ON THE WAVE EQUATION WITH HYPERBOLIC DYNAMICAL BOUNDARY CONDITIONS, INTERIOR AND BOUNDARY DAMPING AND SUPERCRITICAL SOURCES

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Abstract. The aim of this paper is to study the problem

$$\begin{cases} u_{tt} - \Delta u + P(x, u_t) = f(x, u) & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } (0, \infty) \times \Gamma_0, \\ u_{tt} + \partial_\nu u - \Delta_\Gamma u + Q(x, u_t) = g(x, u) & \text{on } (0, \infty) \times \Gamma_1, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with C^1 boundary ($N \geq 2$), $\Gamma = \partial\Omega$, Γ_1 is relatively open on Γ, Δ_Γ denotes the Laplace–Beltrami operator on Γ, ν is the outward normal to Ω, and the terms P and Q represent nonlinear damping terms, while f and g are nonlinear perturbations.

In the paper we establish local and global existence, uniqueness and Hadamard well–posedness results when source terms can be supercritical or super-supercritical.

1. Introduction and main results

1.1. Presentation of the problem and literature overview. We deal with the evolution problem consisting of the wave equation posed in a bounded regular open subset of $\mathbb{R}^N$, supplied with a second order dynamical boundary condition of hyperbolic type, in presence of interior and/or boundary damping terms and sources. More precisely we consider the initial –and–boundary value problem

$$\begin{cases} u_{tt} - \Delta u + P(x, u_t) = f(x, u) & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } (0, \infty) \times \Gamma_0, \\ u_{tt} + \partial_\nu u - \Delta_\Gamma u + Q(x, u_t) = g(x, u) & \text{on } (0, \infty) \times \Gamma_1, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{in } \overline{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$) with C^1 boundary (see [31]). We denote $\Gamma = \partial\Omega$ and we assume $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, Γ_1 being relatively open on Γ (or equivalently $\overline{\Gamma_0} = \Gamma_0$). Moreover, denoting by σ the standard Lebesgue hypersurface measure on Γ, we assume that $\sigma(\overline{\Gamma}_0 \cap \overline{\Gamma}_1) = 0$. These properties of Ω, Γ_0 and Γ_1 will be assumed, without further comments, throughout the paper. Moreover $u = u(t, x)$, $t \geq 0$, $x \in \Omega$, $\Delta = \Delta_x$ denotes the Laplace operator with

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respect to the space variable, while Δ_Γ denotes the Laplace–Beltrami operator on Γ and ν is the outward normal to Ω.

The terms P and Q represent nonlinear damping terms, i.e. $P(x,v)v \geq 0$, $Q(x,v)v \geq 0$, the cases $P \equiv 0$ and $Q \equiv 0$ being specifically allowed, while f and g represent nonlinear source, or sink, terms. The specific assumptions on them will be introduced later on.

Problems with kinetic boundary conditions, that is boundary conditions involving u_{tt} on Γ, or on a part of it, naturally arise in several physical applications. A one dimensional model was studied by several authors to describe transversal small oscillations of an elastic rod with a tip mass on one endpoint, while the other one is pinched. See [3, 17, 18, 32, 38, 37, 41] and also [40] were a piezoelectric stack actuator is modeled.

A two dimensional model introduced in [28] deals with a vibrating membrane of surface density μ, subject to a tension T, both taken constant and normalized here for simplicity. If $u(t,x)$, $x \in \Omega \subset \mathbb{R}^2$ denotes the vertical displacement from the rest state, then (after a standard linear approximation) u satisfies the wave equation $u_{tt} - \Delta u = 0$, $(t,x) \in \mathbb{R} \times \Omega$. Now suppose that a part Γ_0 of the boundary is pinched, while the other part Γ_1 carries a constant linear mass density $m > 0$ and it is subject to a linear tension τ. A practical example of this situation is given by a drumhead with a hole in the interior having a thick border, as common in bass drums. One linearly approximates the force exerted by the membrane on the boundary with $-\partial_\nu u$. The boundary condition thus reads as $mu_{tt} + \partial_\nu u - \tau\Delta_{\Gamma_1} u = 0$. In the quoted paper the case $\Gamma_0 = \emptyset$ and $\tau = 0$ was studied, while here we consider the more realistic case $\Gamma_0 \neq \emptyset$ and $\tau > 0$, with τ and m normalized for simplicity. We would like to mention that this model belongs to a more general class of models of Lagrangian type involving boundary energies, as introduced for example in [23].

A three dimensional model involving kinetic dynamical boundary conditions comes out from [26], where a gas undergoing small irrotational perturbations from rest in a domain $\Omega \subset \mathbb{R}^3$ is considered. Normalizing the constant speed of propagation, the velocity potential ϕ of the gas (i.e. $-\nabla\phi$ is the particle velocity) satisfies the wave equation $\phi_{tt} - \Delta\phi = 0$ in $\mathbb{R} \times \Omega$. Each point $x \in \partial\Omega$ is assumed to react to the excess pressure of the acoustic wave like a resistive harmonic oscillator or spring, that is the boundary is assumed to be locally reacting (see [42, pp. 259–264]). The normal displacement δ of the boundary into the domain then satisfies $m\delta_{tt} + d\delta_t + k\delta + \rho\phi_t = 0$, where $\rho > 0$ is the fluid density and $m, d, k \in C(\partial\Omega)$, $m, k > 0$, $d \geq 0$. When the boundary is nonporous one has $\delta_t = \partial_\nu\phi$ on $\mathbb{R} \times \partial\Omega$, so the boundary condition reads as $m\delta_{tt} + d\partial_\nu\phi + k\delta + \rho\phi_t = 0$. In the particular case $m = k$ and $d = \rho$ (see [26, Theorem 2]) one proves that $\phi_{|\Gamma} = \delta$, so the boundary condition reads as $m\phi_{tt} + d\partial_\nu\phi + k\phi + \rho\phi_t = 0$, on $\mathbb{R} \times \partial\Omega$. Now, if one consider the case in which the boundary is not locally reacting, as in [9], one adds a Laplace–Beltrami term so getting a dynamical boundary condition like in (1.1).

Several papers in the literature deal with the wave equation with kinetic boundary conditions. This fact is even more evident if one takes into account that, plugging the equation in (1.1) into the boundary condition, we can rewrite it as $\Delta u + \partial_\nu u - \Delta_\Gamma u + Q(x,u_t) + P(x,u_t) = f(x,u) + g(x,u)$. Such a condition is usually called a *generalized Wentzell boundary condition*, at least when nonlinear perturbations are not present. We refer to [43], where abstract semigroup techniques are applied to dissipative wave equations, and to [21, 22, 51, 56, 57, 59].

Here we shall consider this type of kinetic boundary condition in connection with nonlinear boundary damping and source terms. These terms have been considered by several authors, but mainly in connection with first order dynamical boundary conditions. See [4, 5, 10, 11, 12, 14, 15, 16, 24, 35, 53, 60]. The competition between interior damping and source terms is methodologically related to the competition between boundary damping and source and it possesses a large literature as well. See [6, 27, 36, 44, 45, 46, 52].

A linear problem strongly related to (1.1) has also been recently studied in [25, 30], and another one in the recent paper [58], dealing with holography, a main theme in theoretical high energy physics and quantum gravity. See also [29].

Problem (1.1) has been studied by the author in the recent paper [55] (see also [54]) when source/sink terms are subcritical, that is when the Nemitskii operators associated to f and g [1] are locally Lischitz, respectively from $H^1(\Omega)$ to $L^2(\Omega)$ and from $H^1(\Gamma)$ to $L^2(\Gamma)$. In particular, using nonlinear semigroup theory, local Hadamard well–posedness, and hence also local existence and uniqueness, has been established in the natural energy space related to the problem. Moreover global existence and well–posedness have been proved when source terms are either sublinear or dominated by corresponding damping terms.

The aim of the present paper is to extend the results of [55] to possibly non–subcritical perturbation terms f and g. As a consequence we do not expect to get new results when all perturbation terms are subcritical, as in the case $N = 2$, but we want to cover the case when one term is subcritical while the other one is non–subcritical. As a byproduct the results in [55] will be a subcase of the more general analysis presented in the sequel.

Our main motivation is constituted by the three dimensional case, in which only the term f can be non–subcritical. Hence our choice to consider also non–subcritical boundary terms g is only of mathematical interest, but it is motivated as follows. At first in dimensions higher than 3 both terms can be non–subcritical. At second this extension is costless, since *all estimates* used in the paper will be explicitly proved only for f (in relation with P) and then trivially transposed to g. Finally to consider terms f and g under the same setting allows to give results in which f and g play a symmetric role, as they do in dimension $N = 4$.

1.2. **A simplified problem, source classification and main assumptions.** To best illustrate our results we shall consider, in this section, the following simplified version of problem (1.1)

$$\begin{cases} u_{tt} - \Delta u + \alpha(x) P_0(u_t) = f_0(u) & \text{in } (0,\infty)\times\Omega, \\ u = 0 & \text{on } (0,\infty)\times\Gamma_0, \\ u_{tt} + \partial_\nu u - \Delta_\Gamma u + \beta(x) Q_0(u_t) = g_0(u) & \text{on } (0,\infty)\times\Gamma_1, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & \text{in } \overline{\Omega}, \end{cases} \tag{1.2}$$

where $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma_1)$, $\alpha, \beta \geq 0$, and the following assumptions hold:

[1] defined on Γ_1 and trivially extended to Γ_0.

(I) P_0 and Q_0 are continuous and monotone increasing in $\mathbb{R}$, $P_0(0)=Q_0(0)=0$, and there are $m,\mu>1$ such that

$$0<\varliminf_{|v|\to\infty}\frac{|P_0(v)|}{|v|^{m-1}}\le\varlimsup_{|v|\to\infty}\frac{|P_0(v)|}{|v|^{m-1}}<\infty,\qquad \varliminf_{|v|\to 0}\frac{|P_0(v)|}{|v|^{m-1}}>0,$$
$$0<\varliminf_{|v|\to\infty}\frac{|Q_0(v)|}{|v|^{\mu-1}}\le\varlimsup_{|v|\to\infty}\frac{|Q_0(v)|}{|v|^{\mu-1}}<\infty,\qquad \varliminf_{|v|\to 0}\frac{|Q_0(v)|}{|v|^{\mu-1}}>0;$$

(II) $f_0,g_0\in C^{0,1}_{\text{loc}}(\mathbb{R})$ and there are exponents $p,q\ge 2$ such that

$$|f_0'(u)|=O(|u|^{p-2})\qquad\text{and}\qquad |g_0'(u)|=O(|u|^{q-2})\quad\text{as } |u|\to\infty.$$

Our model nonlinearities, trivially satisfying assumptions (I–II), are given by

$$\begin{cases} P_0(v)=P_1(v):=a|v|^{\widetilde{m}-2}v+|v|^{m-2}v, & 1<\widetilde{m}\le m, \quad a\ge 0,\\ Q_0(v)=Q_1(v):=b|v|^{\widetilde{\mu}-2}v+|v|^{\mu-2}v, & 1<\widetilde{\mu}\le \mu, \quad b\ge 0,\\ f_0(u)=f_1(u):=\widetilde{\gamma}|u|^{\widetilde{p}-2}u+\gamma|u|^{p-2}u+c_1, & 2\le\widetilde{p}\le p, \quad \widetilde{\gamma},\gamma,c_1\in\mathbb{R},\\ g_0(u)=g_1(u):=\widetilde{\delta}|u|^{\widetilde{q}-2}u+\delta|u|^{q-2}u+c_2, & 2\le\widetilde{q}\le q, \quad \widetilde{\delta},\delta,c_2\in\mathbb{R}.\end{cases}\tag{1.3}$$

We introduce the critical exponents r_{Ω} and r_{Γ} of the Sobolev embeddings of $H^1(\Omega)$ and $H^1(\Gamma)$ into the corresponding Lebesgue spaces, that is

$$r_{\Omega}=\begin{cases}\dfrac{2N}{N-2} & \text{if } N\ge 3,\\ \infty & \text{if } N=2,\end{cases}\qquad r_{\Gamma}=\begin{cases}\dfrac{2(N-1)}{N-3} & \text{if } N\ge 4,\\ \infty & \text{if } N=2,3.\end{cases}$$

The term f_0, or f_1, is usually classified as follows in the literature (see [11, 12]):

(i) f_0 is *subcritical* if $2\le p\le 1+r_{\Omega}/2$, when the Nemitskii operator $\widehat{f_0}$ associated to f_0 is locally Lipschitz from $H^1(\Omega)$ into $L^2(\Omega)$;
(ii) f_0 is *supercritical* if $1+r_{\Omega}/2<p\le r_{\Omega}$, when $\widehat{f_0}$ is no longer locally Lipschitz from $H^1(\Omega)$ into $L^2(\Omega)$ but it still possesses a potential energy in $H^1(\Omega)$;
(iii) f_0 is *super–supercritical* if $p>r_{\Omega}$, when $\widehat{f_0}$ has no potentials in $H^1(\Omega)$.

The analogous classification is made for g_0, depending on q and r_{Γ}.

In [55] the case when (I–II) hold and f_0 and g_0 are both subcritical was studied, while here we are concerned with possibly supercritical and super–supercritical terms. To deal with them we need two further main assumptions:

(III) $\operatorname{essinf}_{\Omega}\alpha>0$ when $p>1+r_{\Omega}/2$, $\operatorname{essinf}_{\Gamma_1}\beta>0$ when $q>1+r_{\Gamma}/2$, and

$$2\le p\le 1+r_{\Omega}/\overline{m}',\quad 2\le q\le 1+r_{\Gamma}/\overline{\mu}',\tag{1.4}$$

where $\overline{m}=\max\{2,m\}$ and $\overline{\mu}=\max\{2,\mu\}$;

(IV) if $1+r_{\Omega}/2<p=1+r_{\Omega}/m'$ then $N\le 4$, $f_0\in C^2(\mathbb{R})$ and $\varlimsup_{|u|\to\infty}\frac{|f_0''(u)|}{|u|^{p-3}}<\infty$;
if $1+r_{\Gamma}/2<q=1+r_{\Gamma}/\mu'$ then $N\le 5$, $g_0\in C^2(\mathbb{R})$ and $\varlimsup_{|u|\to\infty}\frac{|g_0''(u)|}{|u|^{q-3}}<\infty$.

Remark 1.1. Clearly assumption (III) can be equally refereed to our model nonlinearities in (1.3), while they satisfy (IV) if

$N\le 4$ and $\widetilde{p}>3$ or $\widetilde{\gamma}=0$ provided $1+r_{\Omega}/2<p=1+r_{\Omega}/m'$;
$N\le 5$ and $\widetilde{q}>3$ or $\widetilde{\delta}=0$ provided $1+r_{\Gamma}/2<q=1+r_{\Gamma}/\mu'$.

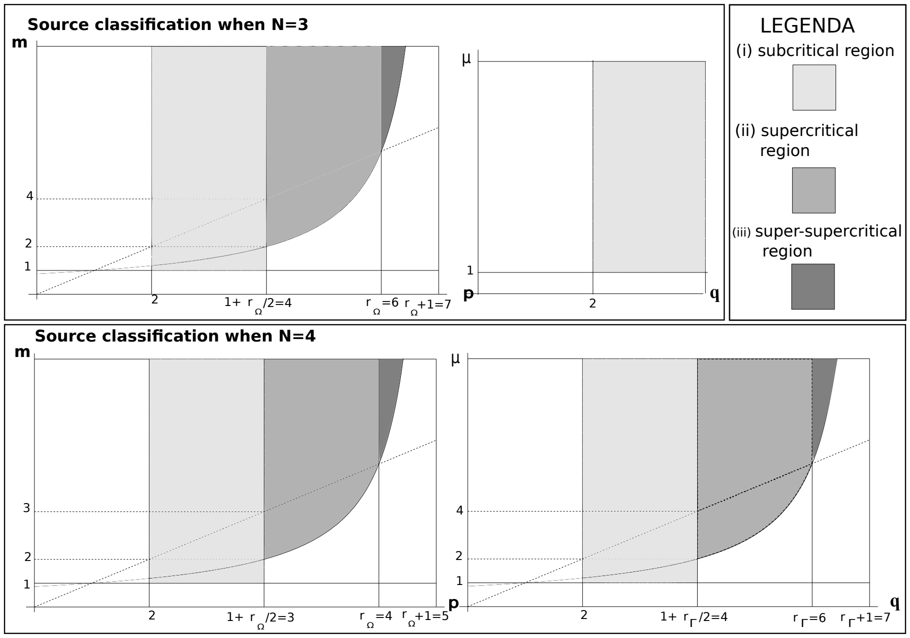

FIGURE 1. The regions covered by (1.4) in dimensions $N = 3, 4$.

Remark 1.2. The sets in the planes (p, m) and (q, μ), for which (1.4) holds, corresponding to the classification above, are illustrated in dimensions $N = 3, 4$ in Figure 1. The subcritical regions were studied in [55], while this paper covers all regions in the picture. In dimensions $N = 3$ and $N = 4$, the first one being of physical relevance, (IV) is a mild strengthening of (II). When $N = 5$ assumption (IV) excludes parameter couples for which $1 + r_\Omega/2 < p = 1 + r_\Omega/m'$, while it is a strengthening of (II) for the term g_0. When $N \geq 6$ assumption (IV) simply excludes parameter couples for which $1 + r_\Omega/2 < p = 1 + r_\Omega/m'$ and $1 + r_\Gamma/2 < q = 1 + r_\Gamma/\mu'$.

To present our results it is useful to subclassify supercritical terms f_0, by distinguishing between *intercritical* ones, when $1 + r_\Omega/2 < p < r_\Omega$, and *Sobolev critical* ones when $p = r_\Omega$. Moreover terms f_0 for which $p < 1 + r_\Omega/m'$ when $m > 2$ will be treated in a different way from those for which $p = 1 + r_\Omega/m'$ when $m > 2$, according to the compactness or non–compactness of $\widehat{f_0}$ from $H^1(\Omega)$ to $L^{m'(p-1)}(\Omega)$. In the latter case we shall say that f_0 *is on the hyperbola*. We shall also use the term *bicritical* for Sobolev critical terms on the hyperbola, when $p = m = r_\Omega$. The same terminology will be adopted for g_0.

1.3. **Add–on assumptions.** We now introduce some further assumptions which will be needed only in some results. In uniqueness and well–posedness results assumption (IV) will be strengthened to the following one:

(IV)$'$ if $p > 1 + r_\Omega/2$ then $N \leq 4$, $f_0 \in C^2(\mathbb{R})$ and $\displaystyle\varlimsup_{|u|\to\infty} \frac{|f_0''(u)|}{|u|^{p-3}} < \infty$;

if $q > 1 + r_\Gamma/2$ then $N \leq 5$, $g_0 \in C^2(\mathbb{R})$ and $\displaystyle\varlimsup_{|u|\to\infty} \frac{|g_0''(u)|}{|u|^{q-3}} < \infty$.

Remark 1.3. The model nonlinearities in (1.3) satisfy assumption (IV)$'$ if

$N \le 4$ and $\widetilde{p} > 3$ or $\widetilde{\gamma} = 0$ provided $p > 1 + r_\Omega/2$;
$N \le 5$ and $\widetilde{q} > 3$ or $\widetilde{\delta} = 0$ provided $q > 1 + r_\Gamma/2$.

Remark 1.4. In dimensions $N = 3, 4$ assumption (IV)$'$ is a mild strengthening of (II). When $N = 5$ it says that f_0 is subcritical while it is a strengthening of (II) for g_0. When $N \ge 6$ it simply says that f_0 and g_0 are both subcritical.

When dealing with well–posedness, we shall restrict to non–bicritical terms f_0, g_0 and we shall use one between the assumptions:

(V) if $p \ge r_\Omega$ then $\operatorname*{essliminf}_{|v|\to\infty} \frac{P_0'(v)}{|v|^{m-2}} > 0$, if $q \ge r_\Gamma$ then $\operatorname*{essliminf}_{|v|\to\infty} \frac{Q_0'(v)}{|v|^{\mu-2}} > 0$;

(V)$'$ if $m > r_\Omega$ then $\operatorname*{essliminf}_{|v|\to\infty} \frac{P_0'(v)}{|v|^{m-2}} > 0$, if $\mu > r_\Gamma$ then $\operatorname*{essliminf}_{|v|\to\infty} \frac{Q_0'(v)}{|v|^{\mu-2}} > 0$.

Remark 1.5. By (1.4), we have $m > r_\Omega$ when $p > r_\Omega$ and $\mu > r_\Omega$ when $q > r_\Omega$, the same implications being true for weak inequalities, hence when sources are non–bicritical we have $m > r_\Omega$ when $p \ge r_\Omega$ and $\mu > r_\Omega$ when $q \ge r_\Omega$. Hence assumption (V)$'$ is stronger than (V).

Remark 1.6. The model nonlinearities in (1.3) trivially satisfy both (V) and (V)$'$.

When looking for global solutions, as shown in [55], one has to restrict to perturbation terms which source part has at most linear growth at infinity or, roughly, it is dominated by the corresponding damping term. Hence, denoting

$$\mathfrak{F}_0(u) = \int_0^u f_0(s)\,ds, \qquad \mathfrak{G}_0(u) = \int_0^u g_0(s)\,ds \qquad \text{for all } u \in \mathbb{R}, \tag{1.5}$$

we shall use the following specific global existence assumption:

(VI) there are p_1 and q_1 satisfying

$$2 \le p_1 \le \min\{p, \max\{2, m\}\} \quad \text{and} \quad 2 \le q_1 \le \min\{q, \max\{2, \mu\}\} \tag{1.6}$$

and such that

$$\varlimsup_{|u|\to\infty} \mathfrak{F}_0(u)/|u|^{p_1} < \infty \quad \text{and} \quad \varlimsup_{|u|\to\infty} \mathfrak{G}_0(u)/|u|^{q_1} < \infty. \tag{1.7}$$

Moreover

$$\operatorname{essinf}_\Omega \alpha > 0 \text{ if } p_1 > 2 \text{ and } \operatorname{essinf}_{\Gamma_1} \beta > 0 \text{ if } q_1 > 2. \tag{1.8}$$

Since $\mathfrak{F}_0(u) = \int_0^1 f_0(su)u\,ds$ (similarly for $\mathfrak{G}_0$), (VI) is a weak version [2] of the following assumption, which is adequate for most purposes and it is easier to verify:

(VI)$'$ there are p_1 and q_1 such that (1.6) holds with (1.8) and

$$\varlimsup_{|u|\to\infty} f_0(u)u/|u|^{p_1} < \infty \qquad \text{and} \qquad \varlimsup_{|u|\to\infty} g_0(u)u/|u|^{q_1} < \infty. \tag{1.9}$$

Remark 1.7. Assumptions (II) and (VI)$'$ are satified by f_0 when it belongs to one among the following classes:

[2] Actually (VI)$'$ is more general than (VI). Indeed, when $f_0(u) = (m+1)|u|^{m-1}u\cos|u|^{m+1}$ and $g_0(u) = (\mu+1)|u|^{\mu-1}u\cos|u|^{\mu+1}$, (1.9) holds only for $p_1 \ge m+1$, $q_1 \ge \mu+1$, while (1.7) does with $p_1 = q_1 = 2$.

(0) f_0 is constant;
(1) f_0 satisfies (II) with $p \le \max\{2,m\}$ and $\operatorname{essinf}_\Omega \alpha > 0$ if $p>2$;
(2) f_0 satisfies (II) and $f_0(u)u \le 0$ in $\mathbb{R}$.

The same remark applies to g_0, *mutatis mutandis*. More generally (II) and (VI)$'$ hold when

$$f_0 = f_0^0 + f_0^1 + f_0^2, \qquad g_0 = g_0^0 + g_0^1 + g_0^2, \tag{1.10}$$

and f_0^i and g_0^i belong to the class (i) for $i = 0,1,2$. [3]

Remark 1.8. One easily checks that f_1 in (1.3) satisfies (II) and (VI) if and only if one among the following cases (the analogous ones applying to g_1) occurs:

(i) $\gamma > 0$, $p \le \max\{2,m\}$ and $\operatorname{essinf}_\Omega \alpha > 0$ if $p > 2$;
(ii) $\gamma \le 0$, $\widetilde{\gamma} > 0$, $\widetilde{p} \le \max\{2,m\}$ and $\operatorname{essinf}_\Omega \alpha > 0$ if $\widetilde{p} > 2$;
(iii) $\gamma, \widetilde{\gamma} \le 0$.

1.4. **Function spaces and auxiliary exponents.** We shall identify $L^2(\Gamma_1)$ with its isometric image in $L^2(\Gamma)$, that is

$$L^2(\Gamma_1) = \{u \in L^2(\Gamma) : u = 0 \text{ a.e. on } \Gamma_0\}. \tag{1.11}$$

We set, for $\rho \in [1,\infty)$, the Banach spaces

$$L^{2,\rho}_\alpha(\Omega) = \{u \in L^2(\Omega) : \alpha^{1/\rho} u \in L^\rho(\Omega)\}, \qquad \|\cdot\|_{2,\rho,\alpha} = \|\cdot\|_2 + \|\alpha^{1/\rho}\cdot\|_\rho,$$
$$L^{2,\rho}_\beta(\Gamma_1) = \{u \in L^2(\Gamma_1) : \beta^{1/\rho} u \in L^\rho(\Gamma_1)\}, \quad \|\cdot\|_{2,\rho,\beta,\Gamma_1} = \|\cdot\|_{2,\Gamma_1} + \|\beta^{1/\rho}\cdot\|_{\rho,\Gamma_1},$$

where $\|\cdot\|_\rho := \|\cdot\|_{L^\rho(\Omega)}$ and $\|\cdot\|_{\rho,\Gamma_1} := \|\cdot\|_{L^\rho(\Gamma_1)}$. We denote by $u_{|\Gamma}$ the trace on Γ of $u \in H^1(\Omega)$. We introduce the Hilbert spaces $H^0 = L^2(\Omega) \times L^2(\Gamma_1)$ and

$$H^1 = \{(u,v) \in H^1(\Omega) \times H^1(\Gamma) : v = u_{|\Gamma}, v = 0 \text{ on } \Gamma_0\}, \tag{1.12}$$

with the norms inherited from the products. For the sake of simplicity we shall identify, when useful, H^1 with its isomorphic counterpart $\{u \in H^1(\Omega) : u_{|\Gamma} \in H^1(\Gamma) \cap L^2(\Gamma_1)\}$, through the identification $(u, u_{|\Gamma}) \mapsto u$, so we shall write, without further mention, $u \in H^1$ for functions defined on Ω. Moreover we shall drop the notation $u_{|\Gamma}$, when useful, so we shall write $\|u\|_{2,\Gamma}$, $\int_\Gamma u$, and so on, for $u \in H^1$. We also introduce, for $\rho, \theta \in [1,\infty)$, the reflexive spaces [4]

$$H^{1,\rho,\theta}_{\alpha,\beta} = H^1 \cap [L^{2,\rho}_\alpha(\Omega) \times L^{2,\theta}_\beta(\Gamma_1)], \; H^{1,\rho,\theta} = H^{1,\rho,\theta}_{1,1} = H^1 \cap [L^\rho(\Omega) \times L^\theta(\Gamma_1)]. \tag{1.13}$$

Remark 1.9. By assumption (III) we have $H^{1,\rho,\theta}_{\alpha,\beta} = H^{1,\rho,\theta}$ when

$$\rho \in \begin{cases} [2, r_\Omega], & \text{if } p \le 1 + r_\Omega/2, \\ [2,\infty), & \text{if } p > 1 + r_\Omega/2, \end{cases} \quad \text{and} \quad \theta \in \begin{cases} [2, r_\Gamma], & \text{if } q \le 1 + r_\Gamma/2, \\ [2,\infty), & \text{if } q > 1 + r_\Gamma/2. \end{cases} \tag{1.14}$$

We introduce the auxiliary exponents $s_\Omega = s_\Omega(p,N)$ and $s_\Gamma = s_\Gamma(q,N)$ by

$$s_\Omega = \max\{2, r_\Omega(p-2)/(r_\Omega - 2)\}, \quad s_\Gamma = \max\{2, r_\Gamma(p-2)/(r_\Gamma - 2)\}, \tag{1.15}$$

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[3] Actually *all* functions verifying (II) and (VI)$'$ are of the form (1.10), where f_0^1 are g_0^1 are sources (that is $f_0^1(u)u \ge 0$ in $\mathbb{R}$). See Remark 7.1.

[4] trivially it is possible to extend the definition of the spaces $H^{1,\rho,\theta}_{\alpha,\beta}$ and $H^{1,\rho,\theta}$ also for $\rho, \theta = \infty$, and actually we shall do in Section 2, but we remark that in the statement of our results presented in this section these values *are not allowed.*

extended by continuity when $r_\Omega, r_\Gamma = \infty$. Trivially one has

$$p \lesseqgtr r_\Omega \quad \Leftrightarrow \quad s_\Omega \lesseqgtr r_\Omega, \qquad q \lesseqgtr r_\Gamma \quad \Leftrightarrow \quad s_\Gamma \lesseqgtr r_\Gamma; \tag{1.16}$$

$$p \geq r_\Omega \quad \Rightarrow \quad p \leq s_\Omega, \qquad q \geq r_\Gamma \quad \Rightarrow \quad q \leq s_\Gamma. \tag{1.17}$$

Moreover, when (IV)$'$ holds, so $r_\Omega \geq 4$ when $p \geq 1 + r_\Omega/2$ and $r_\Gamma \geq 4$ when $q \geq 1 + r_\Gamma/2$, by (1.4) we have [5]

$$p > r_\Omega \quad \Rightarrow \quad r_\Omega < s_\Omega < m, \qquad \text{and} \quad q > r_\Gamma \quad \Rightarrow \quad r_\Gamma < s_\Gamma < \mu, \tag{1.18}$$

the same implications being true with weak inequalities. We also set

$$\sigma_\Omega = \begin{cases} s_\Omega & \text{if } r_\Omega < p = 1 + r_\Omega/m', \\ 2 & \text{otherwise}, \end{cases} \qquad \sigma_\Gamma = \begin{cases} s_\Gamma & \text{if } r_\Gamma < q = 1 + r_\Gamma/\mu', \\ 2 & \text{otherwise}, \end{cases} \tag{1.19}$$

$$l_\Omega = \begin{cases} s_\Omega & \text{if } r_\Omega < p = 1 + r_\Omega/m', \\ p & \text{if } r_\Omega < p < 1 + r_\Omega/m', \\ 2 & \text{if } p \leq r_\Omega, \end{cases} \qquad l_\Gamma = \begin{cases} s_\Gamma & \text{if } r_\Gamma < q = 1 + r_\Gamma/\mu', \\ q & \text{if } r_\Gamma < q < 1 + r_\Gamma/\mu', \\ 2 & \text{if } q \leq r_\Gamma, \end{cases} \tag{1.20}$$

so, by (1.13), (1.15), (1.17), (1.19)–(1.20) and assumption (II),

$$2 \leq \sigma_\Omega \leq l_\Omega \leq s_\Omega, \qquad 2 \leq \sigma_\Gamma \leq l_\Gamma \leq s_\Gamma, \tag{1.21}$$

and

$$H^{1,s_\Omega,s_\Gamma} \hookrightarrow H^{1,l_\Omega,l_\Gamma} \hookrightarrow H^{1,\sigma_\Omega,\sigma_\Gamma} \hookrightarrow H^1.$$

When (IV)$'$ holds, by (1.16), (1.18) and (1.21) we have

$$2 \leq \sigma_\Omega \leq l_\Omega \leq s_\Omega \leq \max\{r_\Omega, m\}, \quad 2 \leq \sigma_\Gamma \leq l_\Gamma \leq s_\Gamma \leq \max\{r_\Gamma, \mu\}. \tag{1.22}$$

1.5. **Local analysis.** Our first main result is the following one.

Theorem 1.1 (**Local existence and continuation**). *Let (I–IV) hold. Then:*

(i) *for any* $(u_0, u_1) \in H^{1,\sigma_\Omega,\sigma_\Gamma} \times H^0$ (1.2) *has a maximal weak solution, that is*

$$\begin{aligned} u &= (u, u_{|\Gamma}) \in L^\infty_{loc}([0, T_{max}); H^1) \cap W^{1,\infty}_{loc}([0, T_{max}); H^0), \\ u' &= (u_t, u_{|\Gamma_t}) \in L^m_{loc}([0, T_{max}); L^{2,m}_\alpha(\Omega)) \times L^\mu_{loc}([0, T_{max}); L^{2,\mu}_\beta(\Gamma_1)), \end{aligned} \tag{1.23}$$

which satisfies (1.2) *in a distribution sense (to be specified later);*

(ii) u *enjoys the regularity*

$$U = (u, u') \in C([0, T_{max}); H^{1,\sigma_\Omega,\sigma_\Gamma} \times H^0) \tag{1.24}$$

and satisfies, for $0 \leq s \leq t < T_{max}$, *the energy identity* [6]

$$\begin{aligned} \tfrac12 \int_\Omega u_t^2 + \tfrac12 \int_{\Gamma_1} {u_{|\Gamma}}_t^2 + \tfrac12 \int_\Omega |\nabla u|^2 + \tfrac12 \int_{\Gamma_1} |\nabla_\Gamma u|_\Gamma^2 \Big|_s^t + \int_s^t\!\!\int_\Omega \alpha P_0(u_t) u_t \\ + \int_s^t \int_{\Gamma_1} \beta Q_0(u_{|\Gamma_t}) u_{|\Gamma_t} = \int_s^t \int_\Omega f_0(u) u_t + \int_s^t \int_{\Gamma_1} g_0(u) u_{|\Gamma_t}; \end{aligned} \tag{1.25}$$

(iii) $\overline{\lim}_{t \to T_{max}^-} \|U(t)\|_{H^1 \times H^0} = \infty$ *provided* $T_{max} < \infty$.

