ([31]; see [33, 32, 34] for the definition), the Musielak-Orlicz $\varphi-$variation ([35, 36, 37, 13]), the Riesz $\varphi-$variation ([38, 39]) and others. In the multidimensional case, results in this direction can be found, for example, in [40] and [41, 42] (nonlinear case) for a new multidimensional concept of $\varphi-$variation by means of the classical convolution integral operators.

The paper is organized as follows: after a preliminary section (Section 2) where the main definitions and notations are presented, together with the assumptions of our study, Section 3 contains the main convergence results. In Section 4 the problem of the rate of approximation is studied while, in Section 5, we prove the characterization of $AC-$functions. Finally, in Section 6, we present some examples of kernel functions to which our results can be applied.

2 Definitions, notations and assumptions

We will work with functions $f : \mathbb{R}^N_+ \longrightarrow \mathbb{R}$ of bounded variation, using the multidimensional variation in the sense of Tonelli introduced in [2], that takes into account of the multiplicative group structure of $\mathbb{R}_+^N$. In order to recall the definition, we now introduce some notations.

Given $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}_+^N$, $I'_j := [\mathtt{a}'_j, \mathtt{b}'_j]$ will denote the $(N-1)-$dimensional interval $I'_j := \prod_{i\neq j}[a_i, b_i]$, obtained deleting by I the $j-$th coordinate, so that $I = [\mathtt{a}'_j, \mathtt{b}'_j] \times [a_j, b_j]$. In a similar way we put $\mathtt{x}'_j := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \in \mathbb{R}_+^{N-1}$, for $\mathtt{x} = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$, so that $\mathtt{x} = (\mathtt{x}'_j, x_j)$ and $f(\mathtt{x}) = f(\mathtt{x}'_j, x_j)$, for $f : \mathbb{R}_+^N \to \mathbb{R}$.

By $L^1_\mu(\mathbb{R}_+^N)$ we will denote the space of all the functions $f : \mathbb{R}_+^N \to \mathbb{R}$ such that $\int_{\mathbb{R}_+^N} |f(\mathtt{t})|\langle \mathtt{t}\rangle^{-1}\, d\mathtt{t} < +\infty$, where $\langle \mathtt{t}\rangle := \prod_{i=1}^N t_i$, $\mathtt{t} = (t_1, \ldots, t_N) \in \mathbb{R}_+^N$ (μ stands for the Haar measure on $\mathbb{R}_+^N$ $\mu(A) := \int_A \langle \mathtt{t}\rangle^{-1}\, d\mathtt{t}$, where A is a Borel subset of $\mathbb{R}_+^N$), in order to point out the difference with the usual Lebesgue space $L^1(\mathbb{R}_+^N)$.

In order to define and compute the multidimensional variation on an interval $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}_+^N$, we first consider the Jordan variation of the $j-$th section of f, namely $V_{[a_j, b_j]}[f(\mathtt{x}'_j, \cdot)]$, $\mathtt{x}'_j \in I'_j$, and then we define the $(N-1)-$dimensional integrals

$$\Phi_j(f, I) := \int_{\mathtt{a}'_j}^{\mathtt{b}'_j} V_{[a_j, b_j]}[f(\mathtt{x}'_j, \cdot)]\, \frac{d\mathtt{x}'_j}{\langle \mathtt{x}'_j\rangle},$$

where $\langle \mathtt{x}'_j\rangle$ denotes the product $\prod_{i=1, i\neq j}^N x_i$.

We now compute the euclidean norm of the vector $(\Phi_1(f, I), \ldots, \Phi_N(f, I))$, i.e.,

$$\Phi(f, I) := \left\{ \sum_{j=1}^N [\Phi_j(f, I)]^2 \right\}^{\frac{1}{2}},$$

where we put $\Phi(f, I) = +\infty$ if $\Phi_j(f, I) = +\infty$ for some $j = 1, \ldots, N$. Then the multidimensional variation of $f : \mathbb{R}_+^N \to \mathbb{R}$ on $I \subset \mathbb{R}_+^N$ is defined as

$$V_I[f] := \sup \sum_{i=1}^m \Phi(f, J_i),$$

where the supremum is taken over all the finite families of $N-$dimensional intervals $\{J_1, \ldots, J_m\}$ which form partitions of I.

If we now pass to the supremum over all the intervals $I \subset \mathbb{R}_+^N$, we obtain the variation of f over the whole space $\mathbb{R}_+^N$, namely

$$V[f] := \sup_{I \subset \mathbb{R}_+^N} V_I[f].$$

Definition 2.1 *By $BV(\mathbb{R}_+^N) := \{f \in L^1_\mu(\mathbb{R}_+^N) : \; V[f] < +\infty\}$ we denote the space of functions of bounded variation on $\mathbb{R}_+^N$.*

Let us notice that, if $f \in L^1_\mu(\mathbb{R}_+^N)$ is of bounded variation on $\mathbb{R}_+^N$, then the sections $f(\mathtt{x}'_j, \cdot)$ are of bounded variation on $\mathbb{R}_+$ and $V_{\mathbb{R}_+}[f(\mathtt{x}'_j, \cdot)] \in L^1_\mu(\mathbb{R}_+^{N-1})$, a.e. $\mathtt{x}'_j \in \mathbb{R}_+^{N-1}$.

Definition 2.2 *A function $f : \mathbb{R}_+^N \to \mathbb{R}$ is said to be absolutely continuous on $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}_+^N$ if, for every $j = 1, 2, \ldots, N$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{\nu=1}^n |f(\mathtt{x}'_j, \beta^\nu) - f(\mathtt{x}'_j, \alpha^\nu)| < \varepsilon,$$

for a.e. $\mathtt{x}'_j \in \mathbb{R}_+^{N-1}$ *and for all finite collections of non-overlapping intervals* $[\alpha^\nu, \beta^\nu] \subset [a_j, b_j]$, $\nu = 1, \dots, n$, *for which* $\sum_{\nu=1}^n (\beta^\nu - \alpha^\nu) < \delta$ *(see, e.g., [43, 44]).*

Now, by $AC(\mathbb{R}_+^N)$ *we denote the space of functions* $f : \mathbb{R}_+^N \to \mathbb{R}$ *which are of bounded variation and absolutely continuous on every* $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}_+^N$.