---

[5]indeed when $p > r_\Omega$ by (1.4) we have $m > r_\Omega$ and, since $r_\Omega \geq 4$ we get $(r_\Omega - 2)m^2 + r_\Omega(1 - r_\Omega)m + {r_\Omega}^2 > 0$ or equivalently $m > \frac{r_\Omega}{r_\Omega - 2}\left(\frac{r_\Omega}{m'} - 1\right)$ so using (1.4) again $m > s_\Omega$. The calculation can be repeated with weak inequalities and with p, m, r_Ω replaced by $q, \mu\, r_\Gamma$.

[6]here ∇_Γ denotes the Riemannian gradient on Γ and $|\cdot|_\Gamma$, the norm associated to the Riemannian scalar product on the tangent bundle of Γ. See Section 2.

Finally the property (ii) is enjoyed by any weak maximal solution of (1.2).

Theorem 1.1 sharply extends the local existence statement in [55, Theorem 1.1] with respect to internal sources/sink when $N = 3$ and to internal and boundary sources/sink when $N \geq 4$ (see Figure 1). Moreover it sharply generalizes all local existence results in the broader literature concerning wave equation with internal/boundary damping and source terms (see [12, 11, 46]).

The proof of Theorem 1.1 presented in the sequel is based on [55, Theorem 1.1] together with a truncation procedure inspired from [12]. Moreover we use a combination of a compactness argument from [46], when (p, m) and (q, μ) are not on the hyperbola, [7]with a key estimate, somewhat simplified and extended to higher dimension, from [11] (see Lemmas 4.1 and 4.3 below) when they are on the hyperbola.

The generality of Theorem 1.1 is best illustrated by some of its corollaries, the first of which concerns data in $H^1 \times H^0$ and involves minimal assumptions on the nonlinearities and no restrictions on N but excludes source/sink terms on the hyperbola, in the spirit of [46].

Corollary 1.1. *Under assumptions (I–III) and*

$$p < 1 + r_\Omega/m' \quad \text{when } m > 2, \qquad q < 1 + r_\Gamma/\mu' \quad \text{when } \mu > 2, \tag{1.26}$$

for any $(u_0, u_1) \in H^1 \times H^0$ *problem* (1.2) *has a maximal weak solution.*

By excluding only super-supercritical sources on the hyperbola but having restrictions on N on the supercritical part of it we get

Corollary 1.2. *If assumptions (I–IV) are satisfied and*

$$p < 1 + r_\Omega/m' \quad \text{when } m > r_\Omega, \qquad q < 1 + r_\Gamma/\mu' \quad \text{when } \mu > r_\Gamma, \tag{1.27}$$

for any $(u_0, u_1) \in H^1 \times H^0$ *problem* (1.2) *has a maximal weak solution.*

In particular the same conclusion holds true when

$$2 \leq p \leq r_\Omega \qquad \text{and} \quad 2 \leq q \leq r_\Gamma. \tag{1.28}$$

Theorem 1.1 and Corollaries 1.1– 1.2 are stated under minimal regularity (or, more properly, integrability) assumptions on u_0. When u_0 is more regular solutions are more regular, as one sees by a trivial time-integration, using (1.23)–(1.24) and Remark 1.9. We explicitly state this remark since it will be crucial in the sequel.

Corollary 1.3. *Under assumptions (I–IV) the conclusions of Theorem 1.1 hold when* $H^{1,\sigma_\Omega,\sigma_\Gamma}$ *is replaced by* $H^{1,\rho,\theta}_{\alpha,\beta}$*, provided* $\rho, \theta \in \mathbb{R}$ *satisfy*

$$(\rho, \theta) \in [\sigma_\Omega, \max\{r_\Omega, m\}] \times [\sigma_\Gamma, \max\{r_\Gamma, \mu\}], \tag{1.29}$$

and by $H^{1,\rho,\theta}$*, provided* ρ *and* θ *satisfy* (1.14) *and* (1.29). *In particular, by* (1.22), *it can be replaced by* H^{1,s_Ω,s_Γ} *when also (IV)′ holds.*

Or second main result asserts also uniqueness when $(u_0, u_1) \in H^{1,s_\Omega,s_\Gamma} \times H^0$.

Theorem 1.2 (**Local existence–uniqueness and continuation).** *Under assumptions (I–III) and (IV)′ the following conclusions hold:*

[7]in this case an alternative approach, only using compactness as in [46], is possible.

(i) *for any* $(u_0,u_1)\in H^{1,s_\Omega,s_\Gamma}\times H^0$ *problem* (1.2) *has a unique maximal weak solution* u *in* $[0,T_{max})$*;*

(ii) u *enjoys the regularity*

$$U=(u,u')\in C([0,T_{max});H^{1,s_\Omega,s_\Gamma}\times H^0) \tag{1.30}$$

and satisfies, for $0\le s\le t<T_{max}$*, the energy identity* (1.25)*;*

(iii) *if* $T_{max}<\infty$ *then*

$$\overline{\lim_{t\to T_{max}^-}}\|U(t)\|_{H^1\times H^0}=\lim_{t\to T_{max}^-}\|U(t)\|_{H^{1,s_\Omega,s_\Gamma}\times H^0}=\infty. \tag{1.31}$$

In particular, when $2\le p\le r_\Omega$ *and* $2\le q\le r_\Gamma$*, we have* $H^{1,s_\Omega,s_\Gamma}=H^1$.

The proof of Theorem 1.2 is based on the key estimate recalled above and on standard arguments. When u_0 is more regular, as in Corollary 1.3, we have

Corollary 1.4. *Under assumptions (I–III) and (IV)′ the main conclusions of Theorem 1.2 hold when the space* H^{1,s_Ω,s_Γ} *is replaced by* $H^{1,\rho,\theta}_{\alpha,\beta}$*, provided* $\rho,\theta\in\mathbb{R}$ *satisfy*

$$(\rho,\theta)\in[s_\Omega,\max\{r_\Omega,m\}]\times[s_\Gamma,\max\{r_\Gamma,\mu\}], \tag{1.32}$$

and by $H^{1,\rho,\theta}$*, provided* ρ *and* θ *satisfy* (1.14) *and* (1.32)*.*

The sequel of our local analysis concerns local Hadamard well–posedness. Unfortunately it is possible to prove this type of result in the same space used in Theorem 1.2 *only* when sources are subcritical or intercritical, in this case being $H^{1,s_\Omega,s_\Gamma}=H^1$.

When $p\ge r_\Omega$ we have to restrict to $u_0\in L^{s_1}(\Omega)$ with $s_1\in(s_\Omega,m]$, while when $q\ge r_\Gamma$ to $u_{0|\Gamma}\in L^{s_2}(\Gamma_1)$ with $s_2\in(s_\Gamma,\mu]$. Since $(s_\Omega,m]=\emptyset$ when $p=m=r_\Omega$ and $(s_\Gamma,\mu]=\emptyset$ when $q=\mu=r_\Gamma$, we have to exclude these cases by assuming

$$(p,m)\neq(r_\Omega,r_\Omega)\qquad\text{and}\qquad(q,\mu)\neq(r_\Gamma,r_\Gamma). \tag{1.33}$$

Consequently we shall consider $u_0\in H^{1,s_1,s_2}\hookrightarrow H^{1,s_\Omega,s_\Gamma}$ where [8]

$$s_1\in\begin{cases}(s_\Omega,r_\Omega] & \text{if } p<r_\Omega,\\ (s_\Omega,\max\{r_\Omega,m\}] & \text{otherwise,}\end{cases}\qquad s_2\in\begin{cases}(s_\Gamma,r_\Gamma] & \text{if } q<r_\Gamma,\\ (s_\Gamma,\max\{r_\Gamma,\mu\}] & \text{otherwise.}\end{cases} \tag{1.34}$$

Trivially, by (1.16) and (1.18), when (1.33) holds we have the implications

$$p\ge r_\Omega\quad\Rightarrow\quad r_\Omega\le s_\Omega<m,\qquad\text{and}\quad q\ge r_\Gamma\quad\Rightarrow\quad r_\Gamma\le s_\Gamma<\mu. \tag{1.35}$$

After these preliminary considerations we can state

Theorem 1.3 (**Local Hadamard well–posedness I**)**.** *Let assumptions (I–III), (IV)′, (V) and* (1.33) *hold. Then problem* (1.2) *is locally well–posed in* $H^{1,s_1,s_2}\times H^0$ *for* s_1 *and* s_2 *satisfying* (1.34)*.*

More explicitly, given $(u_{0n},u_{1n})\to(u_0,u_1)$ *in* $H^{1,s_1,s_2}\times H^0$*, respectively denoting by* u^n *and* u *the unique weak maximal solution of* (1.2) *in* $[0,T^n_{max})$ *and* $[0,T_{max})$ *corresponding to initial data* (u_{0n},u_{1n}) *and* (u_0,u_1)*, which exist by Theorem 1.2,* $U^n=(u^n,\dot u^n)$ *and* $U=(u,\dot u)$*, the following conclusions hold:*

(i) $T_{max}\le\underline{\lim}_n T^n_{max}$ *and*

(ii) $U^n\to U$ *in* $C([0,T^*];H^{1,s_1,s_2}\times H^0)$ *for any* $T^*\in(0,T_{max})$*.*

[8]by (1.16) conditions $s_1>s_\Omega$ when $p<r_\Omega$ and $s_2>s_\Gamma$ when $q<r_\Gamma$ can be trivially skipped.

In particular, when $2 \le p < r_\Omega$ *and* $2 \le q < r_\Gamma$*, since (V) has empty content, problem* (1.2) *is locally well–posed in* $H^1 \times H^0$ *under assumptions (I–III), (IV)*$'$.

Theorem 1.3 covers the supercritical ranges exactly as in [12]. Moreover it is the *first well–posedness result*, in the author's knowledge, dealing with internal/boundary super–supercritical term f_0, g_0. The price paid for this generality is to work in a Banach space smaller than the natural Hilbert energy space $H^1 \times H^0$ and assumption (V). The proof of Theorem 1.3 is based on the key estimate recalled above.

Theorem 1.3 is aimed to get well–posedness in the largest possible space, which turns out to be as close as one likes to $H^{1,s_\Omega,s_\Gamma} \times H^0$, not including it is some cases. The aim of the following variant of Theorem 1.3, is to complete the picture made in Corollaries 1.3–1.4 by showing in which part of the scale of spaces introduced there the problem is locally well-posed.

Theorem 1.4 (**Local Hadamard well–posedness II).** *Let assumptions (I–III), (IV)*$'$*, (V)*$'$*,* (1.33) *hold and* $\rho, \theta \in \mathbb{R}$ *satisfy*

$$(\rho, \theta) \in (s_\Omega, \max\{r_\Omega, m\}] \times (s_\Gamma, \max\{r_\Gamma, \mu\}]. \tag{1.36}$$

Then problem (1.2) *is locally well–posed in* $H^{1,\rho,\theta}_{\alpha,\beta} \times H^0$*, that is the conclusions of Theorem 1.3 hold true, when* H^{1,s_1,s_2} *is replaced by* $H^{1,\rho,\theta}_{\alpha,\beta}$*. In particular it is locally–well posed in* $H^{1,\rho,\theta} \times H^0$ *when* ρ, θ *satisfy* (1.14) *and* (1.36).

1.6. **Global analysis.** As a main application of the local analysis presented above we now state the global–in–time versions of Theorems 1.1–1.4 and their corollaries. When $P_0(v) = Q_0(v) =$, $f_0(u) = |u|^{p-2}u$ and $g_0(u) = |u|^{q-2}u$, $p, q > 2$, solutions of (1.2) blow–up in finite time for suitably chosen initial data, as proved in [55, Theorem 1.5]. Hence in the sequel we shall restrict to terms f_0 and g_0 satisfying assumption (VI) presented above, which excludes this case.

Theorem 1.5 (**Global analysis).** *The following conclusions hold true.*

(i) **(Global existence)** *Under assumptions (I–IV) and (VI), for any* $(u_0, u_1) \in H^{1,l_\Omega,l_\Gamma} \times H^0$ *the weak maximal solution* u *of problem* (1.2) *found in Theorem 1.1 is global in time, that is* $T_{max} = \infty$*, and* $u \in C([0, T_{max}); H^{1,l_\Omega,l_\Gamma})$.

    In particular, when (1.28) *holds, for any* $(u_0, u_1) \in H^1 \times H^0$ *problem* (1.2) *has a global weak solution.*

(ii) **(Global existence–uniqueness)** *Under assumptions (I–III), (IV)*$'$ *and (VI), for any* $(u_0, u_1) \in H^{1,s_\Omega,s_\Gamma} \times H^0$ *the unique maximal solution of problem* (1.2) *found in Theorem 1.2 is global in time, that is* $T_{max} = \infty$*, and* $u \in C([0, \infty); H^{1,s_\Omega,s_\Gamma})$.

    In particular, when (1.28) *holds, for any* $(u_0, u_1) \in H^1 \times H^0$ *problem* (1.2) *has a unique global weak solution.*

(iii) **(Global Hadamard well–posedness)** *Under assumptions (I–III), (IV)*$'$*, (V–VI) and* (1.33) *problem* (1.2) *is globally well–posed in* $H^{1,s_1,s_2} \times H^0$ *for* s_1 *and* s_2 *satisfying* (1.34)*, that is* $T_{max} = \infty$ *in Theorem 1.3.*

    Consequently the semi–flow generated by problem (1.2) *is a dynamical system in* $H^{1,s_1,s_2} \times H^0$.

    In particular, when $2 \le p < r_\Omega$ *and* $2 \le q < r_\Gamma$ *and under assumptions (I–III), (IV)*$'$ *and (VI), problem* (1.2) *is globally well–posed in* $H^1 \times H^0$*, so the semi–flow generated by* (1.2) *is a dynamical system in* $H^1 \times H^0$.

Remark 1.10. Parts (ii) and (iii) of Theorem 1.5 simply follow by combining Theorem 1.5–(i) with, respectively, Theorem 1.2 and Theorem 1.3. We include them in Theorem 1.5 in order to illustrate in parallel the global–in–time counterparts of Theorems 1.1–1.3.

By excluding source/sink terms on the hyperbola, so having no restriction on N, we get the global–in–time counterpart of Corollary 1.1.

Corollary 1.5. *Under assumptions (I–III), (VI) and* (1.26) *for any* $(u_0,u_1)\in H^{1,p.q}\times H^0$ *problem* (1.2) *has a global weak solution* $u\in C([0,\infty);H^{1,p,q})$.

By excluding only super-supercritical sources on the hyperbola but having restrictions on N on the supercritical part of it we get the global–in–time counterpart of Corollary 1.2.

Corollary 1.6. *Under assumptions (I–IV), (VI) and* (1.27) *for any* $(u_0,u_1)\in H^{1,p,q}\times H^0$ *problem* (1.2) *has a global weak solution* $u\in C([0,\infty);H^{1,p,q})$.

The main difference between Corollaries 1.1 and 1.5 (and between Corollaries 1.2 and 1.6) is that the former concerns all data $u_0\in H^1$ while the latter restricts to $u_0\in H^{1,p,q}$. This restriction originates from the use maid, in the proof of Theorem 1.5–(i), of the potentials of $\widehat{f_0}$ and $\widehat{g_0}$. Clearly they are simultaneously defined only in $H^{1,p,q}$. For the same reason Theorem 1.1 concerns $u_0\in H^{1,\sigma_\Omega,\sigma_\Gamma}$ while Theorem 1.5–(i) concerns $u_0\in H^{1,l_\Omega,l_\Gamma}=H^{1,\sigma_\Omega,\sigma_\Gamma}\cap H^{1,p,q}$. Since, by (1.17), $H^{1,s_\Omega,s_\Gamma}\subset H^{1,p,q}$, this restriction do not effects Theorem 1.5–(ii–iii).

We now state, for the reader convenience, the global–in–time version of the more general local analysis made in Corollaries 1.3–1.4 and Theorem 1.4, simply obtained by combining them with Theorem 1.5.

Corollary 1.7 (**Global analysis in the scale of spaces**). *The following conclusions hold true.*

(i) **(Global existence)** *Under assumptions (I–IV) and (VI) the main conclusion of Theorem 1.5–(i) hold when* H^{1,l_Ω,l_Γ} *is replaced by* $H^{1,\rho,\theta}_{\alpha,\beta}$, *provided* $\rho,\theta\in\mathbb{R}$ *satisfy*

$$(\rho,\theta)\in[l_\Omega,\max\{r_\Omega,m\}]\times[l_\Gamma,\max\{r_\Gamma,\mu\}],\tag{1.37}$$

and by $H^{1,\rho,\theta}$, *provided* ρ *and* θ *satisfy* (1.14) *and* (1.37).

(ii) **(Global existence–uniqueness)** *Under assumptions (I–III), (IV)′ and (VI) the main conclusion of Theorem 1.5–(ii) holds when the space* H^{1,s_Ω,s_Γ} *is replaced by* $H^{1,\rho,\theta}_{\alpha,\beta}$, *provided* $\rho,\theta\in\mathbb{R}$ *satisfy* (1.32) *and by* $H^{1,\rho,\theta}$, *provided* ρ *and* θ *satisfy* (1.14) *and* (1.32).

(iii) **(Global Hadamard well–posedness)** *Under assumptions (I–III), (IV)′, (V–VI) and* (1.33) *problem* (1.2) *is locally well–posed in* $H^{1,\rho,\theta}_{\alpha,\beta}\times H^0$ *when* $\rho,\theta\in\mathbb{R}$ *satisfy* (1.36), *that is the conclusions of Theorem 1.5–(iii) hold true when* H^{1,s_1,s_2} *is replaced by* $H^{1,\rho,\theta}_{\alpha,\beta}$. *In particular it is locally–well posed in* $H^{1,\rho,\theta}\times H^0$ *when* ρ,θ *satisfy* (1.14) *and* (1.36).

Theorems 1.1–1.5, with their corollaries, can be easily extended to more general second order uniformly elliptic linear operators, both in Ω and Γ, under suitable regularity assumptions on the coefficients. Here we prefer to deal with the Laplace and Laplace–Beltrami operators for the sake of clearness.

1.7. **Overall conclusions and paper organization.** The presentation of Theorems 1.1–1.3 and 1.5, dealing with results for data in the maximal space, can be simplified and unified by specifing N and slightly strengthening our assumptions set, that is by assuming assumptions (I–II) and the following ones:

1) when $\operatorname{essinf}_\Omega \alpha > 0$ and $\operatorname{essinf}_{\Gamma_1} \beta > 0$, the following properties are satisfied:
 ($\mathcal{P}_1$) $f_0 \in C^2(\mathbb{R})$ and $\overline{\lim}_{|u|\to\infty} |f_0''(u)|/|u|^{p-3} < \infty$ when $p > 1 + r_\Omega/2$;
 ($\mathcal{P}_2$) $g_0 \in C^2(\mathbb{R})$ and $\overline{\lim}_{|u|\to\infty} |g_0''(u)|/|u|^{q-3} < \infty$ when $q > 1 + r_\Gamma/2$;
 ($\mathcal{P}_3$) $\operatorname{essliminf}_{|v|\to\infty} P_0'(v)/|v|^{m-2} > 0$ if $p \ge r_\Omega$;
 ($\mathcal{P}_4$) $\operatorname{essliminf}_{|v|\to\infty} Q_0'(v)/|v|^{\mu-2} > 0$ if $q \ge r_\Gamma$.
2) when $\operatorname{essinf}_\Omega \alpha > 0 = \operatorname{essinf}_{\Gamma_1} \beta$, properties ($\mathcal{P}_1$) and ($\mathcal{P}_3$) are satisfied;
3) when $\operatorname{essinf}_\Omega \alpha = 0 < \operatorname{essinf}_{\Gamma_1} \beta$, properties ($\mathcal{P}_2$) and ($\mathcal{P}_4$) are satisfied;
4) when $\operatorname{essinf}_\Omega \alpha = \operatorname{essinf}_{\Gamma_1} \beta = 0$ no further properties are requested.

Theorems 1.1–1.3 and 1.5 are summarized in Tables 1–4 in the sequel, respectively dealing with the cases $N = 3$, $N = 4$, $N = 5$ and $N \ge 6$, to be read according to the following conventions:

- boxes separated by continuous lines indicate different cases depending on p, q, m and μ, while dashed lines separate different results in the same case;
- depending on α and β, the following parts of Tables 1–4 apply:
  1) when $\operatorname{essinf}_\Omega \alpha > 0$ and $\operatorname{essinf}_{\Gamma_1} \beta > 0$ all the tables;
  2) when $\operatorname{essinf}_\Omega \alpha > 0 = \operatorname{essinf}_{\Gamma_1} \beta$ only the first column;
  3) when $\operatorname{essinf}_\Omega \alpha = 0 < \operatorname{essinf}_{\Gamma_1} \beta$ only the first row;
  4) when $\operatorname{essinf}_\Omega \alpha = \operatorname{essinf}_{\Gamma_1} \beta = 0$ only the first box of the first row;
- by existence, existence–uniqueness and well–posedness in the Banach space V we indicate the corresponding result among Theorems 1.1–1.3 and the corresponding one among parts (i-iii) of Theorem 1.5 which applies for all $(u_0, u_1) \in V \times H^0$, the latter only under the additional assumption (VI); when the space V is different for Theorem 1.1 and part (i) of Theorem 1.5, we shall call the former local existence and the latter global existence;
- since well–posedness yields existence–uniqueness, which in turn yields existence, when two or three results hold in the same space only the strongest result is explicitly written;
- ε denotes a sufficiently small positive number.

The case $N = 2$ is omitted, since in this case, without additional assumptions, we simply get well–posedness in H^1 for all $p, q \in [2, \infty)$.

Tables 1–4 show that when $N \ge 3$ and $\operatorname{essinf}_\Omega \alpha > 0$ or $\operatorname{essinf}_{\Gamma_1} \beta > 0$ the analysis made in the present paper essentially extends the results in [55]. In dimension $N = 3$ we get new results only when $\operatorname{essinf}_\Omega \alpha > 0$, as it is natural due to the essential role played by the damping term, which is even more clear in dimensions $N \ge 4$. Moreover Tables 1–4 suggest that, in dimensions $N = 3, 4$ and partially in dimension $N = 5$, in presence of a couple of effective damping terms, the standard source classification presented above is mainly of technical nature, while Sobolev–criticality and belonging to the hyperbola are essential.

The outcomes of the analysis in the full scale of spaces, contained in Corollaries 1.3–1.4, Theorem 1.4 and Corollary 1.7, are summarized (for simplicity when

TABLE 1. Main results when $N = 3$

<table>
<tr><th></th><th>$2 \leq q < \infty$</th></tr>
<tr><td>$2 \leq p \leq 4$</td><td rowspan="2">well–posedness in H^1</td></tr>
<tr><td>$4 < p < 6$</td></tr>
<tr><td>$6 = p < m$</td><td>existence–uniqueness in H^1
well–posedness in $H^{1,6+\varepsilon,2}$</td></tr>
<tr><td>$6 = p = m$</td><td>existence–uniqueness in H^1</td></tr>
<tr><td>$6 < p < 1 + 6/m'$</td><td>local existence in H^1
global existence in $H^{1,p,2}$
existence–uniqueness in $H^{1,3(p-2)/2,2}$
well–posedness in $H^{1,3(p-2)/2+\varepsilon,2}$</td></tr>
<tr><td>$6 < p = 1 + 6/m'$</td><td>existence–uniqueness in $H^{1,3(p-2)/2,2}$
well–posedness in $H^{1,3(p-2)/2+\varepsilon,2}$</td></tr>
</table>

TABLE 2. Main results when $N = 4$

<table>
<tr><th></th><th>$2 \leq q \leq 4$</th><th>$4 < q < 6$</th><th>$6 = q < \mu$</th><th>$6 = q = \mu$</th><th>$6 < q < 1 + 6/\mu'$</th><th>$6 < q = 1 + 6/\mu'$</th></tr>
<tr><td>$2 \leq p \leq 3$</td><td colspan="2" rowspan="2">well–posedness in H^1</td><td rowspan="2">existence–uniqueness in H^1
well– posedness in $H^{1,2,6+\varepsilon}$</td><td rowspan="3"></td><td rowspan="2">local existence in H^1
global existence in $H^{1,2,q}$
existence–uniqueness in $H^{1,2,3(q-2)/2}$
well–posedness in $H^{1,2,\frac{3}{2}(q-2)+\varepsilon}$</td><td rowspan="2">existence–uniqueness in $H^{1,2,\frac{3}{2}(q-2)}$,
well– posedness in $H^{1,2,\frac{3}{2}(q-2)+\varepsilon}$</td></tr>
<tr><td>$3 < p < 4$</td></tr>
<tr><td>$4 = p < m$</td><td colspan="2">existence–uniqueness in H^1
well–posedness in $H^{1,4+\varepsilon,2}$</td><td>existence–uniqueness in H^1
well–posedness in $H^{1,4+\varepsilon,6+\varepsilon}$</td><td>local existence in H^1
global existence in $H^{1,2,q}$
existence–uniqueness in $H^{1,2,3(q-2)/2}$
well–posedness in $H^{1,4+\varepsilon,3(q-2)/2+\varepsilon}$</td><td>existence–uniqueness in $H^{1,2,3(q-2)/2}$
well–posedness in $H^{1,4+\varepsilon,3(q-2)/2+\varepsilon}$</td></tr>
<tr><td>$4 = p = m$</td><td colspan="4">existence–uniqueness in H^1</td><td>local existence in H^1
global existence in $H^{1,2,q}$
existence–uniqueness in $H^{1,2,3(q-2)/2}$</td><td>existence–uniqueness in $H^{1,2,3(q-2)/2}$</td></tr>
<tr><td>$4 < p < 1 + 4/m'$</td><td colspan="2">local existence in H^1
global existence in $H^{1,p,2}$
existence–uniqueness in $H^{1,2(p-2),2}$
well–posedness in $H^{1,2(p-2)+\varepsilon,2}$</td><td>local existence in H^1
global existence in $H^{1,p,2}$
existence–uniqueness in $H^{1,2(p-2),2}$
well–posedness in $H^{1,2(p-2)+\varepsilon,6+\varepsilon}$</td><td>local existence in H^1
global existence in $H^{1,p,2}$
existence–uniqueness in $H^{1,2(p-2),2}$</td><td>local existence in H^1
global existence in $H^{1,p,q}$
existence–uniqueness in $H^{1,2(p-2),3(q-2)/2}$
well–posedness in $H^{1,2(p-2)+\varepsilon,3(q-2)/2+\varepsilon}$</td><td>local existence in $H^{1,2,3(q-2)/2}$
global existence in $H^{1,p,3(q-2)/2}$
existence–uniqueness in $H^{1,2(p-2),3(q-2)/2}$
well–posedness in $H^{1,2(p-2)+\varepsilon,3(q-2)/2+\varepsilon}$</td></tr>
<tr><td>$4 < p = 1 + 4/m'$</td><td colspan="2">existence–uniqueness in $H^{1,2(p-2),2}$
well–posedness in $H^{1,2(p-2)+\varepsilon,2}$</td><td>existence–uniqueness in $H^{1,2(p-2),2}$
well–posedness in $H^{1,2(p-2)+\varepsilon,6+\varepsilon}$</td><td>existence–uniqueness in $H^{1,2(p-2),2}$</td><td>local existence in $H^{1,2(p-2),2}$
local existence in $H^{1,2(p-2),q}$
existence–uniqueness in $H^{1,2(p-2),3(q-2)/2}$
well–posedness in $H^{1,2(p-2)+\varepsilon,3(q-2)/2+\varepsilon}$</td><td>existence–uniqueness in $H^{1,2(p-2),3(q-2)/2}$
well–posedness in $H^{1,2(p-2)+\varepsilon,3(q-2)/2+\varepsilon}$</td></tr>
</table>

TABLE 3. Main results when $N = 5$

<table>
<tr><th></th><th>$2 \leq q < 4$</th><th>$2 \leq q < 4$</th><th>$4 = q < \mu$</th><th>$4 = q = \mu$</th><th>$4 < q < 1 + 4/\mu'$</th><th>$4 < q = 1 + 4/\mu'$</th></tr>
<tr><td>$2 \leq p \leq \frac{8}{3}$</td><td colspan="2">well–posedness in H^1</td><td>existence–uniqueness in H^1
well– posedness in $H^{1,2,4+\varepsilon}$</td><td></td><td>local existence in H^1
global existence in $H^{1,2,q}$
existence–uniqueness in $H^{1,2,2(q-2)}$
well– posedness in $H^{1,2,2(q-2)+\varepsilon}$</td><td>existence–uniqueness in $H^{1,2,2(q-2)}$
well– posedness in $H^{1,2,2(q-2)+\varepsilon}$</td></tr>
<tr><td>$\frac{8}{3} < p \leq \frac{10}{3}$</td><td colspan="4">existence in H^1</td><td>local existence in H^1
global existence in $H^{1,2,q}$</td><td rowspan="2">existence in $H^{1,p,2(q-2)}$</td></tr>
<tr><td>$\frac{10}{3} < p < 1 + \frac{10}{3m'}$</td><td colspan="4">local existence in H^1
global existence in $H^{1,p,2}$</td><td>local existence in H^1
global existence in $H^{1,p,q}$</td></tr>
<tr><td>$\frac{10}{3} < p = 1 + \frac{10}{3m'}$</td><td colspan="6">no results</td></tr>
</table>

TABLE 4. Main results when $N \geq 6$

<table>
<tr><th></th><th>$2 \leq q \leq 1 + r_\Gamma/2$</th><th>$1 + r_\Gamma/2 < q \leq r_\Gamma$</th><th>$r_\Gamma < q < 1 + r_\Gamma/\mu'$</th><th>$r_\Gamma < q = 1 + r_\Gamma/\mu'$</th></tr>
<tr><td>$2 \leq p \leq 1 + r_\Omega/2$</td><td>well–posedness in H^1</td><td></td><td rowspan="2">local existence in H^1
global existence in $H^{1,2,q}$</td><td rowspan="3"></td></tr>
<tr><td>$1 + r_\Omega/2 < p \leq r_\Omega$</td><td colspan="2">existence in H^1</td></tr>
<tr><td>$r_\Omega < p < 1 + r_\Omega/m'$</td><td colspan="2">local existence in H^1
global existence in $H^{1,p,2}$</td><td>local existence in H^1
global existence in $H^{1,p,q}$</td></tr>
<tr><td>$r_\Omega < p = 1 + r_\Omega/m'$</td><td colspan="4">no results</td></tr>
</table>

$\operatorname{essinf}_\Omega \alpha > 0, \operatorname{essinf}_{\Gamma_1} \beta > 0)$ in Tables 5–8, p. 53–55. In them we follow the same conventions presented above and we denote $x \vee y = \max\{x, y, \}$ for $x, y \in \mathbb{R}$.

The paper is organized as follows:

- in Sections 2–3 we recall some background material from [55], we state our assumptions on f, g, P Q and we give some preliminary results;
- in Section 4 we give a key estimate to be used in the sequel;
- Section 5 is devoted to local existence for problem (1.1), including the proofs of Theorem 1.1 and Corollaries 1.1–1.3;
- Section 6 deals with uniqueness and local well–posedness, including the proofs of Theorems 1.2–1.4 and Corollary 1.4;
- Section 7 is devoted to our global analysis, including the proofs of Theorem 1.5 and Corollaries 1.5–1.7;
- in Appendix A we prove a density result used in the paper.

2. BACKGROUND

2.1. **Notation.** We shall adopt the standard notation for (real) Lebesgue and Sobolev spaces in Ω (see [1]) and Γ (see [31]). As usual ρ' is the Hölder conjugate of ρ, i.e. $1/\rho + 1/\rho' = 1$. Given a Banach space X and its dual X' we shall denote by $\langle\cdot,\cdot\rangle_X$ the duality product between them. Finally, we shall use the standard notation for vector valued Lebesgue and Sobolev spaces in a real interval, with the exception that the derivative of u, a time derivative, will be denoted by $\dot{u}$.

Given $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma)$, $\alpha, \beta \geq 0$ and $\rho \in [1,\infty]$ we shall respectively denote by $(L^\rho(\Omega), \|\cdot\|_\rho)$, $(L^\rho(\Gamma), \|\cdot\|_{\rho,\Gamma})$, $(L^\rho(\Gamma_1), \|\cdot\|_{\rho,\Gamma_1})$, $(L^\rho(\Omega;\lambda_\alpha), \|\cdot\|_{\rho,\alpha})$, $(L^\rho(\Gamma;\lambda_\beta), \|\cdot\|_{\rho,\beta,\Gamma})$ and $(L^\rho(\Gamma_1;\lambda_\beta), \|\cdot\|_{\rho,\beta,\Gamma_1})$ the Lebesgue spaces (and norms) with respect to the following measures: the standard Lebesgue one in Ω, the hypersurface measure σ on Γ and Γ_1, λ_α in Ω defined by $d\lambda_\alpha = \lambda_\alpha\, dx$, λ_β on Γ and Γ_1 defined by $d\lambda_\beta = \lambda_\beta\, d\sigma$. The equivalence classes with respect to the measures λ_α and λ_β will be respectively denoted by $[\cdot]_\alpha$ and $[\cdot]_\beta$.

We recall some well–known preliminaries on the Riemannian gradient, where only the fact that Γ is a C^1 compact manifold endowed with a C^0 Riemannian metric is used. We refer to [50] for more details and proofs, given there for smooth manifolds, and to [49] for a general background on differential geometry on C^k manifolds. We denote by $(\cdot,\cdot)_\Gamma$ the metric inherited from $\mathbb{R}^N$, given in local coordinates $(y_1,\dots,y_{N-1})$ by $(g_{ij})_{i,j=1,\dots,N-1}$, $|\cdot|_\Gamma^2 = (\cdot,\cdot)_\Gamma$, by $d\sigma$ the natural volume element on Γ, given by $\sqrt{\widetilde{g}}\, dy_1 \wedge \ldots \wedge dy_{N-1}$, where $\widetilde{g} = \det(g_{ij})$. We denote by $(\cdot|\cdot)_\Gamma$ the Riemannian (real) inner product on 1-forms on Γ associated to the metric, given in local coordinates by $(g^{ij}) = (g_{ij})^{-1}$, by d_Γ the total differential on Γ and by ∇_Γ the Riemannian gradient, given in local coordinates by $\nabla_\Gamma u = g^{ij}\,\partial_j u\,\partial_i$ for any $u \in H^1(\Gamma)$. It is then clear that $(d_\Gamma u|d_\Gamma v)_\Gamma = (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma$ for $u, v \in H^1(\Gamma)$, so the use of vectors or forms in the sequel is optional. It is well–known (see [50] in the smooth setting, and [34] in the C^1 setting) that the norm $\|u\|^2_{H^1(\Gamma)} = \|u\|^2_{2,\Gamma} + \|\nabla_\Gamma u\|^2_{2,\Gamma}$, where $\|\nabla_\Gamma u\|^2_{2,\Gamma} := \int_\Gamma |\nabla_\Gamma u|^2_\Gamma$, is equivalent in $H^1(\Gamma)$ to the standard one. In the sequel, the notation $d\sigma$ will be dropped from the boundary integrals.

2.2. **Functional setting and weak solutions for a linear problem.** We start by recalling some facts about the spaces $L^{2,\rho}_\alpha(\Omega)$ and $L^{2,\rho}_\beta(\Gamma_1)$, refereing to [55] for more details and proofs. They are reflexive and, making the standard identifications

$$[L^\rho(\Omega)]' \simeq L^{\rho'}(\Omega), \qquad \text{and} \qquad [L^\rho(\Gamma_1)]' \simeq L^{\rho'}(\Gamma_1), \tag{2.1}$$

when $\rho \in [2,\infty)$ we have the two chains of embedding [9]

$$[L^\rho(\Omega,\lambda_\alpha)]' \hookrightarrow [L^{2,\rho}_\alpha(\Omega)]' \hookrightarrow L^{\rho'}(\Omega),\ [L^\rho(\Gamma_1,\lambda_\beta)]' \hookrightarrow [L^{2,\rho}_\beta(\Gamma_1)]' \hookrightarrow L^{\rho'}(\Gamma_1). \tag{2.2}$$

Next, given $\rho, \theta \in [2,\infty)$ and $-\infty \leq a < b \leq \infty$ we introduce the reflexive space

$$L^{2,\rho,\theta}_{\alpha,\beta}(a,b) = L^\rho(a,b\,; L^{2,\rho}_\alpha(\Omega)) \times L^\theta(a,b\,; L^{2,\theta}_\beta(\Gamma_1)),$$

with its dual

$$[L^{2,\rho,\theta}_{\alpha,\beta}(a,b)]' \simeq L^{\rho'}(a,b\,; [L^{2,\rho}_\alpha(\Omega)]') \times L^{\theta'}(a,b\,; [L^{2,\theta}_\beta(\Gamma_1)]'). \tag{2.3}$$

By (2.2)–(2.3) we have the embedding

$$L^{\rho'}(a,b\,; [L^\rho(\Omega,\lambda_\alpha)]') \times L^{\theta'}(a,b\,; [L^\theta(\Gamma_1,\lambda_\beta)]') \hookrightarrow [L^{2,\rho,\theta}_{\alpha,\beta}(a,b)]'. \tag{2.4}$$

The space H^1 introduced in (1.12) is endowed with the norm

$$\|u\|^2_{H^1} = \int_\Omega |\nabla u|^2 + \int_{\Gamma_1} |\nabla_\Gamma u|^2_\Gamma + \int_{\Gamma_1} |u|^2, \tag{2.5}$$

[9] by (2.1) we *can not* identify $[L^\rho(\Omega,\lambda_\alpha)]'$ and $[L^\rho(\Gamma_1,\lambda_\beta)]'$ with $L^{\rho'}(\Omega,\lambda_\alpha)$ and $L^{\rho'}(\Gamma_1,\lambda_\beta)$.

equivalent to the one inherited from the product. The definition of the space $H^{1,\rho,\theta}_{\alpha,\beta}$ given in (1.13) can be extended also for $\rho,\theta=\infty$, loosing reflexivity, and clearly $H^{1,\rho,\theta}_{\alpha,\beta}\hookrightarrow H^1\hookrightarrow H^0$ and $H^{1,\rho,\theta}_{\alpha,\beta}\hookrightarrow L^{2,\rho}_\alpha(\Omega)\times L^{2,\theta}_\beta(\Gamma_1)\hookrightarrow H^0$, which are dense thanks to [55, Lemma 2.1].

Finally we introduce the phase spaces for problem (1.1), that is

$$\mathcal{H}=H^1\times H^0 \qquad \text{and} \qquad \mathcal{H}^{\rho,\theta}=H^{1,\rho,\theta}\times H^0 \quad \text{for } \rho,\theta\in[2,\infty). \tag{2.6}$$

We consider the linear evolution boundary value problem

$$\begin{cases} u_{tt}-\Delta u=\xi & \text{in } (0,T)\times\Omega,\\ u=0 & \text{on } (0,T)\times\Gamma_0,\\ u_{tt}+\partial_\nu u-\Delta_\Gamma u=\eta & \text{on } (0,T)\times\Gamma_1, \end{cases} \tag{2.7}$$

where $0<T<\infty$ and $\xi=\xi(t,x)$, $\eta=\eta(t,x)$ are given forcing terms of the form

$$\begin{cases} \xi=\xi_1+\alpha\xi_2, & \xi_1\in L^1(0,T;L^2(\Omega)), \quad \xi_2\in L^{\rho'}(0,T;L^{\rho'}(\Omega,\lambda_\alpha)),\\ \eta=\eta_1+\beta\eta_2, & \eta_1\in L^1(0,T;L^2(\Gamma_1)), \quad \eta_2\in L^{\theta'}(0,T;L^{\theta'}(\Gamma_1,\lambda_\beta)), \end{cases} \tag{2.8}$$

where $\alpha\in L^\infty(\Omega)$, $\beta\in L^\infty(\Gamma_1)$, $\alpha,\beta\ge 0$ and $\rho,\theta\in[2,\infty)$.

By a *weak solution* of (2.7) in $[0,T]$ we mean

$$u\in L^\infty(0,T;H^1)\cap W^{1,\infty}(0,T;H^0), \qquad \dot u\in L^{2,\rho,\theta}_{\alpha,\beta}(0,T), \tag{2.9}$$

such that the distribution identity

$$\int_0^T\left[-(\dot u,\dot\phi)_{H^0}+\int_\Omega\nabla u\nabla\phi+\int_{\Gamma_1}(\nabla_\Gamma u,\nabla_\Gamma\phi)_\Gamma-\int_\Omega\xi\phi-\int_{\Gamma_1}\eta\phi\right]=0 \tag{2.10}$$

holds for all $\phi\in C_c((0,T);H^1)\cap C^1_c((0,T);H^0)\cap L^{2,\rho,\theta}_{\alpha,\beta}(0,T)$.

When dealing with u satisfying (2.9), we shall *systematically denote in the paper*

$$\dot u=(u_t,u_{|\Gamma_t}) \qquad \text{and} \quad U=(u,\dot u)\in L^\infty([0,T];\mathcal{H}).$$

We shall also denote $A\in\mathcal{L}(H^1,(H^1)')$ defined by

$$\langle Au,v\rangle_{H^1}=\int_\Omega\nabla u\nabla v+\int_{\Gamma_1}(\nabla_\Gamma u,\nabla_\Gamma v)_\Gamma, \qquad \text{for all } u,v\in H^1. \tag{2.11}$$

We recall the following result (see [55, Lemma 2.2]).

Lemma 2.1. *Suppose that* (2.8) *holds. Then any weak solution u of* (2.7) *enjoys the further regularity $U\in C([0,T];\mathcal{H})$ and satisfies the energy identity*

$$\frac12\|\dot u\|^2_{H^0}+\frac12\langle Au,u\rangle_{H^1}\Big|_s^t=\int_s^t\int_\Omega\xi u_t+\int_{\Gamma_1}\eta u_{|\Gamma_t}\,d\tau$$

for $0\le s\le t\le T$. Moreover (2.10) *holds in the generalized form*

$$(\dot u,\phi)_{H^0}\Big|_0^T+\int_0^T\left[-(\dot u,\dot\phi)_{H^0}+\langle Au,\phi\rangle_{H^1}-\int_\Omega\xi\phi-\int_{\Gamma_1}\eta\phi\right]=0$$

for all $\phi\in C([0,T];H^1)\cap C^1([0,T];H^0)\cap L^{2,\rho,\theta}_{\alpha,\beta}(0,T)$.

3. Preliminaries

3.1. **Main assumptions.** With reference to problem (1.1) we suppose that

(PQ1) P and Q are Carathéodory functions, respectively in $\Omega\times\mathbb{R}$ and $\Gamma_1\times\mathbb{R}$, and there are $\alpha\in L^\infty(\Omega)$, $\beta\in L^\infty(\Gamma_1)$, $\alpha,\beta\ge 0$, and $m,\mu>1$, $c_m,c_\mu>0$, such that

$$|P(x,v)|\le c_m\alpha(x)(1+|v|^{m-1}) \text{ for a.a. } x\in\Omega, \text{ all } v\in\mathbb{R}; \tag{3.1}$$

$$|Q(x,v)|\le c_\mu\beta(x)(1+|v|^{\mu-1}) \quad \text{for a.a. } x\in\Gamma_1, \text{ all } v\in\mathbb{R}; \tag{3.2}$$

(PQ2) P (respectively Q) is monotone increasing in v for a.a. $x\in\Omega$ ($x\in\Gamma_1$);
(PQ3) P and Q are coercive, that is there are constants $c'_m,c'_\mu>0$ such that

$$P(x,v)v\ge c'_m\alpha(x)|v|^m \qquad \text{for a.a. } x\in\Omega, \text{ all } v\in\mathbb{R}; \tag{3.3}$$

$$Q(x,v)v\ge c'_\mu\beta(x)|v|^\mu \qquad \text{for a.a. } x\in\Gamma_1, \text{ all } v\in\mathbb{R}. \tag{3.4}$$

Remark 3.1. Trivially (PQ1–3) yield $P(\cdot,0)\equiv 0$ and $Q(\cdot,0)\equiv 0$. Moreover, when $P(x,v)=\alpha(x)P_0(v)$ and $Q(x,v)=\beta(x)Q_0(v)$ with $\alpha\in L^\infty(\Omega)$ and $\beta\in L^\infty(\Gamma_1)$, $\alpha,\beta\ge 0$, (PQ1–3) reduce to assumption (I), p. 4.

We denote $\overline{m}=\max\{2,m\}$, $\overline{\mu}=\max\{2,\mu\}$, and, for $-\infty\le a<b\le\infty$,

$$Y=L^{2,\overline{m}}_\alpha(\Omega)\times L^{2,\overline{\mu}}_\beta(\Gamma_1),\quad X=H^{1,\overline{m},\overline{\mu}}_{\alpha,\beta},\quad Z(a,b)=L^{2,\overline{m},\overline{\mu}}_{\alpha,\beta}(a,b). \tag{3.5}$$

By (PQ1) the Nemitskii operators $\widehat{P}$ and $\widehat{Q}$ (respectively) associated to P and Q are continuous from $L^{\overline{m}}(\Omega)$ to $L^{\overline{m}'}(\Omega))\simeq[L^{\overline{m}}(\Omega))]'$ and from $L^{\overline{\mu}}(\Gamma_1)$ to $L^{\overline{\mu}'}(\Gamma_1))\simeq[L^{\overline{\mu}}(\Gamma_1))]'$, and they can be uniquely extended to $\widehat{P}:L^{2,\overline{m}}_\alpha(\Omega)\to[L^{\overline{m}}(\Omega,\lambda_\alpha)]'$ and $\widehat{Q}:L^{2,\overline{\mu}}_\beta(\Gamma_1)\to[L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)]'$. We denote

$$B=(\widehat{P},\widehat{Q}):Y\to[L^{\overline{m}}(\Omega,\lambda_\alpha)]'\times[L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)]'.$$

We recall (see [55])

Lemma 3.1. *Let (PQ1—2) hold and $(a,b)\subset\mathbb{R}$ is bounded. Then*

(i) *B is continuous and bounded from Y to $[L^{\overline{m}}(\Omega,\lambda_\alpha)]'\times[L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)]'$ and hence, by* (2.3)*, to Y';*
(ii) *B acts boundedly and continuously from $Z(a,b)$ to $L^{\overline{m}'}(a,b\,;[L^{\overline{m}}(\Omega,\lambda_\alpha)]')\times L^{\overline{\mu}'}(a,b\,;[L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)]')$ and hence, by* (2.4)*, to $Z'(a,b)$;*
(iii) *B is monotone in Y and in $Z(a,b)$.*

Our main assumption on f and g is the following one:

(FG1) (F1) f is a Carathéodory function in $\Omega\times\mathbb{R}$ and there are an exponent $p\ge 2$ and constants $c_p,c'_p\ge 0$ such that, for a.a. $x\in\Omega$ and all $u,v\in\mathbb{R}$,

$$|f(x,u)|\le c_p(1+|u|^{p-1}), \tag{3.6}$$

$$|f(x,u)-f(x,v)|\le c'_p|u-v|(1+|u|^{p-2}+|v|^{p-2}); \tag{3.7}$$

(G1) g is a Carathéodory function in $\Gamma_1\times\mathbb{R}$, and there are an exponent $q\ge 2$ and constants $c_q,c'_q\ge 0$ such that, for a.a. $x\in\Gamma_1$ and all

$u, v \in \mathbb{R}$,

$$|g(x,u)| \le c_q(1+|u|^{q-1}) \tag{3.8}$$

$$|g(x,u)-g(x,v)| \le c_q'|u-v|(1+|u|^{q-2}+|v|^{q-2}) \tag{3.9}$$

Remark 3.2. Assumption (FG1) can be equivalently formulated as follows:

(FG1)′ (F1)′ f is a Carathéodory function in $\Omega\times\mathbb{R}$, $f(x,\cdot)\in C^{0,1}_{\text{loc}}(\mathbb{R})$ for a.a. $x\in\Omega$, and there are an exponent $p\ge 2$ and constants $\widetilde{c_p}, \widetilde{c_p}' \ge 0$ such that

$$|f(x,0)| \le \widetilde{c_p}, \qquad \text{for a. a. } x\in\Omega, \tag{3.10}$$

$$|f_u(x,u)| \le \widetilde{c_p}'(1+|u|^{p-2}), \qquad \text{for a.a. } (x,u)\in\Omega\times\mathbb{R}, \tag{3.11}$$

(G1)′ g is a Carathéodory function in $\Omega\times\Gamma_1$, $g(x,\cdot)\in C^{0,1}_{\text{loc}}(\mathbb{R})$ for a.a. $x\in\Gamma_1$, and there are an exponent $q\ge 2$ and constants $\widetilde{c_q}, \widetilde{c_q}' \ge 0$ such that

$$|g(x,0)| \le \widetilde{c_q}, \qquad \text{for a. a. } x\in\Gamma_1, \tag{3.12}$$

$$|g_u(x,u)| \le \widetilde{c_q}'(1+|u|^{q-2}), \qquad \text{for a.a. } (x,u)\in\Gamma_1\times\mathbb{R}, \tag{3.13}$$

Indeed by (3.6) we immediately get (3.10) with $\widetilde{c_p}=c_p$ and by (3.7) we have $f(x,\cdot)\in C^{0,1}_{\text{loc}}(\mathbb{R})$ for a.a. $x\in\Omega$, hence f_u exist a.e. [10] and (3.11) follows from (3.7), with $\widetilde{c_p}'=2c_p'$. In the same way from (G1) we get (G1)′ with $\widetilde{c_q}=c_q$ and $\widetilde{c_q}'=2c_q'$. Conversely by (FG1)′, integrating (3.11) and (3.13) with respect to the second variable in the convenient interval, one gets (FG1), with $c_p=\widetilde{c_p}+2\widetilde{c_p}'$, $c_q=\widetilde{c_q}+2\widetilde{c_q}'$, $c_p'=\widetilde{c_p}'$, $c_q'=\widetilde{c_q}'$. Consequently when $f(x,u)=f_0(u)$ and $g(x,u)=g_0(u)$ assumption (FG1) reduces to (II), p. 4. Other relevant examples of functions f and g satisfying (FG1) are given by

$$\begin{aligned} &f_2(x,u)=\gamma_1(x)|u|^{\widetilde{p}-2}u+\gamma_2(x)|u|^{p-2}u+\gamma_3(x), \quad 2\le\widetilde{p}\le p,\ \gamma_i\in L^\infty(\Omega),\\ &g_2(x,u)=\delta_1(x)|u|^{\widetilde{q}-2}u+\delta_2(x)|u|^{q-2}u+\delta_3(x), \quad 2\le\widetilde{q}\le q,\ \delta_i\in L^\infty(\Gamma_1), \end{aligned} \tag{3.14}$$

and by

$$f_3(x,u)=\gamma(x)f_0(u), \quad g_3(x,u)=\delta(x)g_0(u), \quad \gamma\in L^\infty(\Omega), \quad \delta\in L^\infty(\Gamma_1), \tag{3.15}$$

where f_0 and g_0 satisfy (II).

Beside the structural assumptions (PQ1–3) and (FG1) we introduce the following assumption relating f with P and g with Q

(FGQP1) p, q, m, μ, α and β in (PQ1–3) and (FG1) satisfy (III), p. 4.

Remark 3.3. To simplify several estimates in the sequel we remark that, when $p>1+r_\Omega/2$, so $\alpha_0:=\operatorname{essinf}_\Omega\alpha>0$, by replacing α with α/α_0 and consequently c_m with α_0c_m in (3.1) and c_m' with α_0c_m' in (3.3) we can normalize $\alpha_0=1$. For the same reason when $q>1+r_\Gamma/2$ we shall assume without restriction that $\operatorname{essinf}_{\Gamma_1}\beta=1$.

[10]the fact that measurable functions in an open set, which are locally absolutely continuous with respect to a variable, possess a.e. the partial derivative with respect to that variable is classical, as stated for example in [39, p.297]. However the sceptical reader can prove it by repeating [20, Proof of Proposition 2.1 p. 173] for Carathéodory functions, so getting the measurability of the four Dini derivatives. Hence the set where the derivative does not exist is measurable and finally it has zero measure by Fubini's theorem.

Remark 3.4. Trivially, by (FGQP1), $m=\overline{m}>2$ when $p>1+r_\Omega/2$ and $\mu=\overline{\mu}>2$ when $q>1+r_\Gamma/2$. Moreover, when $p\le 1+r_\Omega/2$ and $q\le 1+r_\Gamma/2$ assumption (FGQP1) can be skipped.

We now introduce the auxiliary exponents

$$m_p=\begin{cases}2 & \text{if } p\le 1+r_\Omega/2,\\ m=\overline{m} & \text{otherwise,}\end{cases}\qquad \mu_q=\begin{cases}2 & \text{if } q\le 1+r_\Gamma/2,\\ \mu=\overline{\mu}, & \text{otherwise,}\end{cases}\tag{3.16}$$

so by (1.4) we have

$$m_p'\le r_\Omega/(p-1)\qquad\text{and}\qquad \mu_q'\le r_\Gamma/(q-1).\tag{3.17}$$

Moreover, by (3.5), (3.16) and assumption (FGQP1), for any $T>0$

$$\|w\|_{L^{m_p}((0,T)\times\Omega)\times L^{\mu_q}((0,T)\times\Gamma_1)}\le \|w\|_{Z(0,T)}\quad\text{for all } w\in Z(0,T).\tag{3.18}$$

The following lemma points out some easy consequences of (PQ1–3), (FG1), (1.4).

Lemma 3.2. *If $\mathfrak{f},\mathfrak{g}$ satisfy (FG1) with constants $\mathfrak{c}_\mathfrak{p}$,$\mathfrak{c}_\mathfrak{p}'$, $\mathfrak{c}_\mathfrak{q}$,$\mathfrak{c}_\mathfrak{q}'$, and $\rho\in[p-1,\infty)$, $\theta\in[q-1,\infty)$, the Nemitskii operators $\widehat{\mathfrak{f}}:L^\rho(\Omega)\to L^{\rho/(p-1)}(\Omega)$ and $\widehat{\mathfrak{g}}:L^\theta(\Gamma_1)\to L^{\theta/(q-1)}(\Gamma_1)$ associated to them are locally Lipschitz and bounded, and there are $k_1,k_2>0$, depending only on Ω, such that for any $R\ge 0$*

$$\begin{aligned}
\|\widehat{\mathfrak{f}}(u)\|_{\frac{\rho}{p-1}} &\le \mathfrak{c}_\mathfrak{p}k_1(1+R^{p-1}), & \|\widehat{\mathfrak{f}}(u)-\widehat{\mathfrak{f}}(v)\|_{\frac{\rho}{p-1}} &\le \mathfrak{c}_\mathfrak{p}'k_1(1+R^{p-2})\|u-v\|_\rho,\\
\|\widehat{\mathfrak{g}}(\widetilde{u})\|_{\frac{\theta}{q-1},\Gamma_1} &\le \mathfrak{c}_\mathfrak{q}k_2(1+R^{q-1}), & \|\widehat{\mathfrak{g}}(\widetilde{u})-\widehat{\mathfrak{g}}(\widetilde{v})\|_{\frac{\theta}{q-1},\Gamma_1} &\le \mathfrak{c}_\mathfrak{q}'k_2(1+R^{q-2})\|\widetilde{u}-\widetilde{v}\|_{\theta,\Gamma_1},
\end{aligned}$$

provided $\|u\|_\rho,\|v\|_\rho,\|\widetilde{u}\|_{\theta,\Gamma_1},\|\widetilde{v}\|_{\theta,\Gamma_1}\le R$. Moreover, if also (PQ1–3) and (1.4) hold then $\widehat{\mathfrak{f}}:H^1(\Omega)\to L^{m_p'}(\Omega)$ and $\widehat{\mathfrak{g}}:H^1(\Gamma)\cap L^2(\Gamma_1)\to L^{\mu_q'}(\Gamma_1)$ enjoy the same properties and there is $k_3>0$, depending only on Ω, such that, for any $R\ge 0$

$$\begin{aligned}
\|\widehat{\mathfrak{f}}(u)\|_{m_p'} &\le \mathfrak{c}_\mathfrak{p}k_3(1+R^{p-1}), & \|\widehat{\mathfrak{f}}(u)-\widehat{\mathfrak{f}}(v)\|_{m_p'} &\le \mathfrak{c}_\mathfrak{p}'k_3(1+R^{p-2})\|u-v\|_{H^1(\Omega)},\\
\|\widehat{\mathfrak{g}}(\widetilde{u})\|_{\mu_q',\Gamma_1} &\le \mathfrak{c}_\mathfrak{q}k_3(1+R^{q-1}), & \|\widehat{\mathfrak{g}}(\widetilde{u})-\widehat{\mathfrak{g}}(\widetilde{v})\|_{\mu_q',\Gamma_1} &\le \mathfrak{c}_\mathfrak{q}'k_3(1+R^{q-2})\|\widetilde{u}-\widetilde{v}\|_{H^1(\Gamma)},
\end{aligned}$$

provided $\|u\|_{H^1(\Omega)},\|v\|_{H^1(\Omega)},\|\widetilde{u}\|_{H^1(\Gamma)},\|\widetilde{v}\|_{H^1(\Gamma)}\le R$.

3.2. **Weak solutions.** We note that by Lemma 3.2 and (3.17), for any u satisfying (2.9), we have

$$\widehat{f}(u)\in L^\infty(0,T;L^{m_p'}(\Omega))\quad\text{and}\quad \widehat{g}(u_{|\Gamma})\in L^\infty(0,T;L^{\mu_q'}(\Gamma_1)).$$

Hence, when $2\le p\le 1+r_\Omega/2$ we get $\widehat{f}(u)\in L^1(0,T;L^2(\Omega))$, while when $p>1+r_\Omega/2$ we get $\widehat{f}(u)\in L^{m'}(0,T;L^{m'}(\Omega))$ and by (FGQP1) we thus have $\widehat{f}(u)\in L^{\overline{m}'}(0,T;[L^{\overline{m}}(\Omega;\lambda_\alpha)]')$. In conclusion in both cases we can write $\widehat{f}(u)$ in the form (2.8) with $\rho=\overline{m}$. Similar arguments show that $\widehat{g}(u)$ can be written in the form (2.8) with $\theta=\overline{\mu}$. Moreover, by Lemma 3.1, $\widehat{P}(u_t)\in L^{\overline{m}'}(0,T;[L^{\overline{m}}(\Omega,\lambda_\alpha)]')$ and $\widehat{Q}(u_{|\Gamma_t})\in L^{\overline{\mu}'}(0,T;[L^{\overline{\mu}}(\Gamma_1,\lambda_\alpha)]')$, so they can be written in the same form. By previous considerations and Lemma 2.1 the following definition makes sense.

Definition 3.1. Let (PQ1–3), (FG1), (FGQP1) hold and $U_0=(u_0,u_1)\in\mathcal{H}$. A weak solution of problem (1.1) in $[0,T]$, $0<T<\infty$, is a weak solution of (2.7) with

$$\xi=\widehat{f}(u)-\widehat{P}(u_t),\quad \eta=\widehat{g}(u_{|\Gamma})-\widehat{Q}(u_{|\Gamma_t}),\quad \rho=\overline{m}\quad\text{and}\quad \theta=\overline{\mu},\tag{3.19}$$

such that $u(0)=u_0$ and $\dot u(0)=u_1$. A weak solution of (1.1) in $[0,T)$, $0<T\le\infty$, is $u\in L^\infty_{\text{loc}}([0,T);H^1)$ which is a weak solution of (1.1) in $[0,T']$ for any $T'\in(0,T)$. Such a solution is called maximal if it has no proper extensions.

Weak solutions enjoy good properties, as shown in the next result.

Lemma 3.3. *Let u be a weak solution of* (1.1) *in dom $u=[0,T]$ or dom $u=[0,T)$. Then*

(i) $u\in C(\textit{dom } u;H^1)\cap C^1(\textit{dom } u;H^0)$, *it satisfies the energy identity*

$$\tag{3.20} \tfrac12\left[\int_\Omega u_t^2+\int_{\Gamma_1} {u_{|\Gamma}}_t^2+\int_\Omega|\nabla u|^2+\int_{\Gamma_1}|\nabla_\Gamma u|_\Gamma^2\right]_s^t+\int_s^t\int_\Omega P(\cdot,u_t)u_t \\ +\int_s^t\left[\int_{\Gamma_1} Q(\cdot,{u_{|\Gamma}}_t){u_{|\Gamma}}_t-\int_\Omega f(\cdot,u)u_t-\int_{\Gamma_1} g(\cdot,u){u_{|\Gamma}}_t\right]=0$$

for all $s,t\in$ dom u, and the distribution identity

$$\tag{3.21} \left[\int_\Omega u_t\phi+\int_{\Gamma_1}{u_{|\Gamma}}_t\phi\right]_0^{T'}+\int_0^{T'}\left[-\int_\Omega u_t\phi_t-\int_{\Gamma_1}{u_{|\Gamma}}_t{\phi_{|\Gamma}}_t+\int_\Omega\nabla u\nabla\phi\right. \\ \left.+\int_{\Gamma_1}(\nabla_\Gamma u,\nabla_\Gamma\phi)_\Gamma+\int_\Omega P(\cdot,u_t)\phi+\int_{\Gamma_1}Q(\cdot,{u_{|\Gamma}}_t)\phi-\int_\Omega f(\cdot,u)\phi-\int_{\Gamma_1}g(\cdot,u)\phi\right]=0$$

for all $T'\in$ dom u and $\phi\in C([0,T'];H^1)\cap C^1([0,T'];H^0)\cap Z(0,T')$;

(ii) *if u is a weak solution of* (1.1) *in $[0,T_u]$, v is a weak solution of* (1.1) *with initial data v_0,v_1 in $[0,T_v]$ and $u(T_u)=v_0$, $\dot u(T_u)=v_1$ then w defined by $w(t)=u(t)$ when $t\in[0,T_u]$, $w(t)=v(t-T_u)$ when $t\in[T_u,T_u+T_v]$, is a weak solution of* (1.1) *in $[0,T_u+T_v]$;*

(iii) *for any $(\rho,\theta)\in[2,\max\{r_\Omega,m\}]\times[2,\max\{r_\Gamma,\mu\}]\cap\mathbb{R}^2$*

$$\tag{3.22} u_0\in H^{1,\rho,\theta}_{\alpha,\beta}\Rightarrow u\in C(\textit{dom } u;H^{1,\rho,\theta}_{\alpha,\beta});$$

(iv) *if dom $u=[0,T)$, $T<\infty$ and $U\in L^\infty(0,T;\mathcal H)$ then u is a weak solution in $[0,T]$ and $U\in C([0,T];\mathcal H)$.*

Proof. Clearly (i) follows from Lemma 2.1 while (ii) follows by (3.21) since (1.1) is autonomous. To prove (iii) let us take $u_0\in H^{1,\rho,\theta}_{\alpha,\beta}$. When $m\le r_\Omega$ then, by the trivial embedding $L^\rho(\Omega)\hookrightarrow L^{2,\rho}_\alpha(\Omega)$ and Sobolev embedding we get $u\in C(\text{dom } u;L^{2,\rho}_\alpha(\Omega))$, while when $m>r_\Omega$, so $\rho\le m$, as $\dot u\in Z(0,T')$ for all $T'\in$ dom u we get $u_t\in L^\rho(0,T';L^{2,\rho}_\alpha(\Omega))$, and hence $u\in W^{1,\rho}(0,T';L^{2,\rho}_\alpha(\Omega))\hookrightarrow C([0,T'];L^{2,\rho}_\alpha(\Omega))$ since $u_0\in L^{2,\rho}_\alpha(\Omega)$ and $u(t)=u_0+\int_0^t u_t(\tau,\cdot)\,d\tau$ in $L^2(\Omega)$. Then $u\in C(\text{dom } u;H^{1,\rho,2}_{\alpha,1})$. Since the same arguments show that $u\in C(\text{dom } u;H^{1,2,\theta}_{1,\beta})$ and $H^{1,\rho,\theta}_{\alpha,\beta}=H^{1,\rho,2}_{\alpha,1}\cap H^{1,2,\theta}_{1,\beta}$ we get (3.22).