We now introduce the family of integral operators that we will use in the present paper in order to obtain approximation results for $BV-$functions.

Given $f \in BV(\mathbb{R}_+^N)$, let us consider the nonlinear Mellin integral operators defined as

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}_+^N} K_w(\mathtt{t}, f(\mathtt{st})) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}, \tag{I}$$

$w > 0$, $\mathtt{s} \in \mathbb{R}_+^N$, where $\mathtt{st} := (s_1 t_1, \dots, s_N t_N)$. Here $\{K_w\}_{w>0}$ is a family of kernels $K_w : \mathbb{R}_+^N \times \mathbb{R} \to \mathbb{R}$ of the form

$$K_w(\mathtt{t}, u) = L_w(\mathtt{t}) H_w(u), \quad \mathtt{t} \in \mathbb{R}_+^N, \ u \in \mathbb{R},$$

where $L_w : \mathbb{R}_+^N \to \mathbb{R}$ and $H_w : \mathbb{R} \to \mathbb{R}$ is such that $H_w(0) = 0$.

We assume that the following assumptions are satisfied:

$\mathbf{K_w.1}$) $L_w : \mathbb{R}_+^N \to \mathbb{R}$ is a measurable essentially bounded function (i.e., $\|L_w\|_{L^\infty} < +\infty$) such that $L_w \in L^1_\mu(\mathbb{R}_+^N)$, $\|L_w\|_{L^1_\mu} \le A$, for an absolute constant $A > 0$, and $\int_{\mathbb{R}_+^N} L_w(\mathtt{t}) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} = 1$, for every $w > 0$;

$\mathbf{K_w.2}$) for every fixed $0 < \delta < 1$,

$$\int_{|\mathbf{1}-\mathtt{t}|>\delta} |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \to 0,$$

as $w \to +\infty$, where $\mathbf{1} = (1, \dots, 1)$ denotes the unit vector of $\mathbb{R}_+^N$;

$\mathbf{K_w.3}$) denoted by $G_w(u) := H_w(u) - u$, $u \in \mathbb{R}$, $w > 0$,

$$\frac{V_J[G_w]}{m(J)} \to 0, \quad \text{as } w \to +\infty,$$

uniformly with respect to every (proper) bounded interval $J \subset \mathbb{R}$, i.e., for every $\varepsilon > 0$, there exists $\overline{w} > 0$ (depending only on ε) such that, for every $w \ge \overline{w}$, $\frac{V_J[G_w]}{m(J)} \le \varepsilon$, for every (proper) bounded interval $J \subset \mathbb{R}$ ($m(J)$ denotes the length of J).

In the following we will say that $\{K_w\}_{w>0} \subset \mathcal{K}_w$ if the above conditions $K_w.1) - K_w.3)$ hold.

Remark 2.3 We point out that assumption $K_w.3)$ implies that $\{H_w\}_{w>0}$ satisfies asymptotically a Lipschitz condition. Indeed, for example in correspondence of $\varepsilon = 1$, by $K_w.3)$ there exists $\overline{w} > 0$ such that for every $w \ge \overline{w}$, $V_J[G_w] \le m(J)$, uniformly with respect to $J \subset \mathbb{R}$. Hence, for every $u, v \in \mathbb{R}$, $v < u$,

$$\begin{aligned} |H_w(u) - H_w(v)| &\le |H_w(u) - u - [H_w(v) - v]| + |u - v| \\ &\le V_{[v,u]}[G_w] + |u - v| \le 2|u - v|, \end{aligned} \tag{1}$$

for every $w \ge \overline{w}$.

3 Estimates and main convergence results

We first prove that, under the above assumptions, the operators $T_w f$ are asymptotically well-defined on $\mathbb{R}_+^N$ for every $f \in L^1_\mu(\mathbb{R}_+^N)$.

Proposition 3.1 *If $f \in L^1_\mu(\mathbb{R}_+^N)$ and $\{K_w\}_{w>0}$ satisfy $K_w.1)$ and $K_w.3)$, then there exists $\overline{w} > 0$ such that, for every $w \geq \overline{w}$, $(T_w f)(\mathtt{s}) < +\infty$, for every $\mathtt{s} \in \mathbb{R}_+^N$. Moreover, $T_w f \in L^1_\mu(\mathbb{R}_+^N)$, for every $w \geq \overline{w}$.*

Proof By $K_w.1)$ and (1), taking into account that $H_w(0) = 0$, there exists $\overline{w} > 0$ such that

$$\begin{aligned} |(T_w f)(\mathtt{s})| &\leq \int_{\mathbb{R}_+^N} |L_w(\mathtt{t})||H_w(f(\mathtt{st}))| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\ &\leq 2 \int_{\mathbb{R}_+^N} |L_w(\mathtt{t})||f(\mathtt{st})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \leq 2\|L_w\|_{L^\infty} \|f\|_{L^1_\mu} < +\infty, \end{aligned}$$

for every $\mathtt{s} \in \mathbb{R}_+^N$ and $w \geq \overline{w}$. Moreover, by the Fubini-Tonelli theorem and $K_w.1)$,

$$\begin{aligned} \int_{\mathbb{R}_+^N} |(T_w f)(\mathtt{s})| \frac{d\mathtt{s}}{\langle \mathtt{s} \rangle} &\leq \int_{\mathbb{R}_+^N} \left(\int_{\mathbb{R}_+^N} |L_w(\mathtt{t})||H_w(f(\mathtt{st}))| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \right) \frac{d\mathtt{s}}{\langle \mathtt{s} \rangle} \\ &\leq 2 \int_{\mathbb{R}_+^N} |L_w(\mathtt{t})| \left(\int_{\mathbb{R}_+^N} |f(\mathtt{st})| \frac{d\mathtt{s}}{\langle \mathtt{s} \rangle} \right) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\ &= 2\|L_w\|_{L^1_\mu} \|f\|_{L^1_\mu} \leq 2A\|f\|_{L^1_\mu} < +\infty. \end{aligned}$$

□

The next estimate proves that, asymptotically, the nonlinear integral operators (I) map $BV(\mathbb{R}_+^N)$ into itself.