To prove (iv), thanks to (i), we just have to prove that if dom $u=[0,T)$, $T<\infty$ and $U\in L^\infty(0,T;\mathcal H)$ then $\dot u\in Z(0,T)$. Set $S=\|U\|_{L^\infty(0,T;\mathcal H)}<\infty$. By the energy identity (3.20) and assumption (PQ3)

$$c'_m \int_0^t \|[u_t]_\alpha\|_{m,\alpha}^m + c'_\mu \int_0^t \|[u_{|\Gamma_t}]_\beta\|_{\mu,\beta,\Gamma_1}^\mu \le \tfrac{1}{2}\|U_0\|_{\mathcal{H}}^2 + \int_0^t \int_\Omega \widehat{f}(u)u_t + \int_0^t \int_{\Gamma_1} \widehat{g}(u)u_{|\Gamma_t} \qquad \text{for } t \in [0,T). \tag{3.23}$$

By Hölder and weighted Young inequalities together with Lemma 3.2 we have

$$\int_0^t \int_\Omega \widehat{f}(u)u_t \le \int_0^t c_p k_3(1+S^{p-1})\|u_t\|_{m_p} \le \tfrac{c'_m}{m_p}\int_0^t \|u_t\|_{m_p}^{m_p} + K_1, \tag{3.24}$$

where $K_1 = (c'_m)^{-1/m_p}[c_p k_3(1+S^{p-1})]^{m'_p}T$. We now distinguish between the cases $2 \le p \le 1 + r_\Omega/2$ and $p > 1 + r_\Omega/2$. In the first one, by (3.16), we have $m_p = 2$ so $\int_0^t \|u_t\|_{m_p}^{m_p} \le S^2 T$, while in the second one $m_p = m$ and, by assumption (FGQP1) and Remark 3.3, we have $\int_0^t \|u_t\|_{m_p}^{m_p} \le \int_0^t \|[u_t]_\alpha\|_{m,\alpha}^m$. Hence

$$\int_0^t \int_\Omega \widehat{f}(u)u_t \le \tfrac{c'_m}{m}\int_0^t \|[u_t]_\alpha\|_{m,\alpha}^m + K_2 \quad \text{for all } t \in [0,T), \tag{3.25}$$

where $K_2 = \{(c'_m)^{-1/m_p}[c_p k_3(1+S^{p-1})]^{m'_p} + c'_m S^2\}T$. Using the same arguments

$$\int_0^t \int_{\Gamma_1} \widehat{g}(u)u_{|\Gamma_t} \le \tfrac{c'_\mu}{\mu}\int_0^t \|[u_{|\Gamma_t}]_\beta\|_{\mu,\beta,\Gamma_1}^\mu + K_3 \quad \text{for all } t \in [0,T), \tag{3.26}$$

where $K_3 = \{(c'_\mu)^{-1/\mu_q}[c_q k_3(1+S^{q-1})]^{\mu'_q} + c'_\mu S^2\}T$. Plugging (3.25)–(3.26) in (3.23) we get

$$\tfrac{c'_m}{m'}\int_0^t \|[u_t]_\alpha\|_{m,\alpha}^m + \tfrac{c'_\mu}{\mu'}\int_0^t \|[u_{|\Gamma_t}]_\beta\|_{\mu,\beta,\Gamma_1}^\mu \le \tfrac{1}{2}\|U_0\|_{\mathcal{H}}^2 + K_2 + K_3 \quad \text{for all } t \in [0,T),$$

from which, since $\dot{u} \in L^\infty(0,T : H^0)$, we get $\dot{u} \in Z(0,T)$, concluding he proof. $\square$

3.3. **Additional assumptions.** The following properties of f and g will be assumed only in connection with (FG1) and for some values of $p, q > 3$ to be precised:

(F2) $f(x,\cdot) \in C^2(\mathbb{R})$ for a.a. $x \in \Omega$ and there is a constant $c''_p \ge 0$ such that

$$|f_u(x,u) - f_u(x,v)| \le c''_p|u-v|(1+|u|^{p-3}+|v|^{p-3}) \quad \text{for a.a. } x \in \Omega \text{ and all } u, v \in \mathbb{R};$$

(G2) $g(x,\cdot) \in C^2(\mathbb{R})$ for a.a. $x \in \Gamma_1$ and there is a constant $c''_q \ge 0$ such that

$$|g_u(x,u) - g_u(x,v)| \le c''_q|u-v|(1+|u|^{q-3}+|v|^{q-3}) \quad \text{for a.a. } x \in \Gamma_1 \text{ and all } u, v \in \mathbb{R}.$$

Remark 3.5. We point out that (F2) and (G2) are respectively equivalent to

(F2)$'$ $f(x,\cdot) \in C^2(\mathbb{R})$ for a.a. $x \in \Omega$ and there is a constant $\widetilde{c_p}'' \ge 0$ such that

$$|f_{uu}(x,u)| \le \widetilde{c_p}''(1+|u|^{p-3}) \quad \text{for a.a. } x \in \Omega \text{ and all } u \in \mathbb{R};$$

(G2)$'$ $g(x,\cdot) \in C^2(\mathbb{R})$ for a.a. $x \in \Gamma_1$ and there is a constant $\widetilde{c_q}'' \ge 0$ such that

$$|g_{uu}(x,u)| \le \widetilde{c_q}''(1+|u|^{q-3}) \quad \text{for a.a. } x \in \Gamma_1 \text{ and all } u \in \mathbb{R}.$$

Moreover (F2) implies (F2)$'$ with $\widetilde{c_p}'' = 2c''_p$, (G2) implies (G2)$'$ with $\widetilde{c_q}'' = 2c''_q$ and conversely (F2)$'$ implies (F2) with $\widetilde{c_p}'' = c''_p$, (G2)$'$ implies (G2) with $\widetilde{c_q}'' = c''_q$. Finally, in the case considered in problem (1.2), that is $f(x,u) = f_0(u)$ and $g(x,u) = g_0(u)$, (F2) means that $f_0 \in \mathcal{F}_p$, while (G2) that $g_0 \in \mathcal{F}_q$.

In the sequel we shall use one between the following two assumptions, the latter being trivially stronger than the former:

(FG2) $N \leq 4$ and (F2) holds when $1 + r_\Omega/2 < p = 1 + r_\Omega/m'$,
$N \leq 5$ and (G2) holds when $1 + r_\Gamma/2 < q = 1 + r_\Gamma/\mu'$;
(FG2)$'$ $N \leq 4$ and (F2) holds when $p > 1 + r_\Omega/2$,
$N \leq 5$ and (G2) holds when $q > 1 + r_\Gamma/2$.

Remark 3.6. By previous remark when $f(x,u) = f_0(u)$ and $g(x,u) = g_0(u)$ clearly (FG2) and (FG2)$'$ reduce to (IV) and (IV)$'$, pp. 4–5.

The following properties of P and Q will be assumed only for some values of p, q, m and μ to be specified later on:

(P4) there are constants $c''_m, M_m > 0$ such that

$$P_v(x,v) \geq c''_m\, \alpha(x)|v|^{m-2} \quad \text{for a.a. } (x,v) \in \Omega \times (\mathbb{R} \setminus (-M_m, M_m)), \tag{3.27}$$

(Q4) there are constants $c''_\mu, M_\mu > 0$ such that

$$Q_v(x,v) \geq c''_\mu\, \beta(x)|v|^{\mu-2} \quad \text{for a.a. } (x,v) \in \Gamma_1 \times (\mathbb{R} \setminus (-M_\mu, M_\mu)). \tag{3.28}$$

Remark 3.7. Since by (PQ1–2) the partial derivatives P_v and Q_v exist almost everywhere (see [20]) and are nonnegative, (3.27)–(3.28) always hold if one allows c''_m and c''_μ to vanish, and (P4) and (Q4) respectively reduce to ask that there is $M_m > 0$ such that one can take $c''_m > 0$ in (3.27) and that there is $M_\mu > 0$ such that one can take $c''_\mu > 0$ in (3.28).

In particular in our well–posedness result we shall use one between the following two assumptions:

(PQ4) if $p \geq r_\Omega$ then (P4) holds, if $q \geq r_\Gamma$ then (Q4) holds;
(PQ4)$'$ if $m > r_\Omega$ then (P4) holds, if $\mu > r_\Gamma$ then (Q4) holds.

Clearly, when (1.33) holds, (PQ4)$'$ is stronger than (PQ4) by (1.35).

Remark 3.8. We remark some trivial consequences of assumptions (PQ1–4) and (PQ1–3)–(PQ4)$'$. Setting $c''_m = 0$ when (P4) is not assumed to hold and $c''_\mu = 0$ when (Q4) is not assumed to hold, since $P_v, Q_v \geq 0$ a.e., from (P4) and (Q4) (when they are assumed) we have

$$P_v(x,v) \geq \alpha(x)\left[c''_m|v|^{m-2} - c'''_m\right] \quad \text{for a.a. } (x,v) \in \Omega \times \mathbb{R}, \tag{3.29}$$

$$Q_v(x,v) \geq \beta(x)\left[c''_\mu|v|^{\mu-2} - c'''_\mu\right] \quad \text{for a.a. } (x,v) \in \Gamma_1 \times \mathbb{R}, \tag{3.30}$$

where $c'''_m = c''_m M_m^{m-2}$, $c'''_\mu = c''_\mu M_\mu^{\mu-2}$. Then, by (PQ2), integrating (3.29) we get, for a.a. $x \in \Omega$ and all $v < w$,

$$P(x,w) - P(x,v) \geq \alpha(x)\left[\frac{c''_m}{m-1}\left(|w|^{m-2}w - |v|^{m-2}v\right) - c'''_m(w-v)\right]. \tag{3.31}$$

Consequently, using the elementary inequality

$$\left(|w|^{m-2}w - |v|^{m-2}v\right)(w-v) \geq \widetilde{c_m}|w-v|^m \qquad \text{for all } v, w \in \mathbb{R},$$

where $\widetilde{c_m}$ is a positive constant, setting $\widetilde{c_m}'' = c''_m\widetilde{c_m}/(m-1)$, from (3.31) we get

$$\widetilde{c_m}''\alpha(x)|v-w|^{\overline{m}} \leq c'''_m\alpha(x)|v-w|^2 + (P(x,w) - P(x,v))(w-v) \tag{3.32}$$

for a.a. $x \in \Omega$ and all $v, w \in \mathbb{R}$, with $\widetilde{c_m}'' > 0$ when $p \geq r_\Omega$ if (PQ4) is assumed and when $m > r_\Omega$ if (PQ4)$'$ is assumed. Using the same arguments we get from (3.30) the existence of $\widetilde{c_\mu}'' \geq 0$ such that

$$\widetilde{c_\mu}''\beta(x)|v-w|^{\overline{\mu}} \leq c_\mu'''\beta(x)|v-w|^2 + (Q(x,w)-Q(x,v))(w-v) \tag{3.33}$$

for a.a. $x \in \Gamma_1$ and all $v, w \in \mathbb{R}$, with $\widetilde{c_\mu}'' > 0$ when $q \geq r_\Gamma$ if (PQ4) is assumed and when $\mu > r_\Gamma$ if (PQ4)$'$ is assumed.

Remark 3.9. When $P(x,v) = \alpha(x)P_0(v)$ and $Q(x,v) = \beta(x)Q_0(v)$ with $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma_1)$, $\alpha, \beta \geq 0$, (PQ4) and (PQ4)$'$ reduce to (V) and (V)$'$, p. 6.

Remark 3.10. In the paper we shall introduce several positive constants depending on Ω, P, Q, f and g, and on the various constants appearing in the assumptions. Since they are fixed we shall denote these constants by k_i, $i \in \mathbb{N}$. We shall denote positive constants (possibly) depending on other objects $\Upsilon_1, \dots, \Upsilon_n$ by $K_i = K_i(\Upsilon_1, \dots, \Upsilon_n)$, $i \in \mathbb{N}$.

4. A key estimate

This section is devoted to give the key estimate which will be used when (FG2) or (FG2)$'$ hold, the Ω – version of which constitutes the content of the following

Lemma 4.1. *Suppose that* $\mathfrak{f}$ *satisfies (F1) with constants* $\mathfrak{c}_\mathfrak{p}$, $\mathfrak{c}_\mathfrak{p}'$, *that* (1.4) *holds and that either* $2 \leq p \leq 1 + r_\Omega/2$ *or* $p > 1 + r_\Omega/2$, $N \leq 4$ *and* $\mathfrak{f}$ *satisfies (F2) with constant* $\mathfrak{c}_\mathfrak{p}''$. *Let* $T \in (0,1]$, $U = (u, \dot{u}), V = (v, \dot{v}) \in C([0,T]; \mathcal{H})$, $\dot{u}, \dot{v} \in Z(0,T)$ *and denote* $W = U - V = (w, \dot{w})$, $U_0 = U(0) = (u_0, u_1)$, $V_0 = V(0) = (v_0, v_1)$, $W_0 = U_0 - V_0 = (w_0, w_1)$, $\boldsymbol{\mathfrak{c}_\mathfrak{p}} = (\mathfrak{c}_\mathfrak{p}, \mathfrak{c}_\mathfrak{p}')$, $\boldsymbol{\mathfrak{c}_\mathfrak{p}}' = (\mathfrak{c}_\mathfrak{p}, \mathfrak{c}_\mathfrak{p}', \mathfrak{c}_\mathfrak{p}'')$. *Suppose moreover that* $u_0, v_0 \in L^{s_\Omega}(\Omega)$ *and take* $R \geq 0$ *such that*

$$\|U\|_{C([0,T];\mathcal{H})},\ \|V\|_{C([0,T];\mathcal{H})},\ \|\dot{u}\|_{Z(0,T)},\ \|\dot{v}\|_{Z(0,T)},\ \|u_0\|_{s_\Omega},\ \|v_0\|_{s_\Omega} \leq R. \tag{4.1}$$

Then given any $\varepsilon > 0$ *there are*

$$K_4 = K_4(R, \boldsymbol{\mathfrak{c}_\mathfrak{p}}) \qquad \text{and} \qquad K_5 = K_5(\varepsilon, R, u_0, v_0, \boldsymbol{\mathfrak{c}_\mathfrak{p}}'),$$

independent on $\mathfrak{f}$ *and increasing in* R, *such that for all* $t \in [0,T]$

$$\begin{aligned} I_\mathfrak{f}(t) := \int_0^t \int_\Omega [\widehat{\mathfrak{f}}(u) - \widehat{\mathfrak{f}}(v)] w_t \leq K_4(\varepsilon + t)\|W(t)\|_\mathcal{H}^2 \\ + K_5 \left[\|W_0\|_\mathcal{H}^2 + \int_0^t (1 + \|u_t\|_{m_p} + \|v_t\|_{m_p}) \|W(\tau)\|_\mathcal{H}^2 \, d\tau \right]. \end{aligned} \tag{4.2}$$

Moreover, if $u_0, v_0 \in L^{s_1}(\Omega)$ *for some* $s_1 > s_\Omega$ *and, in addition to* (4.1), *we have*

$$\|u_0\|_{s_1}, \quad \|v_0\|_{s_1} \leq R, \tag{4.3}$$

then K_5 *is independent on* u_0 *and* v_0, *that is* $K_5(\varepsilon, R, u_0, v_0, \boldsymbol{\mathfrak{c}_\mathfrak{p}}') = K_6(\varepsilon, R, \boldsymbol{\mathfrak{c}_\mathfrak{p}}')$.

To prove Lemma 4.1 we shall use the following well–known abstract version of the Leibnitz formula, which can be proved as in [19, Theorem 2, p. 477].

Lemma 4.2. *Let* $T > 0$, X_1, X_2 *be Banach spaces,* $X_1 \hookrightarrow X_2$ *with dense embedding,* $u \in L^2(0,T;X_1)$, $v \in L^2(0,T;X_2')$, $\dot{u} \in L^2(0,T;X_2)$ *and* $\dot{v} \in L^2(0,T;X_1')$. *Then* $\langle v, u \rangle_{X_1} \in W^{1,1}(0,T)$ *and* $\langle v, u \rangle_{X_1}' = \langle v, \dot{u} \rangle_{X_2} + \langle \dot{v}, u \rangle_{X_1}$.

Proof of Lemma 4.1. We distinguish between the cases $2 \le p \le 1 + r_\Omega/2$ and $p > 1 + r_\Omega/2$ since the first one is trivial. Indeed, as $m_p = 2$, by Lemma 3.2, Hölder and Young inequalities and (4.1) we immediately get

$$\begin{aligned} I_{\mathfrak{f}}(t) \le & \int_0^t \|\widehat{\mathfrak{f}}(u) - \widehat{\mathfrak{f}}(v)\|_2 \|w_t\|_2 \le c_p' k_3 (1 + R^{p-2}) \int_0^t \|w\|_{H^1(\Omega)} \|w_t\|_2 \\ \le & \tfrac{1}{2} c_p' k_3 (1 + R^{p-2}) \int_0^t \|W(\tau)\|_{\mathcal{H}}^2 \, d\tau \end{aligned} \tag{4.4}$$

so getting (4.2) with $K_4 = 1$ and $K_5 = \frac{1}{2} c_p' k_3 (1 + R^{p-2})$.

The case $p > 1 + r_\Omega/2$ (so $r_\Omega < \infty$) is much more involved, and assumption $N \le 4$ and property (F2) will be essentially used. We set $\overline{s}_\Omega = \max\{r_\Omega, s_\Omega\} < \infty$ and we note that, when $p > r_\Omega$, by (1.15)–(1.16) we have $s_\Omega = r_\Omega (p-2)/(r_\Omega - 2) > p$. Hence, as $r_\Omega \ge 4$, for any value of p we have $3 < p \le \overline{s}_\Omega$. Since $m = \overline{m}$ by (1.4), we have

$$[\min\{m, \overline{s}_\Omega\}]' = \max\{m', \overline{s}_\Omega{}'\} \le \frac{\overline{s}_\Omega}{p-1} < \frac{\overline{s}_\Omega}{p-2}, \tag{4.5}$$

hence we can set $l_1 > 1$ by

$$\frac{1}{l_1} := \frac{p-2}{\overline{s}_\Omega} + \frac{1}{m}. \tag{4.6}$$

Moreover we note that, when $p > r_\Omega$ then, by (1.18) (which holds when $r_\Omega \ge 4$), $r_\Omega < s_\Omega < m$ so, since $u_0, v_0 \in L^{s_\Omega}(\Omega)$, $u(t) = u_0 + \int_0^t u_t$, $v(t) = v_0 + \int_0^t v_t$ in $L^2(\Omega)$ and $u_t, v_t \in L^m(0,T; L^m(\Omega))$ by (III), we have $u, v \in C([0,T]; L^{s_\Omega}(\Omega))$. Then, using Sobolev embedding when $p \le r_\Omega$, in general we get

$$u, v \in C([0,T]; L^{\overline{s}_\Omega}(\Omega)). \tag{4.7}$$

From (FG1)$'$ and (F2)$'$ and well–known continuity results for Nemitskii operators (see [2, Theorem 2.2, p.16]), we then get

$$\begin{aligned} &\widehat{\mathfrak{f}}(u), \widehat{\mathfrak{f}}(v) \in C([0,T]; L^{\overline{s}_\Omega/(p-1)}(\Omega)), \quad \widehat{\mathfrak{f}_u}(u), \widehat{\mathfrak{f}_u}(v) \in C([0,T]; L^{\overline{s}_\Omega/(p-2)}(\Omega)), \\ &\qquad \widehat{\mathfrak{f}_{uu}}(u), \widehat{\mathfrak{f}_{uu}}(v) \in C([0,T]; L^{\overline{s}_\Omega/(p-3)}(\Omega)). \end{aligned} \tag{4.8}$$

Moreover, being $u, v \in H^1((0,T) \times \Omega)$ we have $[\widehat{\mathfrak{f}}(u)]_t = \widehat{\mathfrak{f}}_u(u) u_t$ and $[\widehat{\mathfrak{f}}(v)]_t = \widehat{\mathfrak{f}}_u(v) v_t$ a.e. in $(0,T) \times \Omega$, so being $u_t, v_t \in L^m(0,T; L^m(\Omega))$ by (4.6) and (4.8) we have $[\widehat{\mathfrak{f}}(u)]_t, [\widehat{\mathfrak{f}}(v)]_t \in L^m(0,T; L^{l_1}(\Omega))$. By (4.5)–(4.6) we have $\overline{s}_\Omega{}' \le l_1$, so $[\widehat{\mathfrak{f}}(u)]_t, [\widehat{\mathfrak{f}}(v)]_t \in L^m(0,T; L^{\overline{s}_\Omega{}'}(\Omega))$. Hence, setting $X_1 = L^{\overline{s}_\Omega}(\Omega)$ and $X_2 = L^{\min\{m, \overline{s}_\Omega\}}(\Omega)$, by (4.5) we have $\widehat{\mathfrak{f}}(u), \widehat{\mathfrak{f}}(v) \in L^2(0,T; X_2')$, $w \in L^2(0,T; X_1)$, $[\widehat{\mathfrak{f}}(u)]_t, [\widehat{\mathfrak{f}}(v)]_t \in L^m(0,T; X_1')$ and $w_t \in L^2(0,T; X_2)$, so by Lemma 4.2 we can integrate by parts with respect to t in (4.2) to get, for $t \in [0,T]$,

$$I_{\mathfrak{f}}(t) = \int_\Omega [\widehat{\mathfrak{f}}(u) - \widehat{\mathfrak{f}}(v)] w \Big|_0^t - \int_0^t \int_\Omega \widehat{\mathfrak{f}}_u(u) w w_t - \int_0^t \int_\Omega [\widehat{\mathfrak{f}}_u(u) - \widehat{\mathfrak{f}}_u(v)] w v_t. \tag{4.9}$$

Now, since $m, \overline{s}_\Omega > 2$ we can set $l > 1$ by

$$\frac{1}{l} = \frac{1}{\overline{s}_\Omega} + \frac{1}{m}. \tag{4.10}$$

Since $p \le \overline{s}_\Omega$ and, by (1.4), $p \le 1 + \overline{s}_\Omega/m'$, we have

$$[\min\{\overline{s}_\Omega/2, l\}]' = \max\left\{(\overline{s}_\Omega/2)', l'\right\} \le \overline{s}_\Omega/(p-2). \tag{4.11}$$

Moreover, using (4.5) again, we can set $l_2 > 1$ by

$$\frac{1}{l_2} = \frac{p-3}{\overline{s}_\Omega} + \frac{1}{m}. \tag{4.12}$$

Since $[\widehat{\mathfrak{f}}_u(u)]_t = \widehat{\mathfrak{f}}_{uu}(u)u_t$ and $[\widehat{\mathfrak{f}}_u(v)]_t = \widehat{\mathfrak{f}}_{uu}(v)v_t$ a.e. in $(0,T)\times\Omega$, by (4.8) and (4.12) we have $[\widehat{\mathfrak{f}}_u(u)]_t, [\widehat{\mathfrak{f}}_u(v)]_t \in L^m(0,T;L^{l_2}(\Omega))$. Since $p \le 1 + \overline{s}_\Omega/m'$ we also have $(p-3)/\overline{s}_\Omega + 1/m \le (\overline{s}_\Omega - 2)/\overline{s}_\Omega$, hence by (4.12) we get $(\overline{s}_\Omega/2)' = \overline{s}_\Omega/(\overline{s}_\Omega-2) \le l_2$ and consequently $[\widehat{\mathfrak{f}}_u(u)]_t, [\widehat{\mathfrak{f}}_u(v)]_t \in L^m(0,T;L^{(\overline{s}_\Omega/2)'}(\Omega))$. Moreover, by (4.8) and (4.11) we have $\widehat{\mathfrak{f}}_u(u), \widehat{\mathfrak{f}}_u(v) \in C([0,T];L^{(\min\{\overline{s}_\Omega/2,l\})'}(\Omega))$. By Sobolev embedding $w^2 \in C([0,T];L^{\overline{s}_\Omega/2}(\Omega))$ and, as $u_t, v_t \in L^m(0,T;L^m(\Omega))$, $(w^2)_t = 2ww_t \in L^m(0,T;L^l(\Omega)) \hookrightarrow L^m(0,T;L^{\min\{l,\overline{s}_\Omega/2\}}(\Omega))$ by (4.10). From previous considerations we can apply Lemma 4.2 with $X_1 = L^{\overline{s}_\Omega/2}(\Omega)$ and $X_2 = L^{\min\{l,\overline{s}_\Omega/2\}}(\Omega)$ and integrate by parts with respect to t once again in the second addendum of (4.9) to get the final form of $I_{\mathfrak{f}}(t)$ suitable for our estimate, that is, for $t\in[0,T]$,

$$\begin{aligned} I_{\mathfrak{f}}(t) = \int_\Omega [\widehat{\mathfrak{f}}(u) - \widehat{\mathfrak{f}}(v)]w\Big|_0^t - \frac{1}{2}\int_\Omega \widehat{\mathfrak{f}}_u(u)w^2\Big|_0^t \\ + \frac{1}{2}\int_0^t\int_\Omega \widehat{\mathfrak{f}}_{uu}(u)w^2u_t - \int_0^t\int_\Omega [\widehat{\mathfrak{f}}_u(u) - \widehat{\mathfrak{f}}_u(v)]wv_t. \end{aligned}$$

By (FG1), (FG1)$'$, (F2), (F2)$'$ we then derive the preliminary estimate

$$\begin{aligned} I_{\mathfrak{f}}(t) \le & 2\mathfrak{c}_{\mathfrak{p}}{}'\int_\Omega (1+|u_0|^{p-2} + |v_0|^{p-2})w_0^2 + 2\mathfrak{c}_{\mathfrak{p}}{}'\|w(t)\|_2^2 \\ & + 2\mathfrak{c}_{\mathfrak{p}}{}'\int_\Omega (|u(t)|^{p-2} + |v(t)|^{p-2})w^2(t) + \mathfrak{c}_{\mathfrak{p}}{}''\int_0^t\int_\Omega (|u_t|+|v_t|)w^2 \\ & + \mathfrak{c}_{\mathfrak{p}}{}''\int_0^t\int_\Omega (|u|^{p-3} + |v|^{p-3})(|u_t|+|v_t|)w^2 \end{aligned} \tag{4.13}$$

for $t\in[0,T]$. In the sequel we shall estimate the addenda in the right–hand side of (4.13), denoting by $I_{\mathfrak{f}}^i(t)$ the i–th among them, for $i = 1,\dots,5$.

Estimate of $I_{\mathfrak{f}}^1(t)$. Denoting $\nu = \overline{s}_\Omega/(\overline{s}_\Omega - p + 2) > 1$, so $\nu' = \overline{s}_\Omega/(p-2)$, by Hölder inequality we have $\int_\Omega |u_0|^{p-2}w_0^2 \le \|u_0\|_{\overline{s}_\Omega}^{p-2}\|w_0\|_{2\nu}^2$. Since $\overline{s}_\Omega = \max\{r_\Omega, s_\Omega\}$, by (1.15) we have $\nu' \ge r_\Omega/(r_\Omega - 2)$, hence $2\nu \le r_\Omega$ so as $r_\Omega < \infty$ by Sobolev embedding and (4.1) $\int_\Omega |u_0|^{p-2}w_0^2 \le k_4R^{p-2}\|W_0\|_{\mathcal{H}}^2$. Estimating in the same way $\int_\Omega |v_0|^{p-2}w_0^2$

$$I_{\mathfrak{f}}^1(t) \le k_5\mathfrak{c}_{\mathfrak{p}}{}'(1+R^{p-2})\|W_0\|_{\mathcal{H}}^2. \tag{4.14}$$

Estimate of $I_{\mathfrak{f}}^2(t)$. By the trivial estimate

$$\|w(t)\|_2^2 \le 2\|w_0\|_2^2 + 2t\int_0^t \|w_t\|_2^2 \le 2\|w_0\|_2^2 + 2\int_0^t \|w_t\|_2^2 \tag{4.15}$$

where $0\le t\le T\le 1$ was used, we get

$$I_{\mathfrak{f}}^2(t) \le k_6\mathfrak{c}_{\mathfrak{p}}{}'\left(\|W_0\|_{\mathcal{H}}^2 + \int_0^t \|W(\tau)\|_{\mathcal{H}}^2\,d\tau\right). \tag{4.16}$$

Estimate of $I^4_{\mathfrak{f}}(t)$. Since $r_\Omega \ge 4$ so $r_\Omega/(r_\Omega-2)\le 2$, by Hölder inequality, Sobolev embedding and (4.1) we immediately get

$$I^4_{\mathfrak{f}}(t) \le \mathfrak{c}_{\mathfrak{p}}'' \int_0^t \|w\|^2_{r_\Omega}\left(\|u_t\|_{\frac{r_\Omega}{r_\Omega-2}} + \|v_t\|_{\frac{r_\Omega}{r_\Omega-2}}\right) \le k_7 \mathfrak{c}_{\mathfrak{p}}'' R\int_0^t \|W(\tau)\|^2_{\mathcal{H}}\, d\tau. \tag{4.17}$$

Estimate of $I^5_{\mathfrak{f}}(t)$. By Hölder inequality with exponents $r_\Omega/(p-3)$, $r_\Omega/(r_\Omega-p+1)$, $r_\Omega/2$, and Sobolev embedding, since by (1.4) we have $m=m_p\ge r_\Omega/(r_\Omega-p+1)$,

$$\begin{aligned} I^5_{\mathfrak{f}}(t) \le& \mathfrak{c}_{\mathfrak{p}}'' \int_0^t (\|u\|^{p-3}_{r_\Omega} + \|v\|^{p-3}_{r_\Omega})\left(\|u_t\|_{\frac{r_\Omega}{r_\Omega-p+1}} + \|v_t\|_{\frac{r_\Omega}{r_\Omega-p+1}}\right)\|w\|^2_{r_\Omega}\, d\tau\\ \le& k_8 \mathfrak{c}_{\mathfrak{p}}'' R^{p-3}\int_0^t \left(\|u_t\|_{m_p}+\|v_t\|_{m_p}\right)\|W(\tau)\|^2_{\mathcal{H}}\, d\tau. \end{aligned} \tag{4.18}$$

Estimate of $I^3_{\mathfrak{f}}(t)$. We shall distinguish between the two subcases $p<r_\Omega$ and $p\ge r_\Omega$. In the first one, by Hölder inequality with conjugate exponents $r_\Omega/(p-2)$ and $r_\Omega/(r_\Omega-p+2)$ and (4.1) we get

$$\int_\Omega |u(t)|^{p-2}w^2(t) \le \|u(t)\|^{p-2}_{r_\Omega}\|w(t)\|^2_{2r_\Omega/(r_\Omega-p+2)} \le k_9 R^{p-2}\|w(t)\|^2_{2r_\Omega/(r_\Omega-p+2)}.$$

Since $3<p<r_\Omega$ we have $2r_\Omega/(r_\Omega-p+2)\in(2,r_\Omega)$ and then, by interpolation and weighted Young inequalities we get, for any $\varepsilon>0$,

$$\int_\Omega |u(t)|^{p-2}w^2(t) \le k_9 R^{p-2}\|w(t)\|_2^{2\theta_1}\|w(t)\|^{2(1-\theta_1)}_{r_\Omega} \le k_9R^{p-2}\left(\frac{1}{\varepsilon}\|w(t)\|_2^2+\varepsilon\|w(t)\|^2_{r_\Omega}\right),$$

where $\theta_1\in(0,1)$ is given by $\frac{r_\Omega-p+2}{2r_\Omega}=\frac{\theta_1}{2}+\frac{1-\theta_1}{r_\Omega}$, and consequently, using (4.15),

$$\int_\Omega |u(t)|^{p-2}w^2(t) \le \frac14 k_{10}R^{p-2}\left(\frac1\varepsilon\|W_0\|_{\mathcal{H}}+\frac1\varepsilon\int_0^t\|W(\tau)\|^2_{\mathcal{H}}\,d\tau+\varepsilon\|W(t)\|^2_{\mathcal{H}}\right).$$

By estimating the term $\int_\Omega |v|^{p-2}w^2$ in the same way we get our estimate for $I^3_{\mathfrak{f}}(t)$ in case $1+r_\Omega/2<p<r_\Omega$, that is

$$I^3_{\mathfrak{f}}(t)\le k_{10}\mathfrak{c}_{\mathfrak{p}}' R^{p-2}\left(\frac1\varepsilon\|W_0\|^2_{\mathcal{H}}+\frac1\varepsilon\int_0^t\|W(\tau)\|^2_{\mathcal{H}}\,d\tau+\varepsilon\|W(t)\|^2_{\mathcal{H}}\right). \tag{4.19}$$

We now consider the subcase $p\ge r_\Omega$. Trivially

$$\int_\Omega |u(t)|^{p-2}w^2(t)\le 2^{p-3}\left(\int_\Omega |u(t)-u_0|^{p-2}w^2(t)+\int_\Omega |u_0|^{p-2}w^2(t)\right). \tag{4.20}$$

We shall estimate separately the two addenda inside brackets in (4.20). For the first one we use Hölder inequality with conjugate exponents $r_\Omega/(r_\Omega-2)$ and $r_\Omega/2$ to get, recalling that $s_\Omega=r_\Omega(p-2)/(r_\Omega-2)$ in this case,

$$\int_\Omega |u(t)-u_0|^{p-2}w^2\le \|u(t)-u_0\|^{p-2}_{s_\Omega}\|w(t)\|^2_{r_\Omega}\le k_{11}\left(\int_0^t\|u_t\|_{s_\Omega}\right)^{p-2}\|W(t)\|^2_{\mathcal{H}}.$$

Consequently, since by (1.18) we have $s_\Omega\le m$,

$$\int_\Omega |u(t)-u_0|^{p-2}w^2(t)\le k_{12}\left(\int_0^t\|u_t\|_m\right)^{p-2}\|W(t)\|^2_{\mathcal{H}}.$$

Then using Hölder inequality in time, (1.4) and Remark 3.3 we get

$$\int_\Omega |u(t)-u_0|^{p-2}w^2 \le k_{12}t^{(p-2)(m-1)/m}\left(\int_0^t \|[u_t]_\alpha\|_{\overline{m},\alpha}^{\overline{m}}\right)^{(p-2)/m}\|W(t)\|_{\mathcal{H}}^2.$$

Since $0\le t\le T\le 1$, $p\ge r_\Omega\ge 4$ and $m>2$, by (4.1) we get

$$\int_\Omega |u(t)-u_0|^{p-2}w^2 \le k_{12}R^{p-2}t\,\|W(t)\|_{\mathcal{H}}^2. \tag{4.21}$$

We now estimate the second addendum inside brackets in (4.20) when $u_0\in L^{s_1}(\Omega)$, $s_1>s_\Omega$ and (4.3) holds. Using Hölder inequality with conjugate exponents $s_1/(p-2)$, $s_1/(s_1-p+2)$ and (4.3)

$$\int_\Omega |u_0|^{p-2}w^2 \le \|u_0\|_{s_1}^{p-2}\|w(t)\|_{2s_1/(s_1-p+2)}^2 \le R^{p-2}\|w(t)\|_{2s_1/(s_1-p+2)}^2.$$

Since $s_1/(p-2)>s_\Omega/(p-2)=r_\Omega/(r_\Omega-2)$ we have $2s_1/(s_1-p+2)\in(2,r_\Omega)$, hence by interpolation and weighted Young inequalities, for $\varepsilon>0$ we get

$$\int_\Omega |u_0|^{p-2}w^2(t) \le R^{p-2}\|w(t)\|_2^{2\theta_2}\|w(t)\|_{r_\Omega}^{2(1-\theta_2)} \le R^{p-2}\left(\frac{1}{\varepsilon}\|w(t)\|_2^2+\varepsilon\|w(t)\|_{r_\Omega}^2\right),$$

where $\theta_2\in(0,1)$ is given by $\frac{s_1-p+2}{2s_1}=\frac{\theta_2}{2}+\frac{1-\theta_2}{r_\Omega}$. Consequently by (4.15) we get

$$\begin{aligned}\int_\Omega |u_0|^{p-2}w^2(t) \le& R^{p-2}\left[\frac{2}{\varepsilon}\left(\|w_0\|_2^2+\int_0^t\|w_t\|_2^2\right)+\varepsilon\|w(t)\|_{r_\Omega}^2\right]\\ \le& k_{13}R^{p-2}\left(\frac{1}{\varepsilon}\|W_0\|_{\mathcal{H}}^2+\frac{1}{\varepsilon}\int_0^t\|W(\tau)\|_{\mathcal{H}}^2\,d\tau+\varepsilon\|W(t)\|_{\mathcal{H}}^2\right).\end{aligned} \tag{4.22}$$

In the general case $u_0\in L^{s_\Omega}(\Omega)$ a different argument is needed. Since $L^\infty(\Omega)$ is dense in $L^{s_\Omega}(\Omega)$, in correspondence to $\varepsilon>0$ there is $\overline{u}_0=\overline{u_0}(\varepsilon,u_0)\in L^\infty(\Omega)$ such that $\|\overline{u_0}-u_0\|_{s_\Omega}\le\varepsilon^{1/(p-2)}$. Then, using Hölder inequality with conjugate exponents $r_\Omega/(r_\Omega-2)$, $r_\Omega/2$ and (4.15) we get

$$\begin{aligned}\int_\Omega |u_0|^{p-2}w^2(t) \le& 2^{p-3}\left(\int_\Omega |u_0-\overline{u_0}|^{p-2}w^2(t)+\int_\Omega|\overline{u_0}|^{p-2}w^2(t)\right)\\ \le& 2^{p-3}\left(\|\overline{u_0}-u_0\|_{s_\Omega}^{p-2}\|w(t)\|_{r_\Omega}^2+\|\overline{u_0}\|_\infty^{p-2}\|w(t)\|_2^2\right)\\ \le& 2^{p-3}\left[\varepsilon\|w(t)\|_{r_\Omega}^2+\|\overline{u_0}\|_\infty^{p-2}\left(2\|w_0\|_2^2+2\int_0^t\|w_t\|_2^2\right)\right]\\ \le& K_7(\varepsilon,u_0)\left(\|W_0\|_{\mathcal{H}}^2+\int_0^t\|W(\tau)\|_{\mathcal{H}}^2\,d\tau\right)+k_{14}\varepsilon\|W(t)\|_{\mathcal{H}}^2.\end{aligned} \tag{4.23}$$

Comparing (4.22) and (4.23) there are $K_8=K_8(\varepsilon,R,u_0)$ and $K_9=K_9(R)$, increasing in R, such that

$$\int_\Omega |u_0|^{p-2}w^2(t)\le K_8\left(\|W_0\|_{\mathcal{H}}^2+\int_0^t\|W(\tau)\|_{\mathcal{H}}^2\,d\tau\right)+\varepsilon K_9\|W(t)\|_{\mathcal{H}}^2, \tag{4.24}$$

with K_8 independent on u_0 when $u_0\in L^{s_1}(\Omega)$, $s_1>s_\Omega$ and (4.3) holds. Plugging (4.21) and (4.24) in (4.20) and using exactly the same arguments to estimate the term $\int_\Omega |v|^{p-2}w^2$ we then get our final estimate for $I_{\mathfrak{f}}^3(t)$ when $p\ge r_\Omega$, that is

$$I_{\mathfrak{f}}^3(t)\le \mathfrak{c}_{\mathfrak{p}}{}'K_{10}\left(\|W_0\|_{\mathcal{H}}^2+\int_0^t\|W(\tau)\|_{\mathcal{H}}^2\,d\tau\right)+\mathfrak{c}_{\mathfrak{p}}{}'K_{11}(\varepsilon+t)\|W(t)\|_{\mathcal{H}}^2 \tag{4.25}$$

with $K_{10} = K_{10}(\varepsilon, R, u_0, v_0)$, $K_{11} = K_{11}(R)$ increasing in R, K_{10} being independent on u_0, v_0 when $u_0, v_0 \in L^{s_1}(\Omega)$, $s_1 > s_\Omega$ and (4.3) holds. Comparing (4.25) with (4.19) and possibly changing the values of K_{10} and K_{11} we get that actually (4.25) is our final estimate for $I^3_{\mathfrak{f}}(t)$ for any $p > 1 + r_\Omega/2$.