Proposition 3.2 *If $f \in BV(\mathbb{R}_+^N)$ and $\{K_w\}_{w>0}$ satisfy $K_w.1)$ and $K_w.3)$ then there exists $\overline{w} > 0$ such that, for every $w > \overline{w}$,*

$$V[T_w f] \leq 2AV[f].$$

Proof Let $I = \prod_{i=1}^N [a_i, b_i]$ be an interval in $\mathbb{R}_+^N$ and let $\{J_1, \ldots, J_m\}$ be a partition of I, with $J_k = \prod_{j=1}^N [{}^{(k)}a_j, {}^{(k)}b_j]$, $k = 1, \ldots, m$. If $\{s_j^o = {}^{(k)}a_j, \ldots, s_j^\nu = {}^{(k)}b_j\}$ is a partition of $[{}^{(k)}a_j, {}^{(k)}b_j]$, for every $j = 1, \ldots N$, $k = 1, \ldots m$, then, for every $\mathtt{s}'_j \in I'_j$, by (1) there exists $\overline{w} > 0$ such that, for every $w \geq \overline{w}$,

$$\begin{aligned} S_j &:= \sum_{\mu=1}^{\nu} |(T_w f)(\mathtt{s}'_j, s_j^\mu) - (T_w f)(\mathtt{s}'_j, s_j^{\mu-1})| \\ &= \sum_{\mu=1}^{\nu} \left| \int_{\mathbb{R}_+^N} K_w(\mathtt{t}, \tau_\mathtt{t} f(\mathtt{s}'_j, s_j^\mu)) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} - \int_{\mathbb{R}_+^N} K_w(\mathtt{t}, \tau_\mathtt{t} f(\mathtt{s}'_j, s_j^{\mu-1})) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \right| \\ &\leq \sum_{\mu=1}^{\nu} \int_{\mathbb{R}_+^N} |L_w(\mathtt{t})||(H_w(\tau_\mathtt{t} f))(\mathtt{s}'_j, s_j^\mu) - (H_w(\tau_\mathtt{t} f))(\mathtt{s}'_j, s_j^{\mu-1})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\ &\leq 2 \int_{\mathbb{R}_+^N} |L_w(\mathtt{t})| \sum_{\mu=1}^{\nu} |\tau_\mathtt{t} f(\mathtt{s}'_j, s_j^\mu) - \tau_\mathtt{t} f(\mathtt{s}'_j, s_j^{\mu-1})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\ &\leq 2 \int_{\mathbb{R}_+^N} |L_w(\mathtt{t})| \; V_{[{}^{(k)}a_j, {}^{(k)}b_j]} \; [\tau_\mathtt{t} f(\mathtt{s}'_j, \cdot)] \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}, \end{aligned}$$

where $\tau_{\mathtt{t}} f(\mathtt{s}) := f(\mathtt{st})$, $\mathtt{s}, \mathtt{t} \in \mathbb{R}^N_+$, denotes the dilation operator. Hence, passing to the supremum over all the partitions of $[{}^{(k)}a_j, {}^{(k)}b_j]$,

$$V_{[{}^{(k)}a_j, {}^{(k)}b_j]}[(T_w f)(\mathtt{s}'_j, \cdot)] \leq 2 \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})| \; V_{[{}^{(k)}a_j, {}^{(k)}b_j]} \; [\tau_{\mathtt{t}} f(\mathtt{s}'_j, \cdot)] \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}.$$

Applying now the Fubini-Tonelli theorem, for every $j = 1, \ldots, N$,

$$\begin{aligned} \Phi_j(T_w f, J_k) &\leq 2 \int_{{}^{(k)}\mathtt{a}'_j}^{{}^{(k)}\mathtt{b}'_j} \left\{ \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})| V_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\tau_{\mathtt{t}} f(\mathtt{s}'_j, \cdot)] \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \right\} \frac{d\mathtt{s}'_j}{\langle \mathtt{s}'_j \rangle} \\ &= 2 \int_{\mathbb{R}^N_+} \left\{ \int_{{}^{(k)}\mathtt{a}'_j}^{{}^{(k)}\mathtt{b}'_j} V_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\tau_{\mathtt{t}} f(\mathtt{s}'_j, \cdot)] \frac{d\mathtt{s}'_j}{\langle \mathtt{s}'_j \rangle} \right\} |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\ &= 2 \int_{\mathbb{R}^N_+} \Phi_j \; (\tau_{\mathtt{t}} f, J_k) |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}. \end{aligned}$$

By a Minkowski-type inequality, for every $k = 1, \ldots, m$ we have that

$$\begin{aligned} \Phi(T_w f, J_k) &\leq 2 \left\{ \sum_{j=1}^{N} \left(\int_{\mathbb{R}^N_+} \Phi_j(\tau_{\mathtt{t}} f, J_k) |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \right)^2 \right\}^{\frac{1}{2}} \\ &\leq 2 \int_{\mathbb{R}^N_+} \left\{ \sum_{j=1}^{N} [\Phi_j(\tau_{\mathtt{t}} f, J_k)]^2 \right\}^{\frac{1}{2}} |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\ &= 2 \int_{\mathbb{R}^N_+} \Phi(\tau_{\mathtt{t}} f, J_k) |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}. \end{aligned}$$

Finally, summing over $k = 1, \ldots, m$ and passing to the supremum over all the partitions $\{J_1, \ldots, J_m\}$ of I,

$$V_I[T_w f] \leq 2 \int_{\mathbb{R}^N_+} V_I[\tau_{\mathtt{t}} f] |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle},$$

and hence, taking into account of the arbitrariness of $I \subset \mathbb{R}^N_+$ and $K_w.1)$,

$$V[T_w f] \leq 2 \|L_w\|_{L^1_\mu} V[f] \leq 2A \; V[f].$$

□

In order to prove the main convergence theorem we need a result about the convergence in variation of $(H_w \circ f - f)$.