By plugging (4.14)–(4.18) and (4.25) in (4.13) we get (4.2) when $p > 1 + r_\Omega/2$. $\square$

A trivial transposition of the arguments used in the proof of Lemma 4.1 allows to prove the following Γ − version of the estimate.

Lemma 4.3. *Suppose that $\mathfrak{g}$ satisfies (G1) with constants $\mathfrak{c}_\mathfrak{q}$, $\mathfrak{c}_\mathfrak{q}'$, that* (1.4) *holds and that either $2 \le q \le 1 + r_\Gamma/2$ or $q > 1 + r_\Gamma/2$, $N \le 5$ and $\mathfrak{g}$ satisfies (F2) with constant $\mathfrak{c}_\mathfrak{p}''$. Let $T \in (0,1]$, $U = (u, \dot{u}), V = (v, \dot{v}) \in C([0,T]; \mathcal{H})$, $\dot{u}, \dot{v} \in Z(0,T)$ and denote $W = U - V = (w, \dot{w})$, $U_0 = U(0) = (u_0, u_1)$, $V_0 = V(0) = (v_0, v_1)$, $W_0 = U_0 - V_0 = (w_0, w_1)$, $\boldsymbol{\mathfrak{c}}_\mathfrak{q} = (\mathfrak{c}_\mathfrak{q}, \mathfrak{c}_\mathfrak{q}')$, $\boldsymbol{\mathfrak{c}}_\mathfrak{q}' = (\mathfrak{c}_\mathfrak{q}, \mathfrak{c}_\mathfrak{q}', \mathfrak{c}_\mathfrak{q}'')$. Suppose moreover that $u_0, v_0 \in L^{s_\Gamma}(\Gamma_1)$ and take $R \ge 0$ such that*

$$\|U\|_{C([0,T];\mathcal{H})}, \|V\|_{C([0,T];\mathcal{H})}, \|\dot{u}\|_{Z(0,T)}, \|\dot{v}\|_{Z(0,T)}, \|u_0\|_{s_\Gamma,\Gamma_1}\|v_0\|_{s_\Gamma,\Gamma_1} \le R. \tag{4.26}$$

Then given any $\varepsilon > 0$ there are

$$K_{12} = K_{12}(R, \boldsymbol{\mathfrak{c}}_\mathfrak{q}), \qquad \text{and} \quad K_{13} = K_{13}(\varepsilon, R, u_0, v_0, \boldsymbol{\mathfrak{c}}_\mathfrak{q}'),$$

independent on $\mathfrak{g}$ and increasing in R, such that for all $t \in [0,T]$

$$\begin{aligned} \int_0^t \int_{\Gamma_1} [\widehat{\mathfrak{g}}(u) - \widehat{\mathfrak{g}}(v)] w_{|\Gamma_t} \le K_{12}(\varepsilon + t)\|W(t)\|^2_{\mathcal{H}} \\ + K_{13}\left[\|W_0\|^2_{\mathcal{H}} + \int_0^t (1 + \|u_{|\Gamma_t}\|_{\mu_q} + \|v_{|\Gamma_t}\|_{\mu_q})\|W(\tau)\|^2_{\mathcal{H}}\, d\tau\right]. \end{aligned} \tag{4.27}$$

Moreover, if $u_0, v_0 \in L^{s_2}(\Gamma_1)$ for some $s_2 > s_\Gamma$ and, in addition to (4.26), *we have*

$$\|u_{0|\Gamma}\|_{s_2,\Gamma_1}, \quad \|v_{0|\Gamma}\|_{s_2,\Gamma_1} \le R, \tag{4.28}$$

then K_{13} is independent on u_0 and v_0, that is $K_{13}(\varepsilon, R, u_0, v_0, \boldsymbol{\mathfrak{c}}_\mathfrak{q}') = K_{14}(\varepsilon, R, \boldsymbol{\mathfrak{c}}_\mathfrak{q}')$.

5. Local existence

This section is devoted to our local existence result for problem (1.1), that is

Theorem 5.1 (Local existence). *Suppose that (PQ1–3), (FG1–2) and (FGQP1) hold. Then all conclusions of Theorem 1.1 hold true when problem* (1.2) *is generalized to problem* (1.1), *provided the energy identity* (1.25) *is generalized to* (3.20).

To prove Theorem 5.1 we approximate, following a procedure from [12], problem (1.1) with a sequence of problems involving subcritical sources. We start by introducing a suitable cut–off sequence. At first we fix [11] $\eta_1 \in C^\infty_c(\mathbb{R})$ such that $\eta_1 \equiv 1$

[11] η_1 is easily built as follows. Let $\eta_0 \in W^{1,\infty}(\mathbb{R})$ be defined by $\eta_0(\tau) = 1$ for $|\tau| \le 5/4$, $\eta_0(\tau) = 0$ for $|\tau| \ge 7/4$, linear for $5/4 \le |\tau| \le 7/4$, and $(\rho_n)_n$ the standard mollifying sequence in $\mathbb{R}$ defined at [13, p. 108]. Then $\eta_1 = \rho_4 * \eta_0$ satisfies the required properties. Indeed $\|\eta_0'\|_\infty = 2$ so $\|\eta_1'\|_\infty \le 2$. Moreover $\rho_4(x) = 4\rho_1(4x)/\int_{-1}^1 \rho_1$, where $\rho_1(x) = e^{1/(x^2-1)}$ in $(-1,1)$, vanishing outside, $\|\rho_1'\|_\infty \le 2\max_{y\ge 0} y^2 e^{-y} = 8e^{-2}$ and $\int_{-1}^1 \rho_1 \ge 2\int_0^{1/\sqrt{2}} \rho_1 > e^{-2}$ so $\|\rho_4'\|_\infty \le 2^7$ and consequently $\|\eta_1''\|_\infty \le \|\rho_4'\|_\infty \|\eta_0'\|_1 = 2\|\rho_4'\|_\infty \le 2^8$.

in $[-1,1]$, supp $\eta_1 \subset [-2,2]$, $0 \le \eta_1 \le 1$, $\|\eta_1'\|_\infty \le 2$ and $\|\eta_1''\|_\infty \le 2^8$. Then we set the sequence $(\eta_n)_n$ by $\eta_n(x) = \eta_1(x/n)$. For all $n \in \mathbb{N}$ we have

$$\begin{aligned} &\eta_n \in C_c^\infty(\mathbb{R}), \quad \eta_n \equiv 1 \text{ in } [-n,n], \quad \text{supp } \eta_n \subset [-2n,2n],\\ &0 \le \eta_n \le 1, \qquad \|\eta_n'\|_\infty \le 2/n, \qquad \|\eta_n''\|_\infty \le 2^8/n^2. \end{aligned} \tag{5.1}$$

We then define, for f and g satisfying (FG1) and $n \in \mathbb{N}$, the truncated nonlinearities f^n and g^n by setting, for all $u \in \mathbb{R}$, a.a. $x \in \Omega$ and $y \in \Gamma_1$,

$$f^n(x,u) = \eta_n(u) f(x,u), \qquad g^n(y,u) = \eta_n(u) g(y,u). \tag{5.2}$$

By (FG1), Remark 3.2 and (5.1) for each $n \in \mathbb{N}$ we trivially have

$$\begin{aligned} &|f^n(\cdot,0)| \le c_p, \quad |f^n_u(\cdot,u)| \le \left[2c_p'(1+|u|^{p-2}) + 2c_p(1+|u|^{p-1})/n\right] \chi_{A_n}(u)\\ &|g^n(\cdot,0)| \le c_q, \quad |g^n_u(\cdot,u)| \le \left[2c_q'(1+|u|^{q-2}) + 2c_q(1+|u|^{q-1})/n\right] \chi_{A_n}(u) \end{aligned} \tag{5.3}$$

where χ_{A_n} denotes the characteristic function of $A_n = [-2n,2n]$, hence f^n and g^n satisfy assumptions (FG1) with exponents $p = q = 2$ and constants dependent on n. Then, by [55, Theorem 3.1] for each $U_0 \in \mathcal{H}$ and $n \in \mathbb{N}$ the approximating problem

$$\begin{cases} u^n_{tt} - \Delta u^n + P(x,u^n_t) = f^n(x,u^n) & \text{in } (0,\infty)\times\Omega,\\ u^n = 0 & \text{on } (0,\infty)\times\Gamma_0,\\ u^n_{tt} + \partial_\nu u^n - \Delta_\Gamma u^n + Q(x,u^n_t) = g^n(x,u^n) & \text{on } (0,\infty)\times\Gamma_1,\\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & \text{in } \overline{\Omega} \end{cases} \tag{5.4}$$

has a unique weak solution u^n with $U^n \in C([0,\infty);\mathcal{H})$. The strategy of the proof of Theorem 5.1 is to pass to the limit as $n \to \infty$ in (5.4). With this aim we point out the following uniform estimates on f^n, g^n and the Nemitskii operators $\widehat{f^n}$ and $\widehat{g^n}$ associated with them.

Lemma 5.1. *Let (PQ1–3), (FG1) and (FGQP1) hold. Then:*

(i) *for all $n \in \mathbb{N}$ the couples (f^n, g^n) satisfy (FG1) with constants*
$$\overline{\boldsymbol{c}_p} = (\overline{c_p}, \overline{c_p}') = (c_p, 4(c_p + c_p')) \quad \text{and} \quad \overline{\boldsymbol{c}_q} = (\overline{c_q}, \overline{c_q}') = (c_q, 4(c_q + c_q'));$$

(ii) *for any $\rho \in [p-1,\infty)$ and $\theta \in [q-1,\infty)$, $\widehat{f^n} : L^\rho(\Omega) \to L^{\rho/(p-1)}(\Omega)$, $\widehat{f^n} : H^1(\Omega) \to L^{m_p'}(\Omega)$, $\widehat{g^n} : L^\theta(\Gamma_1) \to L^{\theta/(q-1)}(\Gamma_1)$ and $\widehat{g^n} : H^1(\Gamma) \cap L^2(\Gamma_1) \to L^{\mu_q'}(\Gamma_1)$ are locally Lipschitz and bounded, uniformly in $n \in \mathbb{N}$, and for any $R \ge 0$ we have*

$$\begin{aligned} &\|\widehat{f^n}(u)\|_{\rho/(p-1)} &&\le \overline{c_p} k_1 (1 + R^{p-1}),\\ &\|\widehat{g}(\widetilde{u})\|_{\theta/(q-1),\Gamma_1} &&\le \overline{c_q} k_2 (1 + R^{q-1}),\\ &\|\widehat{f^n}(u) - \widehat{f^n}(v)\|_{\rho/(p-1)} &&\le \overline{c_p'} k_1 (1 + R^{p-2}) \|u - v\|_\rho,\\ &\|\widehat{g^n}(\widetilde{u}) - \widehat{g^n}(\widetilde{v})\|_{\theta/(q-1),\Gamma_1} &&\le \overline{c_q'} k_2 (1 + R^{q-2}) \|\widetilde{u} - \widetilde{v}\|_{\theta,\Gamma_1} \end{aligned} \tag{5.5}$$

provided $\|u\|_\rho, \|v\|_\rho, \|\widetilde{u}\|_{\theta,\Gamma_1}, \|\widetilde{v}\|_{\theta,\Gamma_1} \le R$, and

$$\begin{aligned} &\|\widehat{f^n}(u)\|_{m_p'} &&\le \overline{c_p} k_3 (1 + R^{p-1}),\\ &\|\widehat{g^n}(\widetilde{u})\|_{\mu_q',\Gamma_1} &&\le \overline{c_q} k_3 (1 + R^{q-1}),\\ &\|\widehat{f^n}(u) - \widehat{f^n}(v)\|_{m_p'} &&\le \overline{c_p'} k_3 (1 + R^{p-2}) \|u - v\|_{H^1(\Omega)},\\ &\|\widehat{g^n}(\widetilde{u}) - \widehat{g^n}(\widetilde{v})\|_{\mu_q',\Gamma_1} &&\le \overline{c_q'} k_3 (1 + R^{q-2}) \|\widetilde{u} - \widetilde{v}\|_{H^1(\Gamma)}, \end{aligned} \tag{5.6}$$

provided $\|u\|_{H^1(\Omega)}, \|v\|_{H^1(\Omega)}, \|\widetilde{u}\|_{H^1(\Gamma)}, \|\widetilde{v}\|_{H^1(\Gamma)} \le R$;

(iii) *when f and g satisfy also (FG2) or (FG2)′ then f^n and g^n satisfy the same assumption with constants*

$$\overline{c_p}'' = 2^{10}c_p + 16c_p' + 2c_p'', \quad \textit{and} \quad \overline{c_q}'' = 2^{10}c_q + 16c_q' + 2c_q'', \tag{5.7}$$

hence (f^n, g^n) satisfy (FG1–2) or (FG1)–(FG2)′ with constants $\overline{\boldsymbol{c}_p}' = (\overline{\boldsymbol{c}_p}, \overline{c_p}'')$ and $\overline{\boldsymbol{c}_q}' = (\overline{\boldsymbol{c}_q}, \overline{c_q}'')$, independent on $n \in \mathbb{N}$.

Proof. We first note that since $\chi_{A_n}(u)|u| \le 2n$, from (5.3) we also get $|f_u^n(\cdot,u)| \le 2(c_p + c_p') + 4(c_p + c_p')|u|^{p-2}$ and $|g_u^n(\cdot,u)| \le 2(c_q + c_q') + 4(c_q + c_q')|u|^{q-2}$, and by combining them with (5.3) and Remark 3.2 we complete the proof of (i). By combining Lemma 3.2 with (i) we immediately derive (ii). To prove (iii) we note that, when (F2) holds true we have, by (5.1),

$$|f_{uu}^n(\cdot,u)| \le \left[\tfrac{2^8}{n^2}|f(\cdot,u)| + \tfrac{4}{n}|f_u(\cdot,u)| + |f_{uu}(\cdot,u)|\right]\chi_{A_n}(u)$$

and consequently, using (FG1)′ and (F2)′, see Remarks 3.2 and 3.5, we get

$$|f_{uu}^n(\cdot,u)| \le \left[\tfrac{2^8}{n^2}c_p(1+|u^{p-1}) + \tfrac{8}{n}c_p'(1+|u|^{p-2}) + 2c_p''(1+|u|^{p-3})\right]\chi_{A_n}(u),$$

and then (F2) follows since $\chi_{A_n}(u)|u| \le 2n$, using Remark 3.5 again. By the same arguments we get (G2). □

Our first main estimate on the sequence $(u^n)_n$ is the following one.

Lemma 5.2. *If (PQ1–3), (FG1) and (FGQP1) hold there are a decreasing function $T_1 : [0,\infty) \to (0,1]$ and an increasing one $\kappa : [0,\infty) \to [0,\infty)$, such that*

$$\|U^n\|_{C([0,T_1(\|U_0\|_{\mathcal{H}})];\mathcal{H})} \le 1 + 2\|U_0\|_{\mathcal{H}}, \quad \textit{for all } n \in \mathbb{N}, \tag{5.8}$$

$$\|\dot{u}^n\|_{Z(0,T_1(\|U_0\|_{\mathcal{H}}))} \le \kappa(\|U_0\|_{\mathcal{H}}), \quad \textit{for all } n \in \mathbb{N}. \tag{5.9}$$

Proof. We denote $R = 1 + 2\|U_0\|_{\mathcal{H}}$. Since, as already noted, f^n and g^n satisfy assumptions (FG1) with exponents $p = q = 2$ then, by Lemma 3.2, the Nemitskii operators $\widehat{f^n} : H^1(\Omega) \to L^2(\Omega)$ and $\widehat{g^n} : H^1(\Gamma) \cap L^2(\Gamma_1) \to L^2(\Gamma_1)$ are, possibly not uniformly in n, locally Lipschitz. Hence we can introduce, as in [12, 16], their globally Lipschitz truncations $\widehat{f_R^n} : H^1(\Omega) \to L^2(\Omega)$ and $\widehat{g_R^n} : H^1(\Gamma) \cap L^2(\Gamma_1) \to L^2(\Gamma_1)$, given by $\widehat{f_R^n} = \widehat{f^n} \cdot \Pi_{H^1(\Omega),R}$ and $\widehat{g_R^n} = \widehat{g^n} \cdot \Pi_{H^1(\Gamma)\cap L^2(\Gamma_1),R}$, where in any Hilbert space H we denote by $\Pi_{H,R} : H \to B_R(H)$ the projection onto the ball $B_R(H)$ of radius R centered at 0 in H, given for $u \in H$ (see [13, Theorem 5.2]) by

$$\Pi_{H,R}(u) = \begin{cases} u & \text{if } \|u\|_H \le R, \\ Ru/\|u\|_H & \text{otherwise.} \end{cases}$$

Then the operator couple $F_R^n = (\widehat{f_R^n}, \widehat{g_R^n})$ is globally Lipschitz from H^1 to H^0, and consequently by applying [55, Theorem 3.2] the abstract Cauchy problem

$$\begin{cases} \ddot{v}^n + Av^n + B(\dot{v}^n) = F_R^n(v^n) & \text{in } X', \\ v^n(0) = u_0, \quad \dot{v}^n(0) = u_1 \end{cases} \tag{5.10}$$

has a unique global weak solution $v^n \in C([0,\infty);H^1)\cap C^1([0,\infty);H^0)$, which (see [55, Remark 3.6]) satisfies the energy identity

$$\tag{5.11} \begin{aligned} &\tfrac12\|\dot v^n\|_{H^0}^2+\tfrac12\|v^n\|_{H^1}^2\Big|_0^t+\int_0^t\langle B(\dot v^n,\dot v^n\rangle_Y\\ &\quad=\tfrac12\|v^n\|_{2,\Gamma_1}^2\Big|_0^t+\int_0^t\int_\Omega \widehat{f^n_R}(v^n)v^n_t+\int_0^t\int_{\Gamma_1}\widehat{g^n_R}(v^n)v^n_{|\Gamma_1 t},\qquad t\in[0,\infty). \end{aligned}$$

Setting $\varepsilon_1=\min\{1,c'_m,c'_\mu\}>0$, by weighted Young inequality we have

$$\int_0^t\int_\Omega \widehat{f^n_R}(v^n)v^n_t\le \tfrac{\varepsilon_1}{m_p}\int_0^t\|v^n_t\|_{m_p}^{m_p}+\tfrac{\varepsilon_1^{1-m_p'}}{m_p'}\int_0^t\|\widehat{f^n_R}(v^n)\|_{m_p'}^{m_p'}$$

and then, using the definition on $\widehat{f^n_R}$, (5.6), Lemma 5.1 and the fact that $m_p\ge 2$ and $R\ge1$, we get

$$\int_0^t\int_\Omega \widehat{f^n_R}(v^n)v^n_t\le \tfrac{\varepsilon_1}{2}\int_0^t\|v^n_t\|_{m_p}^{m_p}+k_{14}c_p^{m_p'}(1+R^{2p})\,t,\qquad \text{for all } t\in[0,\infty).$$

Using the same arguments to estimate the term $\int_0^t\int_{\Gamma_1}\widehat{g^n_R}(v^n)v^n_{|\Gamma_1 t}$ we obtain

$$\tag{5.12} \begin{aligned} &\int_0^t\int_\Omega \widehat{f^n_R}(v^n)v^n_t+\int_0^t\int_{\Gamma_1}\widehat{g^n_R}(v^n)v^n_{|\Gamma_1 t}\\ &\qquad\le\tfrac{\varepsilon_1}{2}\int_0^t\left(\|v^n_t\|_{m_p}^{m_p}+\|v^n_{|\Gamma_1 t}\|_{\mu_q}^{\mu_q}\right)+k_{15}(1+R^{2(p+q)})\,t. \end{aligned}$$

By Young and Hölder inequality in time we have

$$\tfrac12\|v^n(t)\|_{2,\Gamma_1}^2\le\tfrac12\|v^n(t)\|_{H^0}^2\le\tfrac12\left(\|u_0\|_{H^0}+\int_0^t\|\dot v^n\|_{H^0}\right)^2\le\|u_0\|_{H^0}^2+t\int_0^t\|\dot v^n(\tau)\|_{H^0}^2\,d\tau,$$

so, plugging it and (5.12) in (5.11) and denoting $V^n=(v^n,\dot v^n)$, we get

$$\begin{aligned} &\tfrac12\|V^n(t)\|_{\mathcal H}^2+\int_0^t\langle B(\dot v^n),\dot v^n\rangle_Y\le\tfrac32\|U_0\|_{\mathcal H}^2+\int_0^t\|V^n(\tau)\|_{\mathcal H}^2\,d\tau\\ &\qquad+\tfrac{\varepsilon_1}{2}\int_0^t\left(\|v^n_t\|_{m_p}^{m_p}+\|v^n_{|\Gamma_1 t}\|_{\mu_q}^{\mu_q}\right)+k_{15}(1+R^{2(p+q)})\,t\qquad \text{for } t\in[0,1]. \end{aligned}$$

Denoting

$$\tag{5.13} \begin{aligned} \mathcal I^n_f(t)=&c'_m\int_0^t\|[v^n_t]_\alpha\|_{m,\alpha}^m+\tfrac12\|v^n_t(t)\|_2^2-\tfrac{\varepsilon_1}{2}\|v^n_t(t)\|_{m_p}^{m_p},\\ \mathcal I^n_g(t)=&c'_\mu\int_0^t\|[v^n_{|\Gamma_1 t}(t)]_\beta\|_{\mu,\beta,\Gamma_1}^\mu+\tfrac12\|v^n_{|\Gamma_1 t}(t)\|_{2,\Gamma_1}^2-\tfrac{\varepsilon_1}{2}\|v^n_{|\Gamma_1 t}(t)\|_{\mu_q}^{\mu_q}, \end{aligned}$$

and using assumption (PQ3) in previous formula, we get

$$\tag{5.14} \tfrac12\|V^n(t)\|_{\mathcal H}^2+\mathcal I^n_f(t)+\mathcal I^n_g(t)\le\tfrac32\|U_0\|_{\mathcal H}^2+\tfrac32\int_0^t\|V^n\|_{\mathcal H}^2\,d\tau+k_{15}(1+R^{2(p+q)})\,t$$

for $t\in[0,1]$. Now we remark that, by (3.16), when $p\le 1+r_\Omega/2$ we have $m_p=2$ so, as $\varepsilon_1\le1$, $\mathcal I^n_f(t)\ge c'_m\int_0^t\|[v^n_t]_\alpha\|_{m,\alpha}^m$. When $p>1+r_\Omega/2$ we have $m_p=m>2$ and, by assumption (FGQP1) and Remark 3.3, we have $\|v^n_t\|_m^m\le\|[v^n_t]_\alpha\|_{m,\alpha}^m$ and then, since $\varepsilon_1 \le c'_m$, $\mathcal{I}^n_f(t) \ge \frac12 c'_m \int_0^t \|[v^n_t]_\alpha\|^m_{m,\alpha}$. Using the same arguments to estimate from below $\mathcal{I}^n_g(t)$ we then get from (5.14)

$$(5.15)\quad \begin{aligned} &\tfrac12\|V^n(t)\|^2_{\mathcal{H}} + \tfrac12 c'_m \int_0^t \|[v^n_t]_\alpha\|^m_{m,\alpha} + \tfrac12 c'_\mu \int_0^t \|[(v^n_{|\Gamma_1})_t]_\beta\|^\mu_{\mu,\beta,\Gamma_1} \\ &\le \tfrac32\|U_0\|^2_{\mathcal{H}} + \tfrac32 \int_0^t \|V^n(\tau)\|^2_{\mathcal{H}}\,d\tau + k_{15}(1+R^{2(p+q)})\,t \qquad \text{for } t\in[0,1]. \end{aligned}$$

Disregarding the second and third terms in the left–hand side of (5.15) and using Gronwall inequality (see [47, Lemma 4.2, p. 179]) we get

$$\|V^n(t)\|^2_{\mathcal{H}} \le 3\|U_0\|^2_{\mathcal{H}} e^{3t} + 2e^3 k_{15}(1+R^{2(p+q)})\,t \qquad \text{for } t\in[0,1].$$

Consequently

$$(5.16)\qquad \|V^n(t)\|^2_{\mathcal{H}} \le 1 + 4\|U_0\|^2_{\mathcal{H}} \le (1+2\|U_0\|_{\mathcal{H}})^2 = R^2,$$

provided $3e^{3t}\le 4$ and $2e^3k_{15}(1+R^{2(p+q)})t\le 1$, that is $t\in[0,T_1]$, where

$$T_1 = T_1(\|U_0\|_{\mathcal{H}}) := \min\left\{\tfrac13 \log\tfrac43, [2e^3k_{15}(1+(1+2\|U_0\|_{\mathcal{H}})^{2(p+q)}]^{-1}\right),$$

which is trivially decreasing. By the definitions of $\widehat{f^n_R}$, $\widehat{g^n_R}$ and F^n_R then we have $F^n_R(v^n)(t) = (\widehat{f^n}(v^n)(t), \widehat{g^n}(v^n)(t))$ for $t\in[0,T_1]$, so v^n is a weak solution of (5.4) in $[0,T_1]$. Since weak solutions of (5.4) are unique we get $v^n = u^n$ in $[0,T_1]$, so (5.8) is nothing but (5.16). To prove (5.9) we note that plugging (5.8) in (5.15), since $v^n=u^n$ in $[0,T_1]$ and $2(p+q)\ge 2$, we get

$$(5.17)\qquad c'_m \int_0^{T_1} \|[u^n_t]_\alpha\|^m_{m,\alpha} + c'_\mu \int_0^{T_1} \|[(u^n_{|\Gamma_1})_t]_\beta\|^\mu_{\mu,\beta,\Gamma_1} \le k_{16}\left[1+\|U_0\|^{2(p+q)}_{\mathcal{H}}\right].$$

Recalling that $\overline{m}=\max\{2,m\}$ and $\overline{\mu}=\max\{2,\mu\}$ we have

$$\begin{aligned} \|[u^n_t]_\alpha\|_{\overline{m},\alpha} &\le \|\alpha\|_\infty \|u^n_t\|_2 + \|[u^n_t]_\alpha\|_{m,\alpha}, \\ \|[u^n_{|\Gamma_t}]_\beta\|_{\overline{\mu},\beta,\Gamma_1} &\le \|\beta\|_{\infty,\Gamma_1}\|u^n_{|\Gamma_t}\|_{2,\Gamma_1} + \|[u^n_{|\Gamma_t}]_\beta\|_{\mu,\beta,\Gamma_1} \end{aligned}$$

so by (5.8) and (5.17) we immediately get (5.9), completing the proof. □

To pass to the limit as $n\to\infty$ we shall use the following density result, which is proved in Appendix A for the reader's convenience.

Lemma 5.3. *Let $0<T<\infty$. Then $C^1_c((0,T);H^{1,\infty,\infty})$ is dense in $C_c((0,T);H^1)\cap C^1_c((0,T);H^0)\cap Z(0,T)$ with the norm of $C([0,T];H^1)\cap C^1([0,T];H^0)\cap Z(0,T)$.*

We set

$$(5.18)\quad \widetilde{\sigma_\Omega} = \begin{cases} s_\Omega & \text{if } 1+\frac{r_\Omega}{2} < p = 1+\frac{r_\Omega}{m'}, \\ 2 & \text{otherwise,}\end{cases} \text{ and } \widetilde{\sigma_\Gamma} = \begin{cases} s_\Gamma & \text{if } 1+\frac{r_\Gamma}{2} < q = 1+\frac{r_\Gamma}{\mu'}, \\ 2 & \text{otherwise,}\end{cases}$$

so, by (1.16), we have $H^{1,\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}} = H^{1,\sigma_\Omega,\sigma_\Gamma}$ and consequently $\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}} = \mathcal{H}^{\sigma_\Omega,\sigma_\Gamma}$.

The following result is a main step in the proof of Theorem 5.1.

Proposition 5.1. *Under the assumptions of Theorem 5.1 there is a decreasing function $T_2 : [0,\infty) \to (0,1]$, $T_2 \le T_1$, such that for any $U_0 \in \mathcal{H}^{\sigma_\Omega,\sigma_\Gamma}$ problem* (1.1) *has a weak solution u in $[0, T_2(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}})]$. Moreover*

$$\|U\|_{C([0,T_2(\|U_0\|_{\mathcal{H}})];\mathcal{H})} \le 1 + 2\|U_0\|_{\mathcal{H}}, \tag{5.19}$$

$$\|\dot u\|_{Z(0,T_2(\|U_0\|_{\mathcal{H}}))} \le \kappa(\|U_0\|_{\mathcal{H}}). \tag{5.20}$$

Proof. By Lemmas 3.1 and 5.2 the sequences $(u^n)_n$, $(\dot u^n)_n$ and $(B(\dot u^n))_n$ are bounded, respectively, in $L^\infty(0,T_1;H^1)$, $L^\infty(0,T_1;H^0)\cap Z(0,T_1)$ and $L^{\overline{m}'}(0,T_1;[L^{\overline{m}}(\Omega,\lambda_\alpha)]')\times L^{\overline{\mu}'}(0,T_1;[L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)]')$. Moreover by (1.4) we have $p < 1 + r_\Omega$ and $q < 1 + r_\Gamma$ so by Rellich–Kondrachov theorem (see [13, Theorem 9.16, p. 285] and [33, Theorem 2.9, p. 39]) the embeddings $H^1(\Omega) \hookrightarrow L^{p-1}(\Omega)$ and $H^1(\Gamma) \hookrightarrow L^{q-1}(\Gamma)$ are compact. Then, using Simon's compactness results (see [48, Corollary 5, p. 86]) we get that, up to a subsequence,

$$\begin{cases} u^n \rightharpoonup^* u & \text{in } L^\infty(0,T_1;H^1),\\ \dot u^n \rightharpoonup^* \dot u & \text{in } L^\infty(0,T_1;H^0),\\ u^n \to u & \text{in } C([0,T_1];L^{p-1}(\Omega)\times L^{q-1}(\Gamma_1)),\\ \dot u^n \rightharpoonup \dot u & \text{in } Z(0,T_1),\\ B(\dot u^n) \rightharpoonup \chi & \text{in } L^{\overline{m}'}(0,T_1;[L^{\overline{m}}(\Omega,\lambda_\alpha)]')\times L^{\overline{\mu}'}(0,T_1;[L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)]'), \end{cases} \tag{5.21}$$

where $\to$, $\rightharpoonup$ and $\rightharpoonup^*$ respectively stand for strong, weak and weak * convergence. Hence, by (5.8)–(5.9), estimates (5.19)–(5.20) will be granted for any choice of $T_2 \le T_1$. Since u^n is a weak solution of (5.4) in $[0,T_1]$, by Definition 3.1 we have

$$\begin{gathered}\int_0^{T_1} [-(\dot u^n,\dot\varphi)_{H^0} + \langle Au^n,\varphi\rangle_{H^1} + \langle B(\dot u^n),\varphi\rangle_Y]\\ = \int_0^{T_1}\int_\Omega \widehat{f^n}(u^n)\varphi + \int_{\Gamma_1}\widehat{g^n}(u^n)\varphi \qquad \text{for all } \varphi\in C_c^1((0,T_1);H^{1,\infty,\infty}).\end{gathered} \tag{5.22}$$

We now pass to the limit in (5.3) as $n\to\infty$. By (5.21) we immediately get

$$\begin{gathered}\lim_n \int_0^{T_1} [-(\dot u^n,\dot\varphi)_{H^0} + \langle Au^n,\varphi\rangle_{H^1} + \langle B(\dot u^n),\varphi\rangle_Y]\\ = \int_0^{T_1} [-(\dot u,\dot\varphi)_{H^0} + \langle Au,\varphi\rangle_{H^1} + \langle \chi,\varphi\rangle_Y].\end{gathered} \tag{5.23}$$

To pass to the limit in the first term in right–hand side of (5.22) we note that by combining (5.21) and (5.5) with $\rho = p-1$ we have $\widehat{f^n}(u^n) - \widehat{f^n}(u) \to 0$ in $C([0,T_1];L^1(\Omega))$, hence a fortiori in $L^1(0,T_1;L^1(\Omega))$. Next, by (5.1)—(5.2) and (FG1), we have $f^n(\cdot,u)\to f(\cdot,u)$ a.e. in $(0,T_1)\times\Omega$ and $|f^n(\cdot,u)-f(\cdot,u)| \le c_p|1-\eta_n(u)|(1+|u|^{p-1}) \le c_p(1+|u|^{p-1}) \in L^{m_p'}((0,T_1)\times\Omega)$ by (3.17), hence by Fubini's and Lebesgue dominated convergence theorem we get

$$\widehat{f^n}(u) \to \widehat{f}(u) \quad \text{in } L^{m_p'}(0,T_1;L^{m_p'}(\Omega)). \tag{5.24}$$

A fortiori $\widehat{f^n}(u)\to\widehat f(u)$ in $L^1(0,T_1;L^1(\Omega))$ which, combined with previous remark, yields $\widehat{f^n}(u^n)\to\widehat f(u)$ in $L^1(0,T_1;L^1(\Omega))$. Then, as $\varphi\in C_c^1((0,T_1);H^{1,\infty,\infty})$,

$$\int_0^{T_1}\int_\Omega \widehat{f^n}(u^n)\varphi \to \int_0^{T_1}\int_\Omega \widehat f(u)\varphi. \tag{5.25}$$

By similar arguments we get

$$\int_0^{T_1}\int_{\Gamma_1}\widehat{g^n}(u^n)\varphi \to \int_0^{T_1}\int_{\Gamma_1}\widehat{g}(u)\varphi. \tag{5.26}$$

Combining (5.22)–(5.26) we obtain

$$\int_0^{T_1}\Big[(-\dot u,\dot\phi)_{H^0}+\langle Au,\phi\rangle_{H^1}+\langle\chi,\phi\rangle_Y\Big]=\int_0^{T_1}\int_\Omega \widehat{f}(u)\phi+\int_{\Gamma_1}\widehat{g}(u)\phi \tag{5.27}$$

for all $\varphi\in C^1_c((0,T_1);H^{1,\infty,\infty})$. Using Lemma 5.3 the distribution identity (5.27) holds for all $\varphi\in C_c((0,T);H^1)\cap C^1_c((0,T);H^0)\cap Z(0,T)$. Moreover, denoting $\chi=(\chi_1,\chi_2)$, by the form of the Riesz isomorphism between $L^{\overline{m}'}(\Omega;\lambda_\alpha)$ and $[L^{\overline{m}}(\Omega;\lambda_\alpha)]'$, we have $\chi_1=\alpha\chi_3$ where $\chi_3\in L^{\overline{m}'}(0,T_1;L^{\overline{m}'}(\Omega;\lambda_\alpha))$ and by the same argument $\chi_2=\alpha\chi_4$ where $\chi_4\in L^{\overline{\mu}'}(0,T_1;L^{\overline{\mu}'}(\Gamma_1;\lambda_\beta))$. By the remarks made before Definition 3.1 then u is a weak solution of (2.7) with $\rho=\overline{m}$, $\theta=\overline{\mu}$, $\xi=\widehat{f}(u)-\chi_1$, $\eta=\widehat{g}(u)-\chi_2$ and (2.8) holds. Hence $U\in C([0,T_1];\mathcal{H})$. To complete the proof we then only have to prove that $B(\dot u)=\chi$ in $[0,T_2]$ for a suitable $T_2=T_2(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}})\in(0,T_1]$, this one being the main technical point in the proof.

We claim that there is such a T_2 for which, up to a subsequence,

$$U^n\to U \qquad \text{strongly in } C([0,T_2];\mathcal{H}). \tag{5.28}$$

To prove it we introduce $w^n=u^n-u$, denoting $W^n=U^n-U$, and we note that by (5.22) and (5.27) w^n is a weak solution of (2.7) with $\xi=\widehat{f^n}(u^n)-\widehat{f}(u)-\widehat{P}(u^n_t)+\chi_1$ and $\eta=\widehat{g^n}(u^n)-\widehat{g}(u)-\widehat{Q}(u^n_{|\Gamma_t})+\chi_2$. They verify (2.8) with $\rho=\overline{m}$, $\theta=\overline{\mu}$, so by Lemma 2.1 (as $W^n(0)=0$) the energy identity

$$\begin{aligned}\frac12\|W^n(t)\|^2_{\mathcal{H}}+\int_0^t\langle B(\dot u^n)-\chi,\dot w^n\rangle_Y&=\tfrac12\|w^n(t)\|^2_{2,\Gamma_1}\\&+\int_0^t\int_\Omega[\widehat{f^n}(u^n)-\widehat{f}(u)]w^n_t+\int_0^t\int_{\Gamma_1}[\widehat{g^n}(u^n)-\widehat{g}(u)]w^n_{|\Gamma_t},\end{aligned} \tag{5.29}$$

holds for $t\in[0,T_1]$. Consequently, by Lemma 3.1–(iii) and the trivial estimate $\|w^n(t)\|^2_{2,\Gamma_1}\le\|w^n(t)\|^2_{H^0}\le\int_0^t\|\dot w^n\|^2_{H^0}$ (where $T_1\le 1$ was used) we have

$$\begin{aligned}\frac12\|W^n(t)\|^2_{\mathcal{H}}\le\frac12\int_0^t\|\dot w^n\|^2_{H^0}&+\int_0^{T_1}|\langle B(\dot u)-\chi,\dot w^n\rangle_Y|\\&+\int_0^t\int_\Omega[\widehat{f^n}(u^n)-\widehat{f}(u)]w^n_t+\int_0^t\int_{\Gamma_1}[\widehat{g^n}(u^n)-\widehat{g}(u)]w^n_{|\Gamma_t}.\end{aligned} \tag{5.30}$$

By Lemma 3.1 and (5.21)

$$a_n(U_0):=\int_0^{T_1}|\langle B(\dot u)-\chi,\dot w^n\rangle_Y|\to 0 \qquad \text{as } n\to\infty. \tag{5.31}$$

We are now going to estimate the last two terms in the right–hand side of (5.30)

$$I^f_n(t):=\int_0^t\int_\Omega[\widehat{f^n}(u^n)-\widehat{f}(u)]w^n_t,\quad I^g_n(t):=\int_0^t\int_{\Gamma_1}[\widehat{g^n}(u^n)-\widehat{g}(u)]w^n_{|\Gamma_t}. \tag{5.32}$$

Trivially $I_n^f(t) = I_n^{f,1}(t) + I_n^{f,2}(t)$ and $I_n^g(t) = I_n^{g,1}(t) + I_n^{g,2}(t)$, where

$$
\begin{aligned}
I_n^{f,1}(t) &= \int_0^t\int_\Omega [\widehat{f^n}(u^n) - \widehat{f^n}(u)]w_t^n, \; I_n^{g,1}(t) = \int_0^t\int_{\Gamma_1} [\widehat{g^n}(u^n) - \widehat{g^n}(u)]w^n_{|\Gamma_t}, \\
I_n^{f,2}(t) &= \int_0^t\int_\Omega [\widehat{f^n}(u) - \widehat{f}(u)]w_t^n, \quad I_n^{g,2}(t) = \int_0^t\int_{\Gamma_1} [\widehat{g^n}(u) - \widehat{g}(u)]w^n_{|\Gamma_t}.
\end{aligned} \tag{5.33}
$$

To estimate $I_n^{f,2}(t)$ we note that by Hölder inequality and Fubini's theorem

$$I_n^{f,2}(t) \le b_n(U_0) := \|\widehat{f^n}(u) - \widehat{f}(u)\|_{L^{m_p'}((0,T_1)\times\Omega)} \|w_t^n\|_{L^{m_p}((0,T_1)\times\Omega)}.$$

By (3.18) the sequence $(w_t^n)_n$ is bounded in $L^{m_p}((0,T_1)\times\Omega)$. Then by (5.24)

$$I_n^{f,2}(t) \le b_n(U_0) \to 0 \qquad \text{as } n\to\infty. \tag{5.34}$$

By transposing previous arguments from Ω to Γ_1 we get that, as $n\to\infty$,

$$I_n^{g,2}(t) \le c_n(U_0) := \|\widehat{g^n}(u) - \widehat{g}(u)\|_{L^{\mu_q'}((0,T_1)\times\Gamma_1)} \|w^n_{|\Gamma_t}\|_{L^{\mu_q}((0,T_1)\times\Gamma_1)} \to 0. \tag{5.35}$$

To estimate $I_n^{f,1}(t)$ we shall distinguish between two cases:

(i) $1 + r_\Omega/2 < p < 1 + r_\Omega/m'$,
(ii) $2 \le p \le 1 + r_\Omega/2$ or $1 + r_\Omega/2 < p = 1 + r_\Omega/m'$.

In the first one, by Lemmas 5.1 and 5.2, Hölder inequality and (5.19) , we get

$$I_n^{f,1}(t) \le d_n(U_0) := \int_0^{T_1} \|\widehat{f^n}(u^n) - \widehat{f^n}(u)\|_{m'}\|w_t^n\|_m \le K_{15}\int_0^{T_1} \|w^n\|_{(p-1)m'}\|w_t^n\|_m,$$

where $K_{15} = K_{15}(\|U_0\|_{\mathcal{H}}) = \overline{c_p'}k_3(1 + (1+2\|U_0\|_{\mathcal{H}})^{p-2})$. Since $(p-1)m' < r_\Omega$ the embedding $H^1(\Omega) \hookrightarrow L^{(p-1)m'}(\Omega)$ is compact hence, using (5.21) and the Simon's compactness result recalled above, up to a subsequence we have $w^n \to 0$ strongly in $C([0,T_1]; L^{(p-1)m'}(\Omega))$ and consequently, being w_t^n bounded in $L^m(0,T_1;L^m(\Omega))$ by assumption (FGQP1),

$$I_n^{f,1}(t) \le d_n \to 0 \qquad \text{as } n\to\infty. \tag{5.36}$$

In the second case we are going to apply Lemma 4.1, so let us check its assumptions. By (FG12) and Lemma 5.1 the functions f and f^n satisfy assumptions (FG12) with constants $\overline{c_p}'$ independent on $n\in\mathbb{N}$. Hence (F2) holds when $1 + r_\Omega/2 < p = 1 + r_\Omega/m'$. Moreover, setting $R_1 : [0,\infty) \to [0,\infty)$ by $R_1(\tau) = \max\{1+2\tau, \kappa(\tau)\}$, we note that R_1 is increasing and consequently $1 + 2\|U_0\|_{\mathcal{H}} \le R_1(\|U_0\|_{\mathcal{H}}) \le R_1\left(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}\right)$. Then, since in this case $\widetilde{\sigma_\Omega} = s_\Omega$ by (5.18), setting $R = R_1\left(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}\right)$, by Lemma 5.2 and (5.19)–(5.20) we have

$$\|U^n\|_{C([0,T_1];\mathcal{H})},\; \|U\|_{C([0,T_1];\mathcal{H})},\; \|\dot u^n\|_{Z(0,T_1)},\; \|\dot u\|_{Z(0,T_1)},\; \|u_0\|_{s_\Omega} \le R. \tag{5.37}$$

Then, for any $\varepsilon > 0$, denoting $K_{16} = K_{16}(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}) = K_4(R_1\left(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}\right), \overline{c_p})$, $K_{17} = K_{17}(\varepsilon, U_0) = K_5\left(\varepsilon, R_1\left(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}\right), u_0, u_0, \overline{c_p}'\right)$, by (4.2) we get

$$I_n^{f,1}(t) \le K_{16}(\varepsilon + t)\|W^n(t)\|^2_{\mathcal{H}} + K_{17}\int_0^t (1 + \|u_t^n\|_m + \|u_t\|_m)\|W^n(\tau)\|^2_{\mathcal{H}}\,d\tau. \tag{5.38}$$

Now, by setting $d_n(U_0) = 0$ in case (ii) and $K_{16} = K_{16}(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}) = K_{17} = K_{17}(\varepsilon, U_0) = 1$ in case (i), we combine (5.36) and (5.38) to get

$$I_n^{f,1}(t) \le d_n(U_0) + K_{16}(\varepsilon + t)\|W^n(t)\|^2_{\mathcal{H}} + K_{17}\int_0^t \left(1 + \|u_t^n\|_{m_p} + \|u_t\|_{m_p}\right)\|W^n(\tau)\|^2_{\mathcal{H}}\, d\tau, \tag{5.39}$$

where $d_n(U_0) \to 0$ as $n \to \infty$. Transposing previous arguments to Γ_1, distinguishing between the two cases

(i) $1 + r_\Gamma/2 < q < 1 + r_\Gamma/\mu'$, and
(ii) $2 \le q \le 1 + r_\Gamma/2$ or $1 + r_\Gamma/2 < q = 1 + r_\Gamma/\mu'$,

using Lemma 4.3 instead of Lemma 4.1 we estimate $I_n^{g,1}(t)$ as

$$I_n^{g,1}(t) \le e_n(U_0) + K_{18}(\varepsilon + t)\|W^n(t)\|^2_{\mathcal{H}} + K_{19}\int_0^t \Big(1 + \|u^n_{|\Gamma_t}\|_{\mu_q,\Gamma_1} + \|u_{|\Gamma_t}\|_{\mu_q,\Gamma_1}\Big)\|W^n\|^2_{\mathcal{H}}, \tag{5.40}$$

where $e_n(U_0) \to 0$ as $n \to \infty$ and we denote $K_{18} = K_{18}(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}) = K_{19} = K_{19}(\varepsilon, U_0) = 1$ and $e_n(U_0) = \int_0^{T_1} \|\widehat{g^n}(u^n) - \widehat{g^n}(u)\|_{\mu',\Gamma_1}\|(w^n_{|\Gamma_1})_t\|_{\mu,\Gamma_1}$ in case (i), while $e_n(U_0) = 0$, $K_{18} = K_{18}(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}) = K_{12}(R_1\left(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}\right), \overline{\boldsymbol{c_q}})$ and $K_{19} = K_{19}(\varepsilon, U_0) = K_{13}(\varepsilon, R, u_0, u_0, \overline{\boldsymbol{c_q}}')$ in case (ii).

Hence, denoting $h_n(U_0) = b_n(U_0) + c_n(U_0) + d_n(U_0) + e_n(U_0)$,

$$K_{20} = K_{20}(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}) = K_{16}(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}) + K_{18}(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}})$$

and $K_{21} = K_{21}(\varepsilon, U_0) = K_{17}(\varepsilon, U_0) + K_{19}(\varepsilon, U_0)$, by (5.32)–(5.35), (5.38) and (5.40) we get that $h_n(U_0) \to 0$ as $n \to \infty$ and

$$I_n^f(t) + I_n^g(t) \le h_n(U_0) + K_{20}(\varepsilon + t)\|W^n(t)\|^2_{\mathcal{H}} + K_{21}\int_0^t \Big(1 + \|u_t^n\|_{m_p} + \|u_t\|_{m_p} + \|u^n_{|\Gamma_t}\|_{\mu_q,\Gamma_1} + \|u_{|\Gamma_t}\|_{\mu_q,\Gamma_1}\Big)\|W^n(\tau)\|^2_{\mathcal{H}}\, d\tau \tag{5.41}$$

for all $t \in [0, T_1]$. Plugging it with (5.31) in (5.30) we finally get

$$\frac{1}{2}\|(W^n(t)\|^2_{\mathcal{H}} \le l_n(U_0) + K_{20}(\varepsilon + t)\|W^n(t)\|^2_{\mathcal{H}} + K_{22}\int_0^t \Big(1 + \|u_t^n\|_{m_p} + \|u_t\|_{m_p} + \|u^n_{|\Gamma_t}\|_{\mu_q,\Gamma_1} + \|u_{|\Gamma_t}\|_{\mu_q,\Gamma_1}\Big)\|W^n(\tau)\|^2_{\mathcal{H}}\, d\tau \tag{5.42}$$

where $K_{22} = K_{22}(\varepsilon, U_0) = 1 + K_{21}(\varepsilon, U_0)$ and

$$l_n(U_0) = a_n(U_0) + h_n(U_0) \to 0 \quad \text{as } n \to \infty. \tag{5.43}$$

We now set the function $T_2 : [0, \infty) \to (0, 1]$ by $T_2(\tau) = \min\{T_1(\tau), 1/8K_{20}(\tau)\}$. Hence $T_2 \le T_1$, $K_{20}T_2 \le 1/8$ and T_2 is decreasing. We also choose $\varepsilon = \varepsilon_2 := 1/8K_{20}(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}})$ so that, denoting $K_{23} = K_{23}(U_0) = K_{22}(\varepsilon_2, U_0)$, by (5.42)

$$\frac{1}{4}\|(W^n(t)\|^2_{\mathcal{H}} \le l_n(U_0) + K_{23}\int_0^t \Big(1 + \|u_t^n\|_{m_p} + \|u_t\|_{m_p} + \|u^n_{|\Gamma_t}\|_{\mu_q,\Gamma_1} + \|u_{|\Gamma_t}\|_{\mu_q,\Gamma_1}\Big)\|W^n(\tau)\|^2_{\mathcal{H}}\, d\tau \tag{5.44}$$

for all $t \in [0, T_2(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}})]$, so by the already recalled Gronwall inequality

$$\|W^n(t)\|_{\mathcal{H}}^2 \le 4l_n(U_0)\exp\Big[4K_{23}\int_0^{T_2}\Big(1+\|u_t^n\|_{m_p}+\|u_t\|_{m_p} + \|u^n_{|\Gamma_t}\|_{\mu_q,\Gamma_1}+\|u_{|\Gamma_t}\|_{\mu_q,\Gamma_1}\Big)\,d\tau\Big] \tag{5.45}$$

for all $t \in [0, T_2(\|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}})]$. Since, by (3.18), the sequences $(u_t^n)_n$ and $u^n_{|\Gamma_t}$ are respectively bounded in $L^{m_p}((0,T_1)\times\Omega)$ and in $L^{\mu_q}((0,T_1)\times\Gamma_1)$, by (5.43) we get that (5.28) holds, proving our claim.

Using (5.28), (5.41), the just used boundedness of $(u_t^n)_n$, $u^n_{|\Gamma_t}$ and $\lim_n h_n(U_0)=0$ we get $I_n^f(T_2)+I_n^g(T_2)\to 0$ as $n\to\infty$. Consequently, using (5.28) again in the energy identity (5.29) we get that $\lim_n\int_0^{T_2}\langle B(\dot u^n)-\chi, \dot w^n\rangle_Y = 0$. Since by Lemma 3.1 and [7, Theorem 1.3, p.40] the operator B is maximal monotone in $Z(0,T_2)$ this fact yields, by the classical monotonicity argument (see for example [8, Lemma 1.3, p.49]), that $B(\dot u)=\chi$ in $(0,T_2)$, concluding the proof. □

Proof of Theorem 5.1. By Proposition 5.1 we get the existence of a weak solution in $[0,T_2]$ when $U_0\in\mathcal{H}^{\sigma_\Omega,\sigma_\Gamma}$. The existence of a maximal weak solution u in $[0,T_{\max})$ follows by a standard application of Zorn's lemma. By (1.22) and Lemma 3.3–(i)–(iii) we get (1.24) and the energy identity (3.20), completing the proof of (i–ii). To prove (iii) we suppose by contradiction that $T_{\max}<\infty$ and $\overline{\lim}_{t\to T_{\max}^-}\|U(t)\|_{H^1\times H^0}<\infty$, which by (1.24) implies $U\in L^\infty(0,T_{\max};\mathcal{H})$. By Lemma 3.3-(iii–iv) then u is a weak solution in $[0,T_{\max}]$ and $U\in C([0,T_{\max}];\mathcal{H}^{\sigma_\Omega,\sigma_\Gamma})$. Then, applying Proposition 5.1, problem (1.1) with initial data $U(T_{\max})$ has a weak solution v in $[0,T_2]$. By Lemma 3.3–(ii), $\bar u$ defined by $\bar u(t)=u(t)$ for $t\in[0,T_{\max}]$, $\bar u(t)=v(t-T_{\max})$ for $t\in[T_{\max},T_{\max}+T_2]$ is a weak solution in $[0,T_{\max}+T_2]$, contradicting the maximality of u. □

We now state and prove, for the sake of clearness, some corollaries of Theorem 5.1 which generalize Corollaries 1.1–1.3 in the introduction. The discussion made there applies here as well.

Corollary 5.1. *Under assumptions (PQ1–3), (FG1), (FGQP1) and* (1.26) *the conclusions of Theorem 5.1 hold and* $H^{1,\sigma_\Omega,\sigma_\Gamma}=H^1$.

Proof. Since (1.26) can be written also as $p\neq 1+r_\Omega/m'$ when $p>1+r_\Omega/2$ and $q\neq 1+r_\Gamma/\mu'$ when $q>1+r_\Gamma/2$, clearly assumption (FG2) can be skipped. Moreover, by (1.19), when (1.26) holds we have $H^{1,\sigma_\Omega,\sigma_\Gamma}=H^1$. □

Remark 5.1. Clearly when (1.26) holds the proof of Proposition 5.1 can be simplified since Lemmas 4.1 and 4.3 are not needed.

Corollary 5.2. *If assumptions (PQ1–3), (FG1–2), (FGQP1) are satisfied and* (1.27) *holds, hence in particular when* $2\le p\le r_\Omega$ *and* $2\le q\le r_\Gamma$, *the conclusions of Theorem 5.1 hold and* $H^{1,\sigma_\Omega,\sigma_\Gamma}=H^1$.

Proof. When (1.27) holds by (1.19) we have $\sigma_\Omega=\sigma_\Gamma=2$, hence $H^{1,\sigma_\Omega,\sigma_\Gamma}=H^1$. □

Corollary 5.3. *Under assumptions (PQ1–3), (FG1–2), (FGQP1) the conclusions of Theorem 5.1 hold when the space* $H^{1,\sigma_\Omega,\sigma_\Gamma}$ *is replaced by*

(i) *the space* $H^{1,\rho,\theta}_{\alpha,\beta}$, *provided* (1.29) *holds;*
(ii) *the space* $H^{1,\rho,\theta}$, *provided* (1.14) *and* (1.29) *holds;*
(iii) *the space* H^{1,s_Ω,s_Γ}, *provided also assumption (FG2)′ holds.*

Proof. The statement (i) follows by combining Theorem 5.1 with Lemma 3.3–(iii), (ii) follows by combining (i) with Remark 1.9, while (iii), by (1.18)–(1.22), is a particular case of (ii) when (FG2)′ holds. □

Proof of Theorem 1.1 and Corollaries 1.1–1.3 in Section 1. When

$$P(x,v)=\alpha(x)P_0(v),\quad Q(x,v)=\beta(x)Q_0(v),\quad f(x,u)=f_0(u),\quad g(x,u)=g_0(u),$$

by Remarks 3.1–3.3 assumptions (PQ1–3), (FG1), (FGQP1) reduce to (I–III), while by Remark 3.6 (FG2) and (FG2)′ reduce to (IV) and (IV)′, hence Theorem 1.1 and Corollaries 1.1–1.3 are particular cases of Theorem 5.1 and Corollaries 5.1–5.3. □

6. Uniqueness and local well–posedness

This section is devoted to our uniqueness and well–posedness results for problem (1.1). To get uniqueness of solutions we need to restrict to sources satisfying (FG2)′ and $u_0 \in H^{1,\sigma_\Omega,\sigma_\Gamma}$, as in the last statement of Corollary 5.3.

Theorem 6.1 (Uniqueness). *Suppose that (PQ1–3), (FG1), (FGQP1) and (FG2)′ hold, let* $U_0 \in \mathcal{H}^{\sigma_\Omega,\sigma_\Gamma}$ *and* u, v *are maximal solutions of* (1.1)*. Then* $u=v$.

Proof. We fix u,v, so constants K_i introduced in this proof will depend also on them. We denote dom $u=[0,T^u_{\max})$ and dom $v=[0,T^v_{\max})$. By Lemma 3.3 and (1.18)–(1.22) we have $U\in C([0,T^u_{\max});\mathcal{H}^{\sigma_\Omega,\sigma_\Gamma})$ and $V\in C([0,T^v_{\max});\mathcal{H}^{\sigma_\Omega,\sigma_\Gamma})$. We set $\widetilde{T}_{\max}=\min\{T^u_{\max},T^v_{\max}\}$.

We claim that there is a (possibly small) $\widetilde{T}<\widetilde{T}_{\max}$ such that $u=v$ in $[0,\widetilde{T}]$. To prove our claim we set $T'=\min\{1,\widetilde{T}_{\max}/2\}$, $R_2=R_2(u,v)$ by

$$R_2=\max\left\{\|U\|_{C([0,T'];\mathcal{H}^{\sigma_\Omega,\sigma_\Gamma})},\,\|V\|_{C([0,T'];\mathcal{H}^{\sigma_\Omega,\sigma_\Gamma}},\,\|\dot u\|_{Z(0,T')},\,\|\dot v\|_{Z(0,T')}\right\},\tag{6.1}$$

and we denote $w=u-v$, $W=U-V$. Clearly w is a weak solution of (2.7) with $\xi=\widehat{f}(u)-\widehat{f}(v)-\widehat{P}(u_t)+\widehat{P}(v_t)$, $\eta=\widehat{g}(u)-\widehat{g}(v)-\widehat{Q}(u_t)+\widehat{Q}(v_t)$, ξ,η satisfying (2.8) with $\rho=\overline{m}$, $\theta=\overline{\mu}$ and $W(0)=0$. Hence, by Lemma 2.1, the energy identity

$$\tfrac12\|W(t)\|^2_{\mathcal{H}}+\int_0^t\langle B(\dot u)-B(\dot v),\dot w\rangle_Y = \tfrac12\|w(t)\|^2_{2,\Gamma_1}+\int_0^t\int_\Omega[\widehat{f}(u)-\widehat{f}(v)]w_t+\int_0^t\int_{\Gamma_1}[\widehat{g}(u)-\widehat{g}(v)]w_{|\Gamma}{}_t,$$

holds for $t\in[0,T']$. Consequently, by Lemma 3.1–(iii) and the trivial estimate $\|w(t)\|^2_{2,\Gamma_1}\le\|w(t)\|^2_{H^0}\le\int_0^t\|\dot w\|^2_{H^0}$ (where $T'\le1$ was used) we have, for $t\in[0,T']$,

$$\frac12\|W(t)\|^2_{\mathcal{H}}\le\frac12\int_0^t\|\dot w\|^2_{H^0}+\int_0^t\int_\Omega[\widehat{f}(u)-\widehat{f}(v)]w_t+\int_0^t\int_{\Gamma_1}[\widehat{g}(u)-\widehat{g}(v)]w_{|\Gamma}{}_t.$$

By assumption (FG2)′ we can apply Lemmas 4.1 and 4.3 with R given by (6.1), hence for any $\varepsilon>0$, denoting $K_{24}=K_4(R_2,\boldsymbol{c}_p)+K_{12}(R_2),\boldsymbol{c}_q)$,

$$K_{25}=K_{25}(\varepsilon)=K_5(\varepsilon,R_2,u_0,u_0,\boldsymbol{c}'_p)+K_{13}(\varepsilon,R_2,u_0,u_0,\boldsymbol{c}'_q)+1,$$

$$\boldsymbol{c_p}=(c_p,c_p'),\quad \boldsymbol{c_q}=(c_q,c_q'),\quad \boldsymbol{c_p'}=(c_p,c_p',c_p''),\quad \boldsymbol{c_q'}=(c_q,c_q',c_q''),\tag{6.2}$$

plugging (4.1) and (4.26) in previous estimate we get

$$\tfrac12\|W(t)\|_{\mathcal H}^2\le K_{24}(\varepsilon+t)\|W(t)\|_{\mathcal H}^2+K_{25}\int_0^t\Big(1+\|u_t\|_{m_p}+\|v_t\|_{m_p}\\ +\|u_{|\Gamma_t}\|_{\mu_q,\Gamma_1}+\|v_{|\Gamma_t}\|_{\mu_q,\Gamma_1}\Big)\|W(\tau)\|_{\mathcal H}^2\,d\tau.$$

Choosing $\varepsilon=\varepsilon_3:=1/8K_{24}$ and denoting $K_{26}=K_{25}(\varepsilon_3)$ we have

$$\tfrac14\|W(t)\|_{\mathcal H}^2\le K_{26}\int_0^t\Big(1+\|u_t\|_{m_p}+\|v_t\|_{m_p}+\|u_{|\Gamma_t}\|_{\mu_q,\Gamma_1}\\ +\|v_{|\Gamma_t}\|_{\mu_q,\Gamma_1}\Big)\|W(\tau)\|_{\mathcal H}^2\,d\tau,$$

for $t\in[0,\widetilde T]$ where $\widetilde T:=\min\{T',1/8K_{24}\}$. Since by (3.18) we have $1+\|u_t\|_{m_p}+\|v_t\|_{m_p}+\|u_{|\Gamma_t}\|_{\mu_q,\Gamma_1}+\|v_{|\Gamma_t}\|_{\mu_q,\Gamma_1}\in L^1(0,\widetilde T)$, by the already recalled Gronwall inequality we get $\|W(t)\|_{\mathcal H}^2\le 0$ for $t\in[0,\widetilde T]$, proving our claim.

The statement now follows in a standard way, which is described in the sequel for the reader's convenience. We set $T^*=\sup\{t\in[0,\widetilde T_{\max}):u=v \text{ in } [0,t]\}$. Clearly $T^*\le\widetilde T_{\max}$ and $U=V$ in $[0,T^*)$. Supposing by contradiction that $T^*<\widetilde T_{\max}$ we then have $U,V\in C([0,T^*];\mathcal H^{\sigma_\Omega,\sigma_\Gamma})$, so $U(T^*)=V(T^*)\in\mathcal H^{\sigma_\Omega,\sigma_\Gamma}$. Then, since (1.1) is autonomous, $\tilde u(t):=u(t+T^*)$ and $\tilde v(t):=v(t+T^*)$ are weak solutions of (1.1), with initial data $u(T^*),\dot u(T^*)$, in $[0,\widetilde T_{\max}-T^*)$, so by our claim $\widetilde u=\widetilde v$ in $[0,\tau)$ for some $\tau>0$, i.e. $u=v$ in $[0,T^*+\tau)$ contradicting the definition of T^*. Hence $T^*=\widetilde T_{\max}$ and $U=V$ in $[0,\widetilde T_{\max})$. Finally $T^u_{\max}=T^v_{\max}$ since if $T^u_{\max}<T^v_{\max}$ then v is a proper extension of u, a contradiction. □

Essentially by combining Corollary 5.3 and Theorem 6.1 we get

Theorem 6.2 (Local existence–uniqueness). *Suppose that (PQ1–3), (FG1), (FGQP1) and (FG2)′ hold. Then all conclusions of Theorem 1.2 hold true when problem* (1.2) *is generalized to problem* (1.1)*, provided the energy identity* (1.25) *is generalized to* (3.20)*.*

Proof. By combining Corollary 5.3 and Theorem 6.1 we immediately get statements (i–ii) and $\overline{\lim}_{t\to T^-_{\max}}\|U(t)\|_{H^1\times H^0}=\infty$ when $T_{\max}<\infty$, so we have only to prove that $\lim_{t\to T^-_{\max}}\|U(t)\|_{H^{1,s_\Omega,s_\Gamma}\times H^0}=\infty$ in this case. This fact follows from Proposition 5.1 and a standard procedure, described in the sequel. Since $H^{1,s_\Omega,s_\Gamma}\hookrightarrow H^{1,\sigma_\Omega,\sigma_\Gamma}=H^{1,\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}$ we shall prove that $\lim_{t\to T^-_{\max}}\|U(t)\|_{H^{1,\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}\times H^0}=\infty$. Suppose by contradiction that $M:=\sup_n\{\|U(t_n)\|_{\mathcal H^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}\}<\infty$ for some $t_n\to T^-_{\max}$. Then by Proposition 5.1 for each $n\in\mathbb N$ problem (1.1) with initial data $U(t_n)$ has a weak solution v_n in $[0,T_2(M)]$. Hence, by Lemma 3.3–(ii), w_n defined by $w_n(t)=u(t)$ for $t\in[0,t_n]$ and $w_n(t)=v_n(t-t_n)$ for $t\in[t_n,t_n+T_2(M)]$ is a weak solution of (1.1) in $[0,t_n+T_2(M)]$ and, by Theorem 6.1, $w_n=u$ so, being u maximal, $t_n+T_2(M)<T^-_{\max}$ which, when $n\to\infty$, gives a contradiction. □

Remark 6.1. From the proof of the case (ii) in Proposition 5.1 it is clear that one can give an alternative proof of Theorem 6.2 without using compactness arguments.