Proposition 3.3 *Let us assume that* $f \in AC(\mathbb{R}^N_+)$ *and that* $K_w.3)$ *holds. Then*

$$\lim_{w \to +\infty} V[H_w \circ f - f] = 0.$$

Proof The proof is similar to the proof of Lemma 2 in [30]. □

As an immediate consequence of the previous result we have the following Proposition:

Proposition 3.4 *If $f \in AC(\mathbb{R}^N_+)$ and $K_w.3)$ holds, then $(H_w \circ f)$ are asymptotically equibounded in variation, i.e., there exists $\overline{w} > 0$ such that, for every $w \geq \overline{w}$,*

$$V[H_w \circ f] \leq 2V[f].$$

Proof As a consequence of the additivity of the classical Tonelli variation, it is easy to see that $V[f_1+f_2] \leq V[f_1]+V[f_2]$, for every $f_1, f_2 \in BV(\mathbb{R}^N_+)$. Hence $V[H_w \circ f] \leq V[H_w \circ f - f]+V[f]$ and the thesis follows since, by Proposition 3.3, there exists $\overline{w} > 0$ such that, for every $w \geq \overline{w}$, $V[H_w \circ f - f] \leq V[f]$. □

We are now ready to establish the main convergence result by means of the operators (I).

Theorem 3.5 *If $f \in AC(\mathbb{R}^N_+)$ and $\{K_w\}_{w>0} \in \mathcal{K}_w$, then*

$$\lim_{w \to +\infty} V[T_w f - f] = 0.$$

Proof With the same notations of Proposition 3.2 we can write, by $K_w.1)$,

$$\begin{aligned}
S_j \quad &:= \quad \sum_{\mu=1}^{\nu} |(T_w f)(\mathtt{s}'_j, s_j^{\mu}) - f(\mathtt{s}'_j, s_j^{\mu}) - [(T_w f)(\mathtt{s}'_j, s_j^{\mu-1}) - f(\mathtt{s}'_j, s_j^{\mu-1})| \\
&\leq \quad \sum_{\mu=1}^{\nu} \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})||H_w(f(\mathtt{s}'_j\mathtt{t}'_j, s_j^{\mu} t_j)) - H_w(f(\mathtt{s}'_j, s_j^{\mu})) - H_w(f(\mathtt{s}'_j\mathtt{t}'_j, s_j^{\mu-1} t_j)) + \\
&+ \quad H_w(f(\mathtt{s}'_j, s_j^{\mu-1}))| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} + \sum_{\mu=1}^{\nu} \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})| \Big| H_w(f(\mathtt{s}'_j, s_j^{\mu})) - f(\mathtt{s}'_j, s_j^{\mu}) + \\
&- \quad \Big[H_w(f(\mathtt{s}'_j, s_j^{\mu-1})) - f(\mathtt{s}'_j, s_j^{\mu-1}) \Big] \Big| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\
&\leq \quad \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})| V_{[{}^{(k)}a_j, {}^{(k)}b_j]} [(\tau_{\mathtt{t}}(H_w \circ f) - (H_w \circ f))(\mathtt{s}'_j, \cdot)] \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\
&+ \quad \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})| V_{[{}^{(k)}a_j, {}^{(k)}b_j]} [(H_w \circ f - f)(\mathtt{s}'_j, \cdot)] \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}.
\end{aligned}$$

By the Fubini-Tonelli theorem, for every $j = 1, \ldots, N$, there holds

$$\begin{aligned}
\Phi_j(T_w f - f, J_k) \quad &\leq \quad \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})| \Phi_j((\tau_{\mathtt{t}}(H_w \circ f) - (H_w \circ f)), J_k) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\
&+ \quad \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})| \Phi_j(H_w \circ f - f, J_k) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle},
\end{aligned}$$

and so, by a Minkowski type inequality,

$$\begin{aligned}
\Phi(T_w f - f, J_k) \quad &\leq \quad \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})| \Phi((\tau_{\mathtt{t}}(H_w \circ f) - (H_w \circ f)), J_k) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\
&+ \quad \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})| \Phi(H_w \circ f - f, J_k) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}.
\end{aligned}$$

Hence, summing over k and passing to the supremum over all the possible partitions $\{J_1, \ldots, J_m\}$ of I, and then over all the intervals $I \subset \mathbb{R}^N_+$,

$$\begin{aligned} V[T_w f - f] &\leq \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})| V[\tau_{\mathtt{t}}(H_w \circ f) - (H_w \circ f))] \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\ &+ \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})| V[H_w \circ f - f] \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}. \end{aligned}$$

Taking into account that, by the properties of variation,

$$\begin{aligned} V[\tau_{\mathtt{t}}(H_w \circ f) - (H_w \circ f))] &\leq V[\tau_{\mathtt{t}}(H_w \circ f) - \tau_{\mathtt{t}} f] + V[\tau_{\mathtt{t}} f - f] + V[H_w \circ f - f] \\ &= 2V[H_w \circ f - f] + V[\tau_{\mathtt{t}} f - f], \end{aligned}$$

we have that, for every $0 < \delta < 1$,

$$\begin{aligned} V[T_w f - f] &\leq \int_{\mathbb{R}^N_+} |L_w(\mathtt{t})| \Big(2V[H_w \circ f - f] + V[\tau_{\mathtt{t}} f - f] \Big) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} + AV[H_w \circ f - f] \\ &\leq 3AV[H_w \circ f - f] + \Big(\int_{|\mathbf{1} - \mathtt{t}| \leq \delta} + \int_{|\mathbf{1} - \mathtt{t}| > \delta} \Big) |L_w(\mathtt{t})| V[\tau_{\mathtt{t}} f - f] \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{2}$$

Let us now fix $\varepsilon > 0$. About I_1, by Proposition 3.3 there exists $\overline{w} > 0$ such that, for every $w \geq \overline{w}$, $V[H_w \circ f - f] < \frac{\varepsilon}{9A}$, and so $I_1 < \frac{\varepsilon}{3}$. About I_2, by Theorem 1 of [2], there exists $\overline{\delta} \in]0, 1[$ such that, if $|\mathbf{1} - \mathtt{t}| \leq \overline{\delta}$, $V[\tau_{\mathtt{t}} f - f] < \frac{\varepsilon}{3A}$, and so

$$I_2 < \frac{\varepsilon}{3A} \int_{|\mathbf{1} - \mathtt{t}| \leq \overline{\delta}} |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} < \frac{\varepsilon}{3}.$$

Finally, by K_w.2) there exists $\tilde{w} > 0$ such that, if $w \geq \tilde{w}$, $\int_{|\mathbf{1} - \mathtt{t}| > \overline{\delta}} |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} < \frac{\varepsilon}{6V[f]}$ (w.l.g, $V[f] \neq 0$), and so

$$\begin{aligned} I_3 &\leq \int_{|\mathbf{1} - \mathtt{t}| > \overline{\delta}} |L_w(\mathtt{t})| (V[\tau_{\mathtt{t}} f] + V[f]) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} \\ &= 2V[f] \int_{|\mathbf{1} - \mathtt{t}| > \overline{\delta}} |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} < \frac{\varepsilon}{3}. \end{aligned} \tag{3}$$

Hence we conclude that

$$V[T_w f - f] < \varepsilon,$$

for every $w \geq \max\{\overline{w}, \tilde{w}\}$. □

Remark 3.6 We point out that, in Proposition 3.3, it is actually sufficient to assume that $f \in BV(\mathbb{R}^N_+)$ and that the sections $f(\mathtt{x}'_j, \cdot)$ are continuous for almost every $\mathtt{x}'_j \in \mathbb{R}^{N-1}_+$. We prefer to assume the stronger condition that $f \in AC(\mathbb{R}^N_+)$ since this is the case for the main result (Theorem 3.5) in which such Proposition is used.

4 Order of approximation

We will now study the problem of the order of approximation for the family of integral operators (I). In order to do that, first of all we need to modify the assumptions on kernels. In particular, instead of $K_w.2)$ and $K_w.3)$, we will assume that, for $0 < \alpha \leq 1$,

$\widetilde{\mathbf{K}}_{\mathbf{w}}.\mathbf{2})$ for every $\delta \in]0,1[$

$$\int_{|\mathbf{1}-\mathtt{t}|>\delta} |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} = O(w^{-\alpha}), \; w \to +\infty,$$

(i.e., $\{L_w\}_{w>0}$ are α-*singular*);

$\widetilde{\mathbf{K}}_{\mathbf{w}}.\mathbf{3})$ denoted by $G_w(u) := H_w(u) - u$, $u \in \mathbb{R}$, $w > 0$,

$$\frac{V_J[G_w]}{m(J)} = O(w^{-\alpha}), \quad \text{as } w \to +\infty,$$

uniformly with respect to every (proper) bounded interval $J \subset \mathbb{R}$, i.e., there exist $\overline{w} > 0$ and $N > 0$ such that, for every $w \geq \overline{w}$, $\dfrac{V_J[G_w]}{m(J)} \leq N w^{-\alpha}$, for every (proper) bounded interval $J \subset \mathbb{R}$.

In the following, we will say that $\{K_w\}_{w>0} \subset \widetilde{\mathcal{K}}_w$ if $K_w.1)$, $\widetilde{K}_w.2)$ and $\widetilde{K}_w.3)$ are satisfied.

Moreover, as it is usual in such problems, we will assume that the function f belongs to a Lipschitz class which takes into account of the multidimensional variation and the multiplicative setting of $\mathbb{R}^N_+$, namely,

$$VLip^N(\alpha) := \{f \in AC(\mathbb{R}^N_+) : V[\tau_{\mathtt{t}} f - f] = O(|\log \mathtt{t}|^\alpha), \text{ as } |\mathbf{1} - \mathtt{t}| \to 0\},$$

where we put $\log \mathtt{t} := (\log t_1, \ldots, \log t_N)$, $\mathtt{t} \in \mathbb{R}^N_+$.

We first need to establish a stronger result with respect to Proposition 3.3.

Proposition 4.1 *Let assume that $f \in AC(\mathbb{R}^N_+)$ and that $\widetilde{K}_w.3)$ holds. Then*

$$\lim_{w \to +\infty} V[H_w \circ f - f] = O(w^{-\alpha}), \; w \to +\infty. \tag{4}$$

Proof Again, the proof is similar to the proof of Lemma 2 in [30], taking into account of assumption $\widetilde{K}_w.3)$. □

Theorem 4.2 *Let us assume that $f \in VLip^N(\alpha)$, $\{K_w\}_{w>0} \subset \widetilde{\mathcal{K}}_w$ and that there exists $\widetilde{\delta} \in]0,1[$ such that*

$$\int_{|\mathbf{1}-\mathtt{t}|\leq\widetilde{\delta}} |L_w(\mathtt{t})| |\log \mathtt{t}|^\alpha \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} = O(w^{-\alpha}), \tag{5}$$

as $w \to +\infty$. Then

$$V[T_w f - f] = O(w^{-\alpha}),$$

as $w \to +\infty$.

Proof By (2) and (3) of Theorem 3.5 we have that, for every $\delta \in]0,1[$ and $w > 0$,

$$V[T_w f - f] \leq 3AV[H_w \circ f - f] + \int_{|\mathbf{1}-\mathtt{t}|\leq\delta} V[\tau_{\mathtt{t}} f - f] |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}$$
$$+ 2V[f] \int_{|\mathbf{1}-\mathtt{t}|>\delta} |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}.$$

Since by assumption $f \in VLip^N(\alpha)$, there exist $M > 0$ and $\bar{\delta} \in]0,1[$ such that $V[\tau_{\mathtt{t}} f - f] \leq M|\log \mathtt{t}|^{\alpha}$, for $|\mathbf{1} - \mathtt{t}| < \bar{\delta}$. Hence, if we take $0 < \delta \leq \min\{\widetilde{\delta}, \bar{\delta}\}$, by $\widetilde{K}_w.2)$, (4) and (5), we conclude that

$$V[T_w f - f] \leq 3AV[H_w \circ f - f] + M \int_{|\mathbf{1}-\mathtt{t}|\leq\delta} |L_w(\mathtt{t})| |\log \mathtt{t}|^{\alpha} \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}$$
$$+ 2V[f] \int_{|\mathbf{1}-\mathtt{t}|>\delta} |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} = O(w^{-\alpha}),$$

for sufficiently large $w > 0$, taking into account that $V[f] < +\infty$. □

Remark 4.3 Similar considerations to Remark 3.6 hold for Proposition 4.1 and Theorem 4.2.