We now give a consequence of Theorem 6.2 which generalizes Corollary 1.4 in the introduction, the discussion made there applying as well.

Corollary 6.1. *Under assumptions (PQ1–3), (FG1), (FGQP1) and (FG2)$'$ the conclusions of Theorem 6.2 hold when the space H^{1,s_Ω,s_Γ} is replaced by*

(i) *the space $H^{1,\rho,\theta}_{\alpha,\beta}$, provided* (1.32) *holds, and*
(ii) *the space $H^{1,\rho,\theta}$, provided* (1.14) *and* (1.32) *hold.*

We now give our main local Hadamard well–posedness result for problem (1.1), restricting to damping terms satisfying also assumption (PQ4), to non–bicritical nonlinearities and to $u_0 \in H^{1,s_2,s_2}$ with s_1, s_2 satisfying (1.32).

Theorem 6.3 (**Local Hadamard well–posedness I**)**.** *Suppose that (PQ1–4), (FG1), (FGQP1), (FG2)$'$,* (1.33) *hold and let s_1, s_2 satisfy* (1.34)*. Then all conclusions of Theorem 1.3 hold true when problem* (1.2) *is generalized to* (1.1)*.*

In particular (1.1) *is locally well–posed in $\mathcal{H}$, under assumptions (PQ1–3), (FG1), (FGQP1) and (FG2)$'$, when $2 \le p < r_\Omega$ and $2 \le q < r_\Gamma$.*

Proof. Let $U_{0n} := (u_{0n}, u_{1n}) \to U_0 := (u_0, u_1)$ in $\mathcal{H}^{s_1,s_2}$ and $U^n \in C([0, T^n_{\max}); \mathcal{H}^{s_1,s_2})$, $U \in C([0, T_{\max}); \mathcal{H}^{s_1,s_2})$, s_1 and s_2 be fixed as in the statement. [12] Since $H^{1,s_1,s_2} \hookrightarrow H^{1,s_\Omega,s_\Gamma} \hookrightarrow H^{1,\sigma_\Omega,\sigma_\Gamma} = H^{1,\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}$,

$$M(t) = \max\{\|U\|_{C([0,t];\mathcal{H}^{s_1,s_2})}, \|U\|_{C([0,t];\mathcal{H}^{s_\Omega,s_\Gamma})}, \|U\|_{C([0,t];\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}})}\} \tag{6.3}$$

defines an increasing function $M : [0, T_{\max}) \to [0, \infty)$. Hence

$$T_3(t) = T_2(1 + M(t)), \tag{6.4}$$

where T_2 is the function defined in Proposition 5.1, defines a decreasing function $T_3 : [0, T_{\max}) \to [0, 1)$, with $T_3(t) \le T_1(1 + M(t))$. Now let $T^* \in (0, T_{\max})$. By (6.3) we have $\|U_0\|_{\mathcal{H}^{s_1,s_2}}, \|U_0\|_{\mathcal{H}^{s_\Omega,s_\Gamma}}, \|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}} \le M(T^*)$ and consequently, since $U_{0n} \to U_0$ in $\mathcal{H}^{s_1,s_2}$, there is $n_1 = n_1(T^*) \in \mathbb{N}$, such that for all $n \ge n_1(T^*)$,

$$\begin{aligned} &\|U_{0n}\|_{\mathcal{H}^{s_1,s_2}}, \|U_{0n}\|_{\mathcal{H}^{s_\Omega,s_\Gamma}}, \|U_{0n}\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}} \le 1 + M(T^*),\\ &\|U_0\|_{\mathcal{H}^{s_1,s_2}},\ \|U_0\|_{\mathcal{H}^{s_\Omega,s_\Gamma}},\ \|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}} \le 1 + M(T^*). \end{aligned} \tag{6.5}$$

Since u_n and u are unique maximal solutions by (6.4)–(6.5) and Proposition 5.1

$$T_3(T^*) < T_{\max}, \quad T_3(T^*) < T^n_{\max}, \tag{6.6}$$

$$\begin{aligned} &\|U^n\|_{C([0,T_3(T^*)];\mathcal{H})}, \quad \|U\|_{C([0,T_3(T^*)];\mathcal{H})} \le 1 + 2\|U_0\|_{\mathcal{H}},\\ &\|\dot{u}^n\|_{Z(0,T_3(T^*))}, \qquad \|\dot{u}\|_{Z(0,T_3(T^))} \qquad \le \kappa(\|U_0\|_{\mathcal{H}}) \end{aligned} \tag{6.7}$$

for all $n \ge n_1(T^*)$. Hence, as $\|U_0\|_{\mathcal{H}} \le \|U_0\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}$, setting the increasing function $R_3 : [0, T_{\max}) \to [0, \infty)$ by $R_3(\tau) = \max\{3 + 2M(\tau), \kappa(1 + M(\tau))\}$, by (6.5) and (6.7) we have

$$\begin{aligned} &\|U^n\|_{C([0,T_3(T^*)];\mathcal{H})}, \quad \|U\|_{C([0,T_3(T^*)];\mathcal{H})} \le R_3(T^*),\\ &\|\dot{u}^n\|_{Z(0,T_3(T^*))}, \qquad \|\dot{u}\|_{Z(0,T_3(T^*))} \qquad \le R_3(T^*) \end{aligned} \tag{6.8}$$

for all $n \ge n_1(T^*)$. Since by (FG2)$'$ the property (F2) holds when $p > 1 + r_\Omega/2$, by (1.34) we have $s_\Omega < s_1$, $s_\Gamma < s_2$ and $1 + M(T^*) \le R_3(T^*)$, we can apply the final parts of Lemmas 4.1 and 4.3. Consequently, keeping the notation (6.2) and

[12] functions and constants K_i introduced in this proof will depend also on them.

denoting $w^n = u^n - u$, $W^n = U^n - U$, $W_{0n} = U_{0n} - U_0$, $K_{27} = K_{27}(T^*) = K_4(R_3(T^*), \boldsymbol{c}_p) + K_{12}(R_3(T^*), \boldsymbol{c}_q)$ and

$$K_{28} = K_{28}(\varepsilon, T^*) = K_6(\varepsilon, R_3(T^*), \boldsymbol{c}'_p) + K_{14}(\varepsilon, R_3(T^*), \boldsymbol{c}'_q),$$

we have the estimate

$$\begin{aligned}(6.9)\quad & \int_0^t \int_\Omega [\widehat{f}(u^n) - \widehat{f}(u)] w^n_t + \int_0^t \int_{\Gamma_1} [\widehat{g}(u^n) - \widehat{g}(u)] w^n_{|\Gamma_t} \\ & \le K_{27}(\varepsilon + t)\|W^n(t)\|^2_{\mathcal{H}} + K_{28}\Bigg[\|W_{0n}\|^2_{\mathcal{H}} + \int_0^t \Big(1 + \|u^n_t\|_{m_p} + \|u_t\|_{m_p} \\ & \qquad + \|u^n_{|\Gamma_t}\|_{\mu_q,\Gamma_1} + \|u_{|\Gamma_t}\|_{\mu_q,\Gamma_1}\Big)\|W^n\|^2_{\mathcal{H}}\Bigg] \qquad \text{for all } t \in [0, T_3(T^*)].\end{aligned}$$

Since w^n is a weak solution of (2.7) with $\xi = \widehat{f}(u^n) - \widehat{f}(u) - \widehat{P}(u^n_t) + \widehat{P}(u_t)$ and $\eta = \widehat{g}(u^n) - \widehat{g}(u) - \widehat{Q}(u^n_{|\Gamma_t}) + \widehat{Q}(u_{|\Gamma_t})$ verifying (2.8) with $\rho = \overline{m}$, $\theta = \overline{\mu}$, by Lemma 2.1 the energy identity

$$\begin{aligned}(6.10)\quad & \frac{1}{2}\|W^n(t)\|^2_{\mathcal{H}} + \int_0^t \langle \widehat{B}(\dot{u}^n) - \widehat{B}(\dot{u}), \dot{w}^n\rangle_Y = \tfrac{1}{2}\|W_{0n}\|^2_{\mathcal{H}} - \tfrac{1}{2}\|w_{0n}\|^2_{2,\Gamma_1} \\ & + \tfrac{1}{2}\|w^n(t)\|^2_{2,\Gamma_1} + \int_0^t \int_\Omega [\widehat{f}(u^n) - \widehat{f}(u)] w^n_t + \int_0^t \int_{\Gamma_1} [\widehat{g}(u^n) - \widehat{g}(u)] w^n_{|\Gamma_t},\end{aligned}$$

holds for all $t \in [0, T_3(T^*)]$. By (PQ4) and (3.32)–(3.33) we have

$$\begin{aligned}(6.11)\quad \langle \widehat{B}(\dot{u}^n) - \widehat{B}(\dot{u}), \dot{w}^n\rangle_Y & \ge \widetilde{c_m}''\|[w^n_t]_\alpha\|^m_{m,\alpha} - c'''_m\|\alpha\|_\infty\|w^n_t\|^2_2 \\ & \quad + \widetilde{c_\mu}''\|[w_{|\Gamma}{}^n_t]_\beta\|^\mu_{\mu,\beta,\Gamma_1} - c'''_\mu\|\beta\|_{\infty,\Gamma_1}\|(w^n_{|\Gamma_1})_t\|^2_2 \\ & \ge \widetilde{c_m}''\|[w^n_t]_\alpha\|^m_{m,\alpha} + \widetilde{c_\mu}''\|[w_{|\Gamma}{}^n_t]_\beta\|^\mu_{\mu,\beta,\Gamma_1} - k_{17}\|W^n\|^2_{\mathcal{H}},\end{aligned}$$

with $\widetilde{c_m}'' > 0$ when $p \ge r_\Omega$ and $\widetilde{c_\mu}'' > 0$ when $q \ge r_\Gamma$. Hence, setting $K_{29} = K_{29}(\varepsilon, T^*) = 2 + K_{28}(\varepsilon, T^*)$ and plugging (6.9), (6.11) and the trivial estimate $\frac{1}{2}\|w^n(t)\|^2_{2,\Gamma_1} \le \frac{1}{2}\|w^n(t)\|^2_{H^0} \le \|W_{0n}\|^2_{\mathcal{H}} + \int_0^t \|W^n\|^2_{H^0}$ (where $T_3(T^*) \le 1$ was used) in (6.10) we get

$$\begin{aligned}(6.12)\quad & \frac{1}{2}\|W^n(t)\|^2_{\mathcal{H}} + \int_0^t \widetilde{c_m}''\|[w^n_t]_\alpha\|^m_{m,\alpha} + \widetilde{c_\mu}''\|[w_{|\Gamma}{}^n_t]_\beta\|^\mu_{\mu,\beta,\Gamma_1} \\ & \le K_{27}(\varepsilon + t)\|W^n(t)\|^2_{\mathcal{H}} + K_{29}\Bigg[\|W_{0n}\|^2_{\mathcal{H}} + \int_0^t \Big(1 + \|u^n_t\|_{m_p} + \|u_t\|_{m_p} \\ & \qquad + \|u^n_{|\Gamma_t}\|_{\mu_q,\Gamma_1} + \|u_{|\Gamma_t}\|_{\mu_q,\Gamma_1}\Big)\|W^n\|^2_{\mathcal{H}}\Bigg] \qquad \text{for all } t \in [0, T_3(T^*)].\end{aligned}$$

We now set $T_4(T^*) = \min\{T_3(T^*), 1/8K_{27}(T^*)\}$, so that $T_4 \le T_3$ and $K_{27}T_4 \le 1/8$. We also choose $\varepsilon = \varepsilon_4 := 1/8K_{27}(T^*)$ so that, setting $K_{30} = K_{30}(T^*) =$

$4K_{29}(\varepsilon_4, T^*)$, by (6.12) we have

$$\text{(6.13)}\quad \|W^n(t)\|_{\mathcal{H}}^2 + \int_0^t \widetilde{c_m}''\|[w_t^n]_\alpha\|_{m,\alpha}^m + \widetilde{c_\mu}''\|[w_{|\Gamma}{}_t^n]_\beta\|_{\mu,\beta,\Gamma_1}^\mu$$
$$\le K_{30}\left[\|W_{0n}\|_{\mathcal{H}}^2 + \int_0^t \Big(1+\|u_t^n\|_{m_p}+\|u_t\|_{m_p}+\|u_{|\Gamma}^n{}_t\|_{\mu_q,\Gamma_1}+\|u_{|\Gamma}{}_t\|_{\mu_q,\Gamma_1}\Big)\|W^n\|_{\mathcal{H}}^2\right]$$

for all $t \in [0, T_4(T^*)]$. By disregarding the second term in the left–hand side of (6.13) and applying Gronwall inequality we get

$$\|(W^n(t)\|_{\mathcal{H}}^2 \le K_{30}\|W_{0n}\|_{\mathcal{H}}^2 \exp\left[K_{30}\int_0^{T_3(T^*)} \Big(1+\|u_t^n\|_{m_p}+\|u_t\|_{m_p}\right.$$
$$\left. + \|u_{|\Gamma}^n{}_t\|_{\mu_q,\Gamma_1}+\|u_{|\Gamma}{}_t\|_{\mu_q,\Gamma_1}\Big)\,d\tau\right].$$

Consequently, by Hölder inequality in time, (3.18) and (6.8),

$$\text{(6.14)}\qquad \|W^n(t)\|_{\mathcal{H}}^2 \le K_{30}\|W_{0n}\|_{\mathcal{H}}^2 e^{K_{30}[1+2R_3(T^*)]} \qquad \text{for all } t \in [0, T_4(T^*)],$$

from which we immediately get

$$\text{(6.15)}\qquad W^n \to 0 \quad \text{in } C([0, T_4(T^*)]; \mathcal{H}) \quad \text{as } n \to \infty.$$

To get the stronger (when $p \ge r_\Omega$ or $q \ge r_\Gamma$) convergence in $C([0, T_4(T^*)]; \mathcal{H}^{s_1,s_2})$ we now plug (6.14) into (6.13) and use (3.18), (6.8) and Hölder inequality to get

$$\int_0^{T_4(T^*)} \widetilde{c_m}''\|[w_t^n]_\alpha\|_{m,\alpha}^m + \widetilde{c_\mu}''\|[w_{|\Gamma}{}_t^n]_\beta\|_{\mu,\beta,\Gamma_1}^\mu \le K_{30}\|W_{0n}\|_{\mathcal{H}}^2\Big\{1+K_{30}e^{K_{30}[1+2R_3(T^*)]}$$
$$\times \int_0^{T_3(T^*)} \Big(1+\|u_t^n\|_{m_p}+\|u_t\|_{m_p}+\|u_{|\Gamma}^n{}_t\|_{\mu_q,\Gamma_1}+\|u_{|\Gamma}{}_t\|_{\mu_q,\Gamma_1}\Big)\,d\tau\Big\}$$
$$\le K_{30}\|W_{0n}\|_{\mathcal{H}}^2\Big\{1+K_{30}e^{K_{30}[1+2R_3(T^*)]}[1+2R_3(T^*)]\Big\},$$

from which it immediately follows that

$$\text{(6.16)}\qquad \widetilde{c_m}''\|[w_t^n]_\alpha\|_{L^m(0,T_4(T^*);L^m_\alpha(\Omega)}^m, \widetilde{c_\mu}''\|[w_{|\Gamma}{}_t^n]_{L^\mu(0,T_4(T^*);L^\mu_\beta(\Gamma_1)}^\mu \to 0$$

as $n \to \infty$. When $p \ge r_\Omega$ we have $\widetilde{c_m}'' > 0$ so by (6.16) and (FGQP1) it follows $w_t^n \to 0$ in $L^m(0, T_4(T^*); L^m(\Omega))$. Since by (1.18) and (1.34) in this case we have $s_1 \le m$, we derive $w_t^n \to 0$ in $L^{s_1}(0, T_4(T^*); L^{s_1}(\Omega))$. As $w_{0n} \to 0$ in $L^{s_1}(\Omega)$ we get by a trivial integration in time that $w^n \to 0$ in $C([0, T_4(T^*)]; L^{s_1}(\Omega))$. Using similar arguments we get $w_{|\Gamma}^n \to 0$ in $C([0, T_4(T^*)]; L^{s_2}(\Gamma_1))$ when $q \ge r_\Gamma$. Consequently, using (6.15), Sobolev embeddings and (1.34) when $p < r_\Omega$ and $q < r_\Gamma$ we derive

$$\text{(6.17)}\qquad W^n \to 0 \quad \text{in } C([0, T_4(T^*)]; \mathcal{H}^{s_1,s_2}) \quad \text{as } n \to \infty.$$

We then complete the proof by repeating previous arguments a finite number of times. More explicitly we set the function $\kappa_1 : (0, T_{\max}) \to \mathbb{N}_0$, by $\kappa_1(T^*) = \min\{k \in \mathbb{N}_0 : T^*/T_4(T^*) \le k+1\}$, so that

$$\text{(6.18)}\qquad \kappa_1(T^*)T_4(T^*) < T^* \le [\kappa_1(T^*)+1]T_4(T^*).$$

If $\kappa_1(T^*)=0$, that is if $T^*\le T_4(T^*)$, by (6.17) we have $W^n\to 0$ in $C([0,T^*];\mathcal{H}^{s_1,s_2})$, that is the conclusion (ii) in the statement of Theorem 1.3. Moreover in this case by (6.6) we have

$$T^*\le T_4(T^*)\le T_3(T^*)<T^n_{\max}\qquad\text{for all } n\ge n_1(T^*).$$

Now let $\kappa_1(T^*)\ge 1$, that is $T_4(T^*)<T^*$. By (6.3) we have $\|U(T_4(T^*))\|_{\mathcal{H}^{s_1,s_2}}$, $\|U(T_4(T^*))\|_{\mathcal{H}^{s_\Omega,s_\Gamma}}$, $\|U(T_4(T^*))\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}\le M(T^*)$, and consequently, by (6.17), there is $n_2=n_2(T^*)\in\mathbb{N}$ such that for all $n\ge n_2(T^*)$

$$\begin{aligned}&\|U^n_1\|_{\mathcal{H}^{s_1,s_2}},\quad \|U^n_1\|_{\mathcal{H}^{s_\Omega,s_\Gamma}},\quad \|U^n_1\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}\le 1+M(T^*),\\ &\|U_1\|_{\mathcal{H}^{s_1,s_2}},\quad \|U_1\|_{\mathcal{H}^{s_\Omega,s_\Gamma}},\quad \|U_1\|_{\mathcal{H}^{\widetilde{\sigma_\Omega},\widetilde{\sigma_\Gamma}}}\le 1+M(T^*),\end{aligned}\tag{6.19}$$

where we denote $U^n_1=U^n(T_4(T^*))$ and $U_1=U(T_4(T^*))$. Since problem (1.1) is autonomous, by applying Lemma 3.3–(ii) and Theorem 6.1, starting from (6.19) we can repeat all arguments from (6.5) to (6.17), getting in this way that

$$2T_4(T^*)\le T_3(T^*)+T_4(T^*)<T_{\max},\quad 2T_4(T^*)\le T_3(T^*)+T_4(T^*)<T^n_{\max},\tag{6.20}$$

for all $n\ge n_2(T^*)$, and

$$\begin{aligned}&\widetilde{c_m}''\|[w^n_t]_\alpha\|^m_{L^m(T_4(T^*),2T_4(T^*));L^m_\alpha(\Omega)},\quad \widetilde{c_\mu}''\|[w_{|\Gamma}{}^n_t]^\mu_{L^\mu(T_4(T^*),2T_4(T^*));L^\mu_\beta(\Gamma_1)}\to 0,\\ &W^n\to 0\quad\text{in } C([T_4(T^*),2T_4(T^*)];\mathcal{H}^{s_1,s_2})\text{ as } n\to\infty,\end{aligned}$$

which by (6.17) implies

$$\begin{aligned}&\widetilde{c_m}''\|[w^n_t]_\alpha\|^m_{L^m(0,2T_4(T^*));L^m_\alpha(\Omega)},\quad \widetilde{c_\mu}''\|[w_{|\Gamma}{}^n_t]^\mu_{L^\mu(0,2T_4(T^*));L^\mu_\beta(\Gamma_1)}\to 0,\\ &W^n\to 0\quad\text{in } C([0,2T_4(T^*)];\mathcal{H}^{s_1,s_2})\qquad\text{as } n\to\infty.\end{aligned}\tag{6.21}$$

If $\kappa_1(T^*)=1$, that is if $T^*\le 2T_4(T^*)$, by (6.21) we have

$$\begin{gathered}\widetilde{c_m}''\|[w^n_t]_\alpha\|^m_{L^m(0,T^*);L^m_\alpha(\Omega)}+\widetilde{c_\mu}''\|[w_{|\Gamma}{}^n_t]^\mu_{L^\mu(0,T^*);L^\mu_\beta(\Gamma_1)}\to 0,\\ U^n\to U\quad\text{in } C([0,T^*];\mathcal{H}^{s_1,s_2}),\end{gathered}$$

that is the conclusion (ii) in the statement of Theorem 1.3. Moreover, in this case by (6.20) we have

$$T^*<T^n_{\max}\qquad\text{for all } n\ge n_2(T^*).\tag{6.22}$$

If $\kappa_1(T^*)>1$ we repeat the procedure above $\kappa_1(T^*)-1$ times to get

$$\begin{aligned}&T^*\le(\kappa_1(T^*)+1)T_4(T^*)<T^n_{\max}\qquad\text{for all } n\ge n_{\kappa_1(T^*)+1}(T^*),\\ &\widetilde{c_m}''\|[w^n_t]_\alpha\|^m_{L^m(0,(\kappa_1(T^*)+1)T_4(T^*));L^m_\alpha(\Omega)}\to 0,\\ &\widetilde{c_\mu}''\|[w_{|\Gamma}{}^n_t]^\mu_{L^\mu(0,(\kappa_1(T^*)+1)T_4(T^*));L^\mu_\beta(\Gamma_1)}\to 0,\\ &W^n\to 0\quad\text{in } C([0,(\kappa_1(T^*)+1)T_4(T^*)];\mathcal{H}^{s_1,s_2})\qquad\text{as } n\to\infty.\end{aligned}\tag{6.23}$$

By (6.23) and (6.18) we then get

$$\begin{aligned}&\widetilde{c_m}''\|[w^n_t]_\alpha\|^m_{L^m(0,T^*);L^m_\alpha(\Omega)},\quad \widetilde{c_\mu}''\|[w_{|\Gamma}{}^n_t]^\mu_{L^\mu(0,T^*);L^\mu_\beta(\Gamma_1)}\to 0,\\ &W^n\to 0\quad\text{in } C([0,T^*];\mathcal{H}^{s_1,s_2})\qquad\text{as } n\to\infty,\end{aligned}\tag{6.24}$$

and, being $T^*\in(0,T_{\max})$ arbitrary, $T_{\max}\le\underline{\lim}_n T^n_{\max}$. □

The aim of our final main result is to get well–posedness for spaces in Corollary 5.3.

Theorem 6.4 (**Local Hadamard well–posedness II).** *Suppose that (PQ1–3), (PQ4)′,(FG1), (FGQP1), (FG2)′,* (1.33) *hold and* ρ,θ *satisfy* (1.36). *Then problem* (1.1) *is locally well–posed in* $H^{1,\rho,\theta}_{\alpha,\beta}\times H^0$*, i.e. the conclusions of Theorem 1.3 hold true with* H^{1,s_1,s_2} *replaced by* $H^{1,\rho,\theta}_{\alpha,\beta}$ *and problem* (1.2) *is generalized to problem* (1.1). *In particular it is locally–well posed in* $\mathcal{H}^{\rho,\theta}$ *when* ρ,θ *satisfy* (1.36), $\rho\le r_\Omega$ *if* $p\le 1+r_\Omega/2$ *and* $\theta\le r_\Gamma$ *if* $q\le 1+r_\Gamma/2$.

Proof. At first we remark that, for any ρ,θ satisfying (1.36), setting

$$\overline{s_1}(\rho)=\begin{cases}2, & \text{if } p<r_\Omega,\\ \rho & \text{if } p\ge r_\Omega,\end{cases}\qquad \overline{s_2}(\theta)=\begin{cases}2, & \text{if } q<r_\Gamma,\\ \theta & \text{if } q\ge r_\Gamma,\end{cases}\tag{6.25}$$

when $p\ge r_\Omega$ we have $\rho\ge r_\Omega$ so, by (FGQP1), $L^{2,\rho}_\alpha(\Omega)=L^\rho(\Omega)$, while by the same arguments when $q\ge r_\Gamma$ we have $L^{2,\rho}_\beta(\Gamma_1)=L^\rho(\Gamma_1)$. Hence $H^{1,\rho,\theta}_{\alpha,\beta}\hookrightarrow H^{1,\overline{s_1}(\rho),\overline{s_2}(\theta)}$.

Consequently, given a sequence $U_{0n}\to U_0$ in $H^{1,\rho,\theta}_{\alpha,\beta}\times H^0$, we have $U_{0n}\to U_0$ in $\mathcal{H}^{\overline{s_1}(\rho),\overline{s_2}(\theta)}$, hence, since $(\overline{s_1}(\rho),\overline{s_2}(\theta))$ satisfies (1.34), by (6.24) we get

$$\begin{aligned}&\widetilde{c_m}''\|[w^n_t]_\alpha\|^m_{L^m(0,T^*);L^m_\alpha(\Omega)},\quad \widetilde{c_\mu}''\|[w_{|\Gamma}{}^n_t]_\beta\|^\mu_{L^\mu(0,T^*);L^\mu_\beta(\Gamma_1)}\to 0,\\ &W^n\to 0\quad \text{in } C([0,T^*];\mathcal{H})\quad \text{as } n\to\infty,\end{aligned}\tag{6.26}$$

so to prove that $W^n\to 0$ in $C([0,T^*];H^{1,\rho,\theta}_{\alpha,\beta}\times H^0)$, by Sobolev embeddings, reduces to prove the following two facts: if $\rho>r_\Omega$ then $w_n\to 0$ in $C[0,T^*];L^{2,\rho}_\alpha(\Omega))$, and, if $\theta>r_\Gamma$, then $w_n\to 0$ in $C[0,T^*];L^{2,\rho}_\beta(\Gamma_1))$. We prove the first one. When $\rho>r_\Omega$, by (1.36) we also have $m>r_\Omega$. Then, by (PQ4)′, see Remark 3.8, we have $\widetilde{c_m}''>0$, so by (6.26) we get $w^n_t\to 0$ in $L^m(0,T^*;L^{2,m}_\alpha(\Omega))$. Consequently, since $\rho\le m$ by (1.36), $w^n_t\to 0$ in $L^\rho(0,T^*;L^{2,\rho}_\alpha(\Omega))$. Since $w_{0n}\to 0$ in $L^{2,\rho}_\alpha(\Omega)$ then $w_n\to 0$ in $C[0,T^*];L^{2,\rho}_\alpha(\Omega))$. The proof of the second fact uses similar arguments and it is omitted. Finally, when (ρ,θ) satisfy (1.36), $\rho\le r_\Omega$ if $p\le 1+r_\Omega/2$ and $\theta\le r_\Gamma$ if $q\le 1+r_\Gamma/2$, by Remark 1.9 we have $H^{1,\rho,\theta}=H^{1,\rho,\theta}_{\alpha,\beta}$. □

Proof of Theorems 1.2–1.4 and Corollary 1.4 in Section 1. When

$$P(x,v)=\alpha(x)P_0(v),\quad Q(x,v)=\beta(x)Q_0(v),\quad f(x,u)=f_0(u),\quad g(x,u)=g_0(u),$$

by Remarks 3.1–3.3 assumptions (PQ1–3), (FG1), (FGQP1) reduce to (I–III). Moreover by Remark 3.6 (FG2) and (FG2)′ reduce to (IV) and (IV)′, while by Remark 3.9 (PQ4) and (PQ4)′ reduce to (V) and (V)′. Hence Theorems 1.2–1.4 and Corollary 1.4 are particular cases of Theorem 6.2–6.4 and Corollary 6.1. □

7. Global existence

In this section we shall prove that when the source parts of the perturbation terms f and g has at most linear growth at infinity, uniformly in the space variable, or, roughly, it is dominated by the corresponding damping term, then weak solutions of (1.1) found in Theorem 5.1 are global in time provided $u_0\in H^{1,p,q}$.

To precise our statement we introduce, the assumption (FG1) being in force, the primitives of the functions f and g by

$$\mathfrak{F}(x,u)=\int_0^u f(x,s)\,ds,\qquad \text{and}\quad \mathfrak{G}(y,u)=\int_0^u g(y,s)\,ds,\tag{7.1}$$

for a.a. $x \in \Omega$, $y \in \Gamma_1$ and all $u \in \mathbb{R}$. Moreover we shall make the following specific assumption:

(FGQP2) there are p_1 and q_1 verifying (1.6) and constants $C_{p_1}, C_{q_1} \ge 0$ such that
$$\mathfrak{F}(x,u) \le C_{p_1}\left[1+u^2+\alpha(x)|u|^{p_1}\right], \quad \mathfrak{G}(y,u) \le C_{q_1}\left[1+u^2+\beta(y)|u|^{q_1}\right]$$
for a.a. $x \in \Omega$, $y \in \Gamma_1$ and all $u \in \mathbb{R}$.

Since $\mathfrak{F}(\cdot,u) = \int_0^1 f(\cdot,su)u\,ds$ (and similarly $\mathfrak{G}$), assumption (FGQP2) is a weak version of of the following one:

(FGQP2)$'$ there are p_1 and q_1 verifying (1.6) and constants $C'_{p_1}, C'_{q_1} \ge 0$ such that
$$f(x,u)u \le C'_{p_1}\left[|u|+u^2+\alpha(x)|u|^{p_1}\right], \quad g(y,u)u \le C'_{q_1}\left[|u|+u^2+\beta(y)|u|^{q_1}\right]$$
for a.a. $x \in \Omega$, $y \in \Gamma_1$ and all $u \in \mathbb{R}$.

Remark 7.1. Assumptions (FG1) and (FGQP2)$'$ hold provided
$$f = f^0+f^1+f^2, \qquad g = g^0+g^1+g^2, \tag{7.2}$$
where f^i, g^i satisfy the following assumptions:

(i) f^0 and g^0 are a.e. bounded and independent on u;
(ii) f^1 and g^1 satisfy (FG1) with exponents p_1 and q_1 satisfying (1.6), and
  (a) when $p_1 > 2$ and $\operatorname{essinf}_\Omega \alpha = 0$ there is a constant $\overline{c_{p_1}} \ge 0$ such [13] that
  $$|f^1(x,u)| \le \overline{c_{p_1}}\left[1+|u|+\alpha(x)|u|^{p_1-1}\right]$$
  for a.a. $x \in \Omega$ and all $u \in \mathbb{R}$;
  (b) when $q_1 > 2$ and $\operatorname{essinf}_{\Gamma_1} \beta = 0$ there is a constant $\overline{c_{q_1}} \ge 0$ such that
  $$|g^1(y,u)| \le \overline{c_{q_1}}\left[1+|u|+\beta(y)|u|^{q_1-1}\right]$$
  for a.a. $y \in \Gamma_1$ and all $u \in \mathbb{R}$;
(iii) f^2 and g^2 satisfy (FG1), $f^2(x,u)u \le 0$ and $g^2(y,u)u \le 0$ for a.a. $x \in \Omega$, $y \in \Gamma_1$ and all $u \in \mathbb{R}$.

Conversely any couple of functions f and g satisfying (FG1) and (FGQP2)$'$ admits a decomposition of the form (7.2)–(i–iii) with f^1 and g^1 being source terms. Indeed one can set $f^0 = f(\cdot,0)$,
$$f^1(\cdot,u) = \begin{cases} [f(\cdot,u)-f^0]^+ & \text{if } u>0,\\ 0 & \text{if } u=0,\\ -[f(\cdot,u)-f^0]^- & \text{if } u<0,\end{cases} \text{ and } f^2(\cdot,u) \begin{cases} -[f(\cdot,u)-f^0]^- & \text{if } u>0,\\ 0 & \text{if } u=0,\\ [f(\cdot,u)-f^0]^+ & \text{if } u<0,\end{cases}$$
and define g^0, g^1, g^2 in the analogous way.