5 A characterization of the absolute continuity

In this Section we will prove that, in case of regular (AC) kernel functions, the converse of the main convergence result (Theorem 3.5) holds. In order to do this, we will use the following equivalent concept of absolute continuity, introduced in [1]:

Definition 5.1 *A function $f : \mathbb{R}^N_+ \to \mathbb{R}$ is said to be log-absolutely continuous on $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}^N_+$ if, for every $j = 1, 2, \ldots, N$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{\nu=1}^{n} |f(\mathtt{x}'_j, \beta^{\nu}) - f(\mathtt{x}'_j, \alpha^{\nu})| < \varepsilon,$$

for a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}_+$ and for all finite collections of non-overlapping intervals $[\alpha^{\nu}, \beta^{\nu}] \subset [a_j, b_j]$, $\nu = 1, \ldots, n$, for which $\sum_{\nu=1}^n (\log(\beta^{\nu}) - \log(\alpha^{\nu})) < \delta$.

As an immediate consequence of Proposition 3.5 of [1], we have that $f : \mathbb{R}^N_+ \to \mathbb{R}$ is log-absolutely continuous on $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}^N_+$ if and only if it is absolutely continuous on I, hence the two notions of AC-functions are equivalent.

The advantage of using the previous concept of absolute continuity, with respect to the classical one, is that the theory becomes significantly simplified. Indeed, taking into account of the Haar measure μ, the definition of the log-absolute continuity reveals to be more natural and more suitable in order to study some problems for Mellin integral operators in the setting of the multiplicative group structure of $\mathbb{R}^N_+$ equipped with the logarithmic measure μ.

In particular, using the log-absolute continuity, we are able to prove that, if the kernel functions $\{L_w\}_{w>0}$ are absolutely continuous, so are, asymptotically, the integral operators (I).

Proposition 5.2 *If $f \in BV(\mathbb{R}^N_+)$, $\{L_w\}_{w>0}$ are absolutely continuous on every interval $I \subset \mathbb{R}^N_+$ and $K_w.1)$ and $K_w.3)$ are satisfied, then $T_w f \in AC(\mathbb{R}^N_+)$, for sufficiently large $w > 0$.*

Proof Let us notice that, by a simple change of variables, we may write

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}^N_+} L_w \left(\frac{\mathtt{t}}{\mathtt{s}}\right) H_w(f(\mathtt{t})) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}. \tag{6}$$

We first prove that, for $w > 0$ large enough, $T_w f$ is log-absolutely continuous on every $I = \prod_{i=1}^N [a_i, b_i] \subset R^N_+$. Let us notice that, by the equivalence of the two concepts of absolute continuity, $\{L_w\}_{w>0}$, are log-absolutely continuous on I. Therefore, let us fix $\varepsilon > 0$ and a collection of nonoverlapping intervals in $[a_j, b_j]$, $\{[\alpha^\nu, \beta^\nu]\}_{\nu=1}^n$, such that $\sum_{\nu=1}^n (\log(\beta^\nu) - \log(\alpha^\nu)) < \delta$, where δ is the number of the log-absolute continuity of $L_w(\mathtt{x}'_j, \cdot)$ in correspondence to $\overline{\varepsilon} := \frac{\varepsilon}{2\|f\|_{L^1_\mu}}$, a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}_+$, for $j = 1, \dots, N$: here, without any loss of generality, we assume that $\|f\|_{L^1_\mu} \neq 0$, since the other case is trivial. Then, using (6), it is possible to write

$$\sum_{\nu=1}^n \quad |(T_w f)(\mathtt{x}'_j, \beta^\nu) - (T_w f)(\mathtt{x}'_j, \alpha^\nu)| \leq$$
$$\leq \int_{\mathbb{R}^N_+} |H_w(f(\mathtt{t}))| \sum_{\nu=1}^n \left| L_w \left(\frac{\mathtt{t}'_j}{\mathtt{x}'_j}, \frac{t_j}{\beta^\nu}\right) - L_w \left(\frac{\mathtt{t}'_j}{\mathtt{x}'_j}, \frac{t_j}{\alpha^\nu}\right) \right| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}.$$

Taking into account that $\sum_{\nu=1}^n \left| \log\left(\frac{t_j}{\beta^\nu}\right) - \log\left(\frac{t_j}{\alpha^\nu}\right) \right| < \delta$, by the log-absolute continuity of L_w on I,

$$\sum_{\nu=1}^n \left| L_w \left(\frac{\mathtt{t}'_j}{\mathtt{x}'_j}, \frac{t_j}{\beta^\nu}\right) - L_w \left(\frac{\mathtt{t}'_j}{\mathtt{x}'_j}, \frac{t_j}{\alpha^\nu}\right) \right| < \frac{\varepsilon}{2\|f\|_{L^1_\mu}}.$$

Now, by (1), which is implied by K_w.3), and taking into account that $H_w(0) = 0$, there exists $\overline{w} > 0$ such that, for every $w \geq \overline{w}$, $|H_w(f(\mathtt{t}))| \leq 2|f(\mathtt{t})|$, and so

$$\sum_{\nu=1}^n |(T_w f)(\mathtt{x}'_j, \beta^\nu) - (T_w f)(\mathtt{x}'_j, \alpha^\nu)| \leq \varepsilon,$$

a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}_+$: this means that, for every $w \geq \overline{w}$, $(T_w f)(\mathtt{x}'_j, \cdot)$ is log-absolutely continuous on I, and hence absolutely continuous on I. The thesis follows taking into account that, by Proposition 3.2, $T_w f \in BV(\mathbb{R}^N_+)$. □

Another step in order to get the characterization is to prove that the space of the absolutely continuous functions is a closed subspace of $BV(\mathbb{R}^N_+)$ with respect to the convergence in variation.

Proposition 5.3 *$AC(\mathbb{R}^N_+)$ is a closed subspace of $BV(\mathbb{R}^N_+)$ with respect to the topology generated by the convergence in variation.*

Proof We have to prove that, if $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions in $AC(\mathbb{R}^N_+)$ such that $\lim_{n\to+\infty} V[f_n - f] = 0$, then $f \in AC(\mathbb{R}^N_+)$. Since, by the properties of variation, $V[f] \leq V[f_n - f] + V[f_n] < +\infty$, for some $n \in \mathbb{N}$, it remains to prove that f is absolutely continuous on every $I \subset \mathbb{R}^N_+$, i.e., for every $I \subset \mathbb{R}^N_+$ and $j = 1, \dots, N$,

$f(\mathtt{x}'_j, \cdot)$ is absolutely continuous a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}_+$. Let us fix $I = \prod_{j=1}^N [a_j, b_j] \subset \mathbb{R}^N_+$ and $j = 1, \dots, N$, and notice that

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^{N-1}_+} V_{[a_j,b_j]}[(f_n - f)(\mathtt{x}'_j, \cdot)] \frac{d\mathtt{x}'_j}{\langle \mathtt{x}'_j \rangle} = \sup_{I'_j \subset \mathbb{R}^{N-1}_+} \int_{I'_j} V_{[a_j,b_j]}[(f_n - f)(\mathtt{x}'_j, \cdot)] \frac{d\mathtt{x}'_j}{\langle \mathtt{x}'_j \rangle} \\ &= \sup_{I'_j \subset \mathbb{R}^{N-1}_+} \Phi_j(f_n - f, I) \leq \sup_{I \subset \mathbb{R}^N_+} \Phi(f_n - f, I) \\ &\leq \sup_{I \subset \mathbb{R}^N_+} \sup_{\mathcal{D}} \sum_{k=1}^p \Phi(f_n - f, J_k) = V[f_n - f], \end{aligned}$$

where $\mathcal{D}$ represents the set of all the possible partitions $\{J_1, \dots, J_p\}$ of the interval I. Hence we have that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^{N-1}_+} V_{[a_j,b_j]}[(f_n - f)(\mathtt{x}'_j, \cdot)] \frac{d\mathtt{x}'_j}{\langle \mathtt{x}'_j \rangle} = 0, \tag{7}$$

which implies that, for every $\varepsilon > 0$ there exists $\overline{n} \in \mathbb{N}$ such that, for every $n \geq \overline{n}$,

$$V_{[a_j,b_j]}[(f_n - f)(\mathtt{x}'_j, \cdot)] < \varepsilon, \tag{8}$$

a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}_+$. Indeed if, by contradiction, there exists a set of positive measure $A'_j \subset \mathbb{R}^{N-1}_+$ and $\varepsilon > 0$ such that, for every $\overline{n} \in \mathbb{N}$, there exists $n \geq \overline{n}$ for which $V_{[a_j,b_j]}[(f_n - f)(\mathtt{x}'_j, \cdot)] > \varepsilon$ for every $\mathtt{x}'_j \in A'_j$, then

$$\int_{\mathbb{R}^{N-1}_+} V_{[a_j,b_j]}[(f_n - f)(\mathtt{x}'_j, \cdot)] \frac{d\mathtt{x}'_j}{\langle \mathtt{x}'_j \rangle} \geq \int_{A'_j} V_{[a_j,b_j]}[(f_n - f)(\mathtt{x}'_j, \cdot)] \frac{d\mathtt{x}'_j}{\langle \mathtt{x}'_j \rangle} > \varepsilon \mu(A'_j),$$

which is in contradiction with (7).

Now, let us fix $\varepsilon > 0$: by (8) there exists $n \in \mathbb{N}$ such that $V_{[a_j,b_j]}[(f_n - f)(\mathtt{x}'_j, \cdot)] < \frac{\varepsilon}{2}$, a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}_+$. Moreover, since $f_n \in AC(\mathbb{R}^N_+)$, there exists $\delta > 0$ such that, if $\{[\alpha^\nu, \beta^\nu]\}_{\nu=1}^p$ is a family of nonoverlapping intervals in $[a_j, b_j]$ such that $\sum_{\nu=1}^p (\beta^\nu - \alpha^\nu) < \delta$, then $\sum_{\nu=1}^p |f_n(\mathtt{x}'_j, \beta^\nu) - f_n(\mathtt{x}'_j, \alpha^\nu)| < \frac{\varepsilon}{2}$, a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}_+$. Therefore

$$\begin{aligned} \sum_{\nu=1}^p |f(\mathtt{x}'_j, \beta^\nu) - f(\mathtt{x}'_j, \alpha^\nu)| &\leq \sum_{\nu=1}^p |(f - f_n)(\mathtt{x}'_j, \beta^\nu) - (f - f_n)(\mathtt{x}'_j, \alpha^\nu)| + \\ &+ \sum_{\nu=1}^p |f_n(\mathtt{x}'_j, \beta^\nu) - f_n(\mathtt{x}'_j, \alpha^\nu)| \\ &\leq V_{[a_j,b_j]}[(f_n - f)(\mathtt{x}'_j, \cdot)] + \sum_{\nu=1}^p |f_n(\mathtt{x}'_j, \beta^\nu) - f_n(\mathtt{x}'_j, \alpha^\nu)| < \varepsilon, \end{aligned}$$

a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}_+$, that is, $f(\mathtt{x}'_j, \cdot)$ is absolutely continuous on $[a_j, b_j]$, a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}_+$.□

We are now ready to prove the equivalence between the absolute continuity and the convergence in variation by means of the operators (I), for AC-kernels.

Theorem 5.4 *Let $f \in BV(\mathbb{R}^N_+)$ and let $\{K_w\}_{w>0} \in \mathcal{K}_w$ be such that $\{L_w\}_{w>0}$ are absolutely continuous on every interval $I \subset \mathbb{R}^N_+$. Then $f \in AC(\mathbb{R}^N_+)$ if and only if*

$$\lim_{w \to +\infty} V[T_w f - f] = 0.$$

Proof Taking into account of Theorem 3.5, we have just to prove the sufficient part. Now, by Proposition 5.2, $T_w f \in AC(\mathbb{R}^N_+)$, for sufficiently large $w > 0$, and therefore, if $\lim_{w\to+\infty} V[T_w f - f] = 0$, f turns out to be absolutely continuous, by Proposition 5.3. □

The previous result shows that, also in the case of nonlinear Mellin-type integral operators, the situation is the same as, for example, in the case of the classical convolution operators, where it is possible to get the equivalence between convergence in variation and absolute continuity by directly using the usual notion of AC-functions (see, e.g., [30]).

Remark 5.5 We point out that, assuming as it is usual in the nonlinear setting (see, e.g., [13, 30, 42]) a strong Lipschitz condition on $\{H_w\}_{w>0}$, then Proposition 5.2 and all the estimates of Section 3 hold for every $w > 0$, with suitable constants, without assuming $K_w.3$).

6 Examples

We point out that is not difficult to find examples of kernel functions which fulfill all the assumptions of the previous theory.

For example, let us consider the kernel functions $H_w(u)$ defined as

$$H_w(u) = \begin{cases} u + e^{\frac{u}{w}} - 1, & 0 \le u < 1, \\ u + e^{\frac{1}{wu}} - 1, & u \ge 1, \end{cases}$$

(we extend the definition of $H_w(u)$ in odd-way for $u < 0$). Then

$$G_w(u) = \begin{cases} e^{\frac{u}{w}} - 1, & 0 \le u < 1, \\ e^{\frac{1}{wu}} - 1, & u \ge 1, \end{cases}$$

and obviously $G_w(u)$ is increasing in $[0, 1]$, decreasing in $[1, +\infty)$. Hence, for every $[a, b] \subset [0, 1]$, taking into account that the exponential function is Lipschitzian with Lipschitz constant e on $[0, 1]$,

$$\frac{V_{[a,b]}[G_w]}{m([a, b])} = \frac{e^{\frac{b}{w}} - e^{\frac{a}{w}}}{b - a} \le \frac{e}{w} \to 0$$

as $w \to +\infty$. Moreover, since $e^{\frac{1}{u}}$ is also Lipschitzian with Lipschitz constant e on $[1, +\infty)$, for every $[a, b] \subset [1, +\infty)$,

$$\frac{V_{[a,b]}[G_w]}{m([a, b])} = \frac{e^{\frac{1}{aw}} - e^{\frac{1}{bw}}}{b - a} \le \frac{e}{wab} \le \frac{e}{w} \to 0$$

as $w \to +\infty$. If $[a, b]$ is such that $0 \le a < 1 < b$, it is sufficient to notice that $V_{[a,b]}[G_w] = V_{[a,1]}[G_w] + V_{[1,b]}[G_w]$. This implies that $K_w.3$) holds. Obviously, such kernels satisfy also assumption $\widetilde{K}_w.3$) with $\alpha = 1$.

About $\{L_w\}_{w>0}$, surely one can consider an approximate identity, so that $K_w.1$) and $K_w.2$) are satisfied. Besides this example, there is also another important class of kernels $\{L_w\}_{w>0}$ for which all the assumptions for the rate of approximation are satisfied, i.e., the Fejér-type kernels with finite absolute moments of order α ($0 < \alpha \leq 1$). Such kernels are of the form

$$L_w(\mathtt{t}) = w^N L(\mathtt{t}^w), \quad \mathtt{t} \in \mathbb{R}^N_+, \; w > 0, \tag{9}$$

where $L \in L^1_\mu(\mathbb{R}^N_+)$ is such that $\int_{\mathbb{R}^N_+} L(\mathtt{t}) \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} = 1$ and $\mathtt{t}^w := (t_1^w, \ldots, t_N^w)$.

This condition is the natural reformulation (see also [31]), in the multiplicative setting of $\mathbb{R}^N_+$ with the Haar measure, of the classical Fejér-type kernels on $\mathbb{R}^N$.

The case of Fejér-type kernels is important since, in the classical frame of $\mathbb{R}^N$ equipped with the Lebesgue measure, all the assumptions for the rate of approximation are implied by the finiteness of the absolute moments of order α ($0 < \alpha \leq 1$).
Also in the present frame it is easy to see that assumptions $K_w.1)$ and $\widetilde{K}_w.2)$ are satisfied if $m(L, \alpha) < +\infty$, where the absolute moments of order α are defined by

$$m(L, \alpha) := \int_{\mathbb{R}^N_+} |\log \mathtt{t}|^\alpha |L(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}.$$

Indeed the following Proposition ([2]) holds:

Proposition 6.1 *If $\{L_w\}_{w>0}$ are of the form* (9) *and $m(L, \alpha) < +\infty$, $0 < \alpha \leq 1$, then*

(a) $\displaystyle\int_{|\mathbf{1}-\mathtt{t}|>\delta} |L_w(\mathtt{t})| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} = O(w^{-\alpha})$, *as $w \to +\infty$, for every $\delta \in]0, 1[$;*

(b) $\displaystyle\int_{|\mathbf{1}-\mathtt{t}|\leq\delta} |L_w(\mathtt{t})| |\log \mathtt{t}|^\alpha \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle} = O(w^{-\alpha})$, *as $w \to +\infty$, for every $\delta \in]0, 1[$.*

We finally point out that there are several classes of Fejér-type kernels for which the absolute moments are finite. Among them, the Mellin Gauss-Weierstrass kernels (see [2] and, e.g., [45, 13] for their classical version), defined as

$$G_w(\mathtt{t}) := \frac{w^N}{\pi^{\frac{N}{2}}} e^{-w^2 |\log \mathtt{t}|^2}, \quad \mathtt{t} \in \mathbb{R}^N_+, \; w > 0$$

(see Fig. 1(a)), the Mellin Picard kernels, defined as

$$P_w(\mathtt{t}) := \frac{w^N}{2\pi^{\frac{N}{2}}} \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} e^{-w|\log \mathtt{t}|}, \quad \mathtt{t} \in \mathbb{R}^N_+, \; w > 0,$$

(see Fig. 1(b)) where Γ is the Euler function (see [45, 13] for their classical version), and others.

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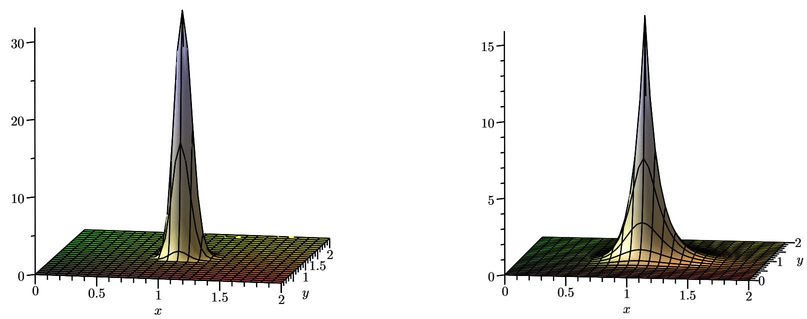

(a) Mellin Gauss-Weierstrass kernels $G_{10}(x, y)$ (b) Mellin Picard kernels $P_{10}(x, y)$

Figure 1: Example of kernels on R_+^2

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