Remark 7.2. When dealing with problem (1.2) assumption (FGQP2) reduces to (VI). The function $f \equiv f_2$ defined in (3.14) satisfies (FGQP2) provided one among the following cases occurs:

(i) $\gamma_1^+ = \gamma_2^+ \equiv 0$,
(ii) $\gamma_2^+ \equiv 0$, $\gamma_1^+ \not\equiv 0$, $\widetilde{p} \le \max\{2,m\}$ and $\gamma_1 \le c'_1\alpha$ a.e. in Ω when $\widetilde{p} > 2$
(iii) $\gamma_1^+ \ne 0$, $\gamma_2^+ \not\equiv 0$, $p \le \max\{2,m\}$, $\gamma_1 \le c'_1\alpha$ when $\widetilde{p} > 2$ and $\gamma_2 \le c'_2\alpha$ when $p > 2$, a.e. in Ω,

[13] that is $\overline{\lim}_{|u|\to\infty} |f_1(\cdot,u)|/|u|^{p_1-1} \le \overline{c_{p_1}}\alpha$ a.e. uniformly in Ω.

where $c_1', c_2' \geq 0$ denote suitable constants. The analogous cases (j–jjj) occurs when $g \equiv g_2$, so that (f_2, g_2) satisfies (FGQP2) provided any combination between the cases (i–iii) and (j–jjj) occurs. In particular then a damping term can be localized provided the corresponding source is equally localized.

Finally when $f \equiv f_3$ and $g \equiv g_3$ as in (3.15), assumption (FGQP2) holds provided f_0 and g_0 satisfy assumption (VI) (where we conventionally take $f_0 \equiv 0$ when $\gamma \equiv 0$ and $g_0 \equiv 0$ when $\delta \equiv 0$), $\gamma \leq \alpha$ when $p_1 > 2$ and $\delta \leq \beta$ when $q_1 > 2$.

We can now state the main result of this section.

Theorem 7.1 (**Global analysis**). *The following conclusions hold true.*

(i) **(Global existence)** *Suppose that (PQ1–3), (FG1–2) and (FGQP1–2) hold. Then for any* $(u_0, u_1) \in \mathcal{H}^{l_\Omega, l_\Gamma}$ *the weak maximal solution* u *of problem* (1.1) *found in Theorem 5.1 is global in time, that is* $T_{max} = \infty$*, and* $u \in C([0, T_{max}); H^{1, l_\Omega, l_\Gamma})$.

In particular, when (1.28) *holds, for any* $(u_0, u_1) \in \mathcal{H}$ *problem* (1.1) *has a global weak solution.*

(ii) **(Global existence–uniqueness)** *Suppose that (PQ1–3), (FG1), (FG2)′ and (FGQP1–2) hold. Then for any* $(u_0, u_1) \in H^{1, s_\Omega, s_\Gamma} \times H^0$ *the unique maximal solution of problem* (1.1) *found in Theorem 6.2 is global in time, that is* $T_{max} = \infty$*, and* $u \in C([0, \infty); H^{1, s_\Omega, s_\Gamma})$.

In particular, when (1.28) *holds, for any* $(u_0, u_1) \in H^1 \times H^0$ *problem* (1.1) *has a unique global weak solution.*

(iii) **(Global Hadamard well–posedness)** *Suppose that (PQ1–4), (FG1), (FG2)′, (FGQP1–2) and* (1.33) *hold. Then problem* (1.1) *is globally well–posed in* $H^{1, s_1, s_2} \times H^0$ *for* s_1 *and* s_2 *satisfying* (1.34)*, that is* $T_{max} = \infty$ *in Theorem 6.3.*

Consequently the semi–flow generated by problem (1.2) *is a dynamical system in* $H^{1, s_1, s_2} \times H^0$.

In particular, when $2 \leq p < r_\Omega$ *and* $2 \leq q < r_\Gamma$ *and under assumptions (PQ1–3), (FG1), (FG2)′, (FGQP1–2), problem* (1.1) *is globally well–posed in* $H^1 \times H^0$*, so the semi–flow generated by* (1.1) *is a dynamical system in* $H^1 \times H^0$.

To prove Theorem 7.1 we shall use following abstract version of the classical chain rule, which proof is given for the reader's convenience.

Lemma 7.1. *Let* X_1 *and* Y_1 *be real Banach spaces such that* $X_1 \hookrightarrow Y_1$ *with dense embedding, so that* $Y_1' \hookrightarrow X_1'$*, and let* I *be a bounded real interval.*

Then for any $J_1 \in C^1(X_1)$ *having Frèchet derivative* $J_1' \in C(X_1; Y_1')$ *and any* $w \in W^{1,1}(I; Y_1) \cap C(\overline{I}; X_1)$ *we have* $J_1 \cdot w \in W^{1,1}(I)$ *and* $(J_1 \cdot w)' = \langle J_1' \cdot w, w' \rangle_{Y_1}$ *almost everywhere in* I*, where* $\cdot$ *denotes the composition product.*

Proof. We first note that, when $w \in C^1(\mathbb{R}; X_1)$, by the chain rule for the Frèchet derivative (see [2, Proposition 1.4, p. 12]), we have $J_1 \cdot w \in C^1(\mathbb{R})$ and

$$(J_1 \cdot w)' = \langle J_1' \cdot w, w' \rangle_{X_1} = \langle J_1' \cdot w, w' \rangle_{X_1} \qquad \text{in } \mathbb{R}.$$

When $w \in W^{1,1}(I; Y_1) \cap C(\overline{I}; X_1)$ we first extend it by reflexion to $w \in W^{1,1}(\mathbb{R}; Y_1) \cap C(\mathbb{R}; X_1)$ as in [13, Theorem 8.6, p. 209]). Then, denoting by $(\rho_n)_n$ a standard sequence of mollifiers and by $*$ the standard convolution product in $\mathbb{R}$, we set $w_n=\rho_n * w$, so $w_n\in C^1(\mathbb{R};X_1)$ and, as in [13, Proposition 4.21, p. 108 and proof of Theorem 8.7, p. 211]), we have $w_{n|I}\to w$ in $W^{1,1}(I;Y_1)\cap C(\overline{I};X_1)$. By previous remark

$$\int_I J_1\cdot w_n\varphi' = -\int_I \langle J_1'\cdot w_n, w_n'\rangle_{Y_1}\phi \qquad \text{for all } \phi\in C^1_c(I) \text{ and } n\in\mathbb{N}. \tag{7.3}$$

We now claim that $C_0:=\bigcup_{i=0}^\infty w_{k_i}(\overline{I})$ (where we denoted $w_0=w$) is compact in X_1. Indeed, given any sequence $(x_n)_n$ in C_0, either there is $N_0\in\mathbb{N}$ such that $x_n\in\bigcup_{i=0}^{N_0} w_{k_i}(\overline{I})$ for all $n\in\mathbb{N}$, and hence $(x_n)_n$ has a convergent subsequence since this set is compact, or there are sequences $(t_n)_n$ in $\overline{I}$ and $(k_n)_n$ in $\mathbb{N}_0$ such that $x_n=w_{k_n}(t_n)$ for all $n\in\mathbb{N}$ and $k_n\to\infty$. Then $t_n\to\overline{t}\in\overline{I}$, up to a subsequence, and consequently

$$\|x_n-w(\overline{t})\|_{X_1}\le \|w_{k_n}(t_n)-w(t_n)\|_{X_1}+\|w(t_n)-w(\overline{t})\|_{X_1}\to 0,$$

proving our claim.

We now pass to the limit in (7.3). By the continuity of J_1 and our claim we get that $J_1\cdot w_n\to J_1\cdot w$ in I and that $J_1(C_0)$ is compact, and hence bounded, in X_1. Consequently $(J_1\cdot w_n)_n$ is uniformly bounded in I and we get $\lim_n\int_I J_1\cdot w_n\varphi' = \int_I J_1\cdot w\,\varphi'$. Moreover, up to a subsequence, there is $\psi\in L^1(I)$ such that $w_n'\to w'$ and $\|w_n'\|_{Y_1}\le\psi$ a.e. in I. By the continuity of J_1' and our claim we get that $J_1'\cdot w_n\to J_1'\cdot w$ in I and that $J_1'(C_0)$ is compact in Y_1'. Consequently

$$M:=\sup\{\|J_1'\cdot w_n(t)\|_{Y_1'}, n\in\mathbb{N}, t\in I\}<\infty.$$

It follows that $\langle J_1'\cdot w_n, w_n'\rangle_{Y_1}\to\langle J_1'\cdot w, w'\rangle_{Y_1}$ a.e in I and that $\langle J_1'\cdot w_n, w_n'\rangle_{Y_1}\le M\psi$, so $\lim_n\int_I\langle J_1'\cdot w_n, w_n'\rangle_{Y_1}=\int_I\langle J_1'\cdot w, w'\rangle_{Y_1}$. □

Proof of Theorem 7.1. We first remark that, since $\mathcal{H}^{s_\Omega,s_\Gamma}\subset\mathcal{H}^{l_\Omega,l_\Gamma}$, parts (ii) and (iii) simply follow by combining Theorems 6.2–6.3 with part (i), hence in the sequel we are just going to prove it. Since $\mathcal{H}^{l_\Omega,l_\Gamma}\subset\mathcal{H}^{\sigma_\Omega,\sigma_\Gamma}$ by Theorem 5.1 problem (1.1) has a maximal weak solution u in $[0,T_{\max})$ and

$$\lim_{t\to T_{\max}^-}\|u(t)\|_{H^1}+\|u'(t)\|_{H^0}=\infty. \tag{7.4}$$

provided $T_{\max}<\infty$. Moreover, by (1.20) and (1.22), the couple (l_Ω,l_Γ) satisfies (1.14) and (1.29), so $u\in C([0,T_{\max});H^{1,l_\Omega,l_\Gamma})$ by Corollary 5.3. We no suppose by contradiction that $T_{\max}<\infty$, so (7.4) holds.

By (FG1) and Sobolev embedding theorem we can set the potential operator $J: H^{1,p,q}\to\mathbb{R}$ by

$$J(v)=\int_\Omega\mathfrak{F}(\cdot,v)+\int_{\Gamma_1}\mathfrak{G}(\cdot,v_{|\Gamma}) \qquad \text{for all } v\in H^{1,p,q}, \tag{7.5}$$

and, using standard results on Nemitskii operators (see [2, pp. 16–22]) one easily gets that $J\in C^1(H^{1,p,q})$, with Frèchet derivative $J'=(\widehat{f},\widehat{g})$. Moreover, by Lemma 3.2, $\widehat{f}\in C(L^p(\Omega);L^{p'}(\Omega))$, $\widehat{f}\in C(H^1(\Omega);L^{m_p'}(\Omega))$, $\widehat{g}\in C(L^q(\Gamma_1);L^{q'}(\Gamma_1))$

and $\widehat{g} \in C(H^1(\Gamma)\cap L^2(\Gamma_1); L^{\mu_q'}(\Gamma_1))$. Hence, setting the auxiliary exponents

$$\widetilde{m_p} = \begin{cases} m_p = 2 & \text{if } p \le 1 + r_\Omega/2,\\ m_p = m & \text{if } p > \max\{m, 1 + r_\Omega/2\},\\ p & \text{if } 1 + r_\Omega/2 < p \le m, \end{cases} \widetilde{\mu_q} = \begin{cases} \mu_q = 2 & \text{if } q \le 1 + r_\Gamma/2,\\ \mu_q = \mu & \text{if } q > \max\{\mu, 1 + r_\Gamma/2\},\\ q & \text{if } 1 + r_\Gamma/2 < q \le \mu, \end{cases}$$

we have $J' = (\widehat{f}, \widehat{g}) \in C(H^{1,p,q}; L^{\widetilde{m_p}'}(\Omega) \times L^{\widetilde{\mu_q}'}(\Gamma_1))$. We also introduce the functional $\mathcal{I} : H^{1,p,q} \to \mathbb{R}_0^+$ given by

$$\mathcal{I}(v) = C_{p_1}\int_\Omega \alpha|v|^{p_1} + C_{q_1}\int_{\Gamma_1}\beta|v|^{q_1}. \tag{7.6}$$

Since by (1.6) we have $p_1 \le p$ and $q_1 \le q$, the functions

$$\mathfrak{f}(x,u) = p_1 C_{p_1}\alpha(x)|u|^{p_1-2}u \qquad \text{and} \qquad \mathfrak{g}(x,u) = q_1 C_{q_1}\beta(x)|u|^{q_1-2}u \tag{7.7}$$

satisfy assumption (FG1) with exponents p and q, hence by repeating previous arguments $\mathcal{I} \in C^1(H^{1,p,q})$, with Frèchet derivative $\mathcal{I}' = (\mathfrak{f}, \mathfrak{g}) \in C(H^{1,p,q}; L^{\widetilde{m_p}'}(\Omega) \times L^{\widetilde{\mu_q}'}(\Gamma_1))$.

We are now going to apply Lemma 7.1, with $X_1 = H^{1,p,q}$ and $Y_1 = L^{\widetilde{m_p}}(\Omega) \times L^{\widetilde{\mu_q}}(\Gamma_1)$, to the potential operators $J_1 = J$ and $J_1 = \mathcal{I}$, to $w = u$ and to $I = [s,t] \subset [0, T_{\max})$. By the definition of $\widetilde{m_p}$, $\widetilde{\mu_q}$ and [55, Lemma 2.1] we have $H^{1,p,q} \hookrightarrow L^{\widetilde{m_p}}(\Omega) \times L^{\widetilde{\mu_q}}(\Gamma_1)$ with dense embedding. Moreover, since $H^{1,l_\Omega,l_\Gamma} \hookrightarrow H^{1,p,q}$, we have $u \in C([0,T_{\max}); H^{1,p,q})$. Next, since by definition $\widetilde{m_p} \le m_p$ and $\widetilde{\mu_q} \le \mu_q$, by (3.18) we have $u' \in L^1_{\text{loc}}([0,T_{\max}); L^{\widetilde{m_p}}(\Omega) \times L^{\widetilde{\mu_q}}(\Gamma_1))$.

Then, by Lemma 7.1, we have $J \cdot u$, $\mathcal{I} \cdot u \in W^{1,1}_{\text{loc}}([0,T_{\max}))$ and

$$J(u(t)) - J(u(s)) = \int_s^t \left[\int_\Omega f(\cdot,u)u_t + \int_{\Gamma_1} g(\cdot,u_{|\Gamma})(u_{|\Gamma})_t\right], \tag{7.8}$$

$$\mathcal{I}(u(t)) - \mathcal{I}(u(s)) = \int_s^t \left[\int_\Omega \mathfrak{f}(\cdot,u)u_t + \int_{\Gamma_1} \mathfrak{g}(\cdot,u_{|\Gamma})(u_{|\Gamma})_t\right], \tag{7.9}$$

for all $s,t \in [0,T_{\max})$, where $\mathfrak{f}$ and $\mathfrak{g}$ are given by (7.7).

We also introduce the energy functional $\mathcal{E} \in C^1(\mathcal{H}^{p,q})$ defined for $(v,w) \in \mathcal{H}^{p,q}$ by

$$\mathcal{E}(v,w) = \frac{1}{2}\|w\|^2_{H^0} + \tfrac{1}{2}\int_\Omega |\nabla v|^2 + \tfrac{1}{2}\int_{\Gamma_1} |\nabla_\Gamma v|^2_\Gamma - J(v), \tag{7.10}$$

and the energy function associated to u by

$$\mathcal{E}_u(t) = \mathcal{E}(u(t), u'(t)), \qquad \text{for all } t \in [0,T_{\max}). \tag{7.11}$$

By (7.8) and (7.10) the energy identity (3.20) can be rewritten as

$$\mathcal{E}_u(t) - \mathcal{E}_u(s) + \int_s^t \langle B(u'), u'\rangle_Y = 0 \quad \text{for all } s,t \in [0,T_{\max}). \tag{7.12}$$

Consequently, by (2.5), (7.10) and (7.11), for $t \in [0,T_{\max})$ we have

$$\tfrac{1}{2}\|u'(t)\|^2_{H^0} + \tfrac{1}{2}\|u(t)\|^2_{H^1} \le \mathcal{E}_u(0) + \tfrac{1}{2}\|u(t)\|^2_{H^0} + J(u(t)) - \int_0^t \langle B(u'), u'\rangle_Y. \tag{7.13}$$

The proof can then be completed, starting from (7.13), as in [55, Proof of Theorem 6.2], since (7.13) is nothing but (the correct form of) formula [55, (6.18)]. For the reader's convenience we repeat in the sequel the arguments used there.

We introduce an auxiliary function associated to u by

$$\Upsilon(t)=\tfrac12\|u'(t)\|_{H^0}^2+\tfrac12\|u(t)\|_{H^1}^2+\mathcal{I}(u(t)),\qquad \text{for all } t\in[0,T_{\max}). \tag{7.14}$$

By (7.13) and (7.14) we have

$$\Upsilon(t)\le \mathcal{E}_u(0)+\tfrac12\|u(t)\|_{H^0}^2+J(u(t))+\mathcal{I}(u(t))-\int_0^t\langle B(u'),u'\rangle_W. \tag{7.15}$$

By (7.6) and assumption (FGQP1) we get

$$J(v)\le [C_{p_1}|\Omega|+C_{q_1}\sigma(\Gamma)]\left(1+\|v\|_{H^0}^2\right)+\mathcal{I}(v)\qquad \text{for all } v\in H^1. \tag{7.16}$$

By (7.15)– (7.16) we thus obtain

$$\Upsilon(t)\le \mathcal{E}_u(0)+k_{18}+k_{18}\|u(t)\|_{H^0}^2+2\mathcal{I}(u(t))-\int_0^t\langle B(u'),u'\rangle_W, \tag{7.17}$$

where $k_{18}=C_{p_1}|\Omega|+C_{q_1}\sigma(\Gamma)+1/2$. Writing $\|u(t)\|_{H^0}^2=\|u_0\|_{H^0}^2+2\int_0^t(u',u)_{H^0}$ in (7.17) and using (7.6) and (7.9) we get

$$\begin{aligned}\Upsilon(t)\le &K_{31}+\int_0^t\Bigg[2k_{18}(u',u)_{H^0}-\langle B(u'),u'\rangle_W\\ &+2p_1C_{p_1}\int_\Omega\alpha|u|^{p_1-2}uu_t+2q_1C_{q_1}\int_{\Gamma_1}\beta|u|^{q_1-2}u(u_{|\Gamma})_t\Bigg],\end{aligned} \tag{7.18}$$

where $K_{31}=K_{31}(u_0,u_1)=\mathcal{E}_u(0)+2\mathcal{I}(u_0)+k_{18}(1+\|u_0\|_{H^0}^2)$. Consequently, by assumption (PQ3), Cauchy–Schwartz and Young inequalities, we get the preliminary estimate

$$\begin{aligned}\Upsilon(t)\le &K_{31}+\int_0^t\Big[-c_m'\|[u_t]_\alpha\|_{m,\alpha}^m-c_\mu'\|[(u_\Gamma)_t]_\beta\|_{\mu,\beta}^\mu+k_{18}\left(\|u'\|_{H^0}^2+\|u\|_{H^0}^2\right)\\ &+2p_1C_{p_1}\int_\Omega\alpha|u_t||u|^{p-1}|u_t|+2q_1C_{q_1}\int_{\Gamma_1}\beta|u|^{q-1}|(u_{|\Gamma})_t||(u_{|\Gamma})_t|\Big]\end{aligned} \tag{7.19}$$

for all $t\in[0,T_{\max})$. We now estimate, a.e. in $[0,T_{\max})$, the last three integrands in the right–hand side of (7.19). By (7.14) we get

$$k_{18}\|u'\|_{H^0}^2\le 2k_{18}\Upsilon. \tag{7.20}$$

Moreover, by the embedding $H^1(\Omega;\Gamma)\hookrightarrow L^2(\Omega)\times L^2(\Gamma)$,

$$\|u\|_{H^0}^2\le k_{19}\|u\|_{H^1}^2. \tag{7.21}$$

Consequently, by (7.14),

$$k_{18}\|u\|_{H^0}^2\le k_{20}\Upsilon. \tag{7.22}$$

To estimate the addendum $2p_1C_{p_1}\int_\Omega\alpha|u|^{p_1-1}|u_t|$ we now distinguish between the cases $p_1=2$ and $p_1>2$. When $p_1=2$, by (7.14), (7.22) and Young inequality,

$$2p_1C_{p_1}\int_\Omega\alpha|u|^{p-1}|u_t|\le p_1C_{p_1}\|\alpha\|_\infty(\|u\|_{H^0}^2+\|u'\|_{H^0}^2)\le k_{21}\Upsilon, \tag{7.23}$$

where $k_{21}=2p_1C_{p_1}\|\alpha\|_\infty(1+k_{19})$.

When $p_1>2$, for any $\varepsilon\in(0,1]$ to be fixed later, by weighted Young inequality

$$2p_1C_{p_1}\int_\Omega\alpha|u|^{p_1-1}|u_t|\le 2(p_1-1)C_{p_1}\varepsilon^{1-p_1'}\int_\Omega\alpha|u|^{p_1}+2\varepsilon C_{p_1}\int_\Omega\alpha|u_t|^{p_1}. \tag{7.24}$$

By (7.14) we have

$$2(p_1-1)C_{p_1}\varepsilon^{1-p_1'}\int_\Omega \alpha|u|^{p_1} \le 2(p_1-1)\varepsilon^{1-p_1'}\Upsilon. \tag{7.25}$$

Moreover by (1.6) we have $p_1 \le m = \overline{m}$ and consequently $|u_t|^{p_1} \le 1+|u_t|^m$ a.e. in Ω, which yields

$$\int_\Omega \alpha|u_t|^{p_1} \le \int_\Omega \alpha + \int_\Omega \alpha|u_t|^m \le \|\alpha\|_\infty|\Omega| + \|[u_t]_\alpha\|_{m,\alpha}^m. \tag{7.26}$$

Plugging (7.25) and (7.26) in (7.24) we get, as $\varepsilon \le 1$,

$$2p_1C_{p_1}\int_\Omega \alpha|u|^{p_1-1}|u_t| \le k_{22}\left(\varepsilon^{1-p_1'}\Upsilon + \varepsilon\|[u_t]_\alpha\|_{m,\alpha}^m + 1\right). \tag{7.27}$$

Comparing (7.21) and (7.27) we get that for $p \ge 2$ we have

$$2p_1C_{p_1}\int_\Omega \alpha|u|^{p_1-1}|u_t| \le k_{23}\left[(1+\varepsilon^{1-p_1'})\Upsilon + \varepsilon\|[u_t]_\alpha\|_{m,\alpha}^m + 1\right]. \tag{7.28}$$

We estimate the last integrand in the right–hand side of (7.19) by transposing from Ω to Γ_1 the arguments used to get (7.28). At the end we get

$$2q_1C_{q_1}\int_{\Gamma_1} \beta|u|^{q_1-1}|(u_{|\Gamma})_t| \le k_{24}\left[(1+\varepsilon^{1-q_1'})\Upsilon + \varepsilon\|[(u_{|\Gamma})_t]_\beta\|_{\mu,\beta,\Gamma_1}^\mu + 1\right]. \tag{7.29}$$

Plugging estimates (7.20), (7.22), (7.28) and (7.29) into (7.19) we get

$$\begin{aligned}\Upsilon(t) \le K_{31} &+ \int_0^t \left[(k_{23}\varepsilon - c_m')\|[u_t]_\alpha\|_{m,\alpha}^m + (k_{24}\varepsilon - c_\mu')\|[(u_\Gamma)_t]_\beta\|_{\mu,\beta}^\mu\right] \\ &+ k_{25}\int_0^t \left[(1+\varepsilon^{1-p_1'}+\varepsilon^{1-q_1'})\Upsilon + 1\right] \qquad \text{for all } t\in[0,T_{\max}).\end{aligned} \tag{7.30}$$

Fixing $\varepsilon = \varepsilon_1$, where $\varepsilon_1 = \min\{1, c_m'/k_{23}, c_\mu'/k_{24}\}$, and setting $K_{32} = K_{32}(u_0,u_1) = K_{31}(u_0,u_1) + k_{25}(1+\varepsilon_1^{1-p_1'}+\varepsilon_1^{1-q_1'})$, the estimate (7.30) reads as

$$\Upsilon(t) \le K_{32}(1+t) + K_{32}\int_0^t \Upsilon(s)\,ds \qquad \text{for all } t\in[0,T_{\max}).$$

Then, since $T_{\max} < \infty$, by Gronwall Lemma (see [47, Lemma 4.2, p. 179]), Υ is bounded in $[0,T_{\max})$, getting, by (7.4) and (7.14), the desired contradiction. □

We now state and prove, for the sake of clearness, two corollaries of Theorem 7.1–(i) which generalize Corollaries 1.5–1.6 in the introduction. The discussion made there applies here as well.

Corollary 7.1. *Suppose that (PQ1–3), (FG1), (FGQP1–2) and* (1.26) *hold. Then for any* $(u_0,u_1)\in H^{1,p,q}\times H^0$ *problem* (1.1) *has a global weak solution* $u\in C([0,\infty);H^{1,p,q})$.

Proof. Since (1.26) can be written also as $p \ne 1 + r_\Omega/m'$ when $p > 1 + r_\Omega/2$ and $q \ne 1 + r_\Gamma/\mu'$ when $q > 1 + r_\Gamma/2$, clearly assumption (FG2) can be skipped. Moreover, by (1.20), when (1.26) holds we have $H^{1,\sigma_\Omega,\sigma_\Gamma} = H^{1,p,q}$. □

Corollary 7.2. *Suppose that (PQ1–3), (FG1–2), (FGQP1–2) and* (1.27) *hold. Then for any* $(u_0,u_1)\in H^{1,p,q}\times H^0$ *problem* (1.1) *has a global weak solution* $u\in C([0,\infty);H^{1,p,q})$.

Proof. When (1.27) holds by (1.20) we have $H^{1,l_\Omega,l_\Gamma} = H^{1,p,q}$. □

We now state, for the reader convenience, the global–in–time version of the more general local analysis made in Corollaries 5.3, 6.1 and Theorem 6.4, simply obtained by combining them with Theorem 7.1.

Corollary 7.3 (Global analysis in the scale of spaces). *The following conclusions hold true.*

- (i) **(Global existence)** *Under assumptions (PQ1–3), (FG1–2), (FGQP1–2) the main conclusion of Theorem 7.1–(i) hold when H^{1,l_Ω,l_Γ} is replaced by*
  - (i.1) $H^{1,\rho,\theta}_{\alpha,\beta}$, *provided ρ and θ satisfy* (1.37)*, and by*
  - (i.2) $H^{1,\rho,\theta}$, *provided ρ and θ satisfy* (1.14) *and* (1.37)*.*
- (ii) **(Global existence–uniqueness)** *Under assumptions (PQ1–3), (FG1), (FG2)′, (FGQP1–2) the main conclusion of Theorem 7.1–(ii) holds when the space H^{1,s_Ω,s_Γ} is replaced by*
  - (ii.1) $H^{1,\rho,\theta}_{\alpha,\beta}$, *provided ρ and θ satisfy* (1.32) *and by*
  - (ii.2) $H^{1,\rho,\theta}$, *provided ρ and θ satisfy* (1.14) *and* (1.32)*.*
- (iii) **(Global Hadamard well–posedness)** *Under assumptions (PQ1–3), (PQ4)′, (FG1), (FGQP1), (FG2)′ and* (1.33) *problem* (1.1) *is locally well–posed*
  - (iii.1) *in $H^{1,\rho,\theta}_{\alpha,\beta}\times H^0$ when ρ and θ satisfy* (1.36) *(that is the conclusions of Theorem 7.1–(iii) hold true when H^{1,s_1,s_2} is replaced by $H^{1,\rho,\theta}_{\alpha,\beta}$), and*
  - (iii.2) $H^{1,\rho,\theta}\times H^0$ *when ρ,θ satisfy* (1.14) *and* (1.36) *(that is the conclusions of Theorem 7.1–(iii) hold true when H^{1,s_1,s_2} is replaced by $H^{1,\rho,\theta}$).*

Proof of Theorem 1.5 and Corollaries 1.5–1.7 in Section 1. When

$$P(x,v)=\alpha(x)P_0(v),\quad Q(x,v)=\beta(x)Q_0(v),\quad f(x,u)=f_0(u),\quad g(x,u)=g_0(u),$$

by Remarks 3.1–3.3 assumptions (PQ1–3), (FG1), (FGQP1) reduce to (I–III). Moreover by Remark 3.6 (FG2) and (FG2)′ reduce to (IV) and (IV)′. By Remark 3.9 (PQ4) and (PQ4)′ reduce to (V) and (V)′, while by Remark 7.2 assumption (FGQP2) reduces to (VI). Hence Theorem 1.5 and Corollaries 1.5–1.6 are particular cases of Theorem 7.1 and Corollaries 7.1–7.3. □

Appendix A. Proof of Lemma 5.3

To prove Lemma 5.3 we first recall the following elementary result, which proof is given only for the reader's convenience.

Lemma A.1. *Let $0<T<\infty$ and Y_1,Y_2 be two Banach spaces with Y_2 densely embedded in Y_1. Then $C^1_c((0,T);Y_2)$ is dense in $C^1_c((0,T);Y_1)$ with respect to the norm of $C^1([0,T];Y_1)$.*

Proof. Let $u\in C^1_c((0,T);Y_1)$ and $\eta\in(0,T/2)$ such that $\operatorname{supp} u\subset[\eta,T-\eta]$. Since u is uniformly continuous and Y_2 is dense in Y_1 one can easily build a sequence of piecewise linear functions $(v_n)_n$ in $C_c((0,T);Y_2)$ such that $v_n\to\dot u$ in $C([0,T];Y_1)$. Hence setting $w_n(t)=\int_0^t v_n(\tau)\,d\tau$ for $\tau\in[0,T]$ we have $w_n\in C^1([0,T];Y_2)$ and, since $w_n(0)=u(0)=0$, $w_n\to u$ in $C^1([0,T];Y_1)$. Consequently $w_n\to 0$ in $C^1([0,\eta]\cup[T-\eta,T];Y_1)$.

Taking a standard cut–off function $\xi \in C_c^\infty(\mathbb{R})$ such that $0 \le \xi \le 1$, $\xi = 1$ in $[\eta, T-\eta]$ and $\xi = 0$ in $(-\infty, \eta/2) \cup (T-\eta/2, \infty)$ and setting $u_n = \xi w_n$ we trivially have $u_n \in C^1([0,T]; Y_2)$, $\operatorname{supp} u_n \subset [\eta/2, T-\eta/2]$. Moreover $u_n = w_n \to u$ in $C^1([\eta, T-\eta]; Y_1)$, while $u_n \to 0$ in $C^1([0,\eta] \cup [T-\eta, T]; Y_1)$. Since $u = 0$ in $[0,\eta] \cup [T-\eta, T]$ we get $u_n \to u$ in $C^1([0,T]; Y_1)$. □

Proof of Lemma 5.3. Given $\varphi \in C_c((0,T); H^1) \cap C_c^1((0,T); H^0) \cap Z(0,T)$, trivially extended to $\varphi \in C_c(\mathbb{R}; H^1) \cap C_c^1(\mathbb{R}; H^0) \cap Z(-\infty, \infty)$, by standard time regularization we build a sequence $(\psi_n)_n$ in $C_c^1((0,T;X)$ such that $\psi_n \to \varphi$ in the norm of $C([0,T]; H^1) \cap C^1([0,T]; H^0) \cap Z(0,T)$, where $X = H^{1,\overline{m},\overline{\mu}}_{\alpha,\beta}$ (see (3.5)). Since by [55, Lemma 2.1] $H^{1,\infty,\infty}$ is dense in X it follows from Lemma A.1 that $C_c^1((0,T); H^{1,\infty,\infty})$ is dense in $C_c^1((0,T); X)$ with respect to the norm of $C^1([0,T]; X)$ and then, since $X = H^1 \cap [L^{2,\overline{m}}_\alpha(\Omega) \times L^{2,\overline{\mu}}_\beta(\Gamma_1)]$, with respect to the norm of $C([0,T]; H^1) \cap C^1([0,T]; H^0) \cap Z(0,T)$. □

TABLE 5. Further results when $N = 3$ and $\operatorname{essinf}_\Omega \alpha > 0$

	$2 \le q < \infty$
$2 \le p \le 4$	well–posedness in $H^{1,\rho,2}$ for $\rho \in [6, 6 \vee m]$
$4 < p < 6$	
$6 = p < m$	existence–uniqueness in $H^{1,\rho,2}$ for $\rho \in [6, m]$
	well–posedness in $H^{1,\rho,2}$ for $\rho \in (6, m]$
$6 = p = m$	existence–uniqueness in H^1
$6 < p < 1 + 6/m'$	local existence in $H^{1,\rho,2}$ for $\rho \in [6, m]$
	global existence in $H^{1,\rho,2}$ for $\rho \in [p, m]$
	existence–uniqueness in $H^{1,\rho,2}$ for $\rho \in [3(p-2)/2, m]$
	well–posedness in $H^{1,\rho,2}$ for $\rho \in (3(p-2)/2, m]$
$6 < p = 1 + 6/m'$	existence–uniqueness in $H^{1,\rho,2}$ for $\rho \in [3(p-2)/2, m]$
	well–posedness in $H^{1,\rho,2}$ for $\rho \in (3(p-2)/2, m]$

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TABLE 6. Further results when $N = 4$ and $\operatorname{essinf}_\Omega \alpha, \operatorname{essinf}_{\Gamma_1} \beta > 0$

<table>
<tr><th></th><th>$2 \le q \le 4$</th><th>$4 < q < 6$</th><th>$6 = q < \mu$</th><th>$6 = q = \mu$</th><th>$6 < q < 1 + 6/\mu'$</th><th>$6 < q = 1 + 6/\mu'$</th></tr>
<tr><td>$2 \le p \le 3$</td><td colspan="2" rowspan="2">well–posedness in $H^{1,\rho,\theta}$ for $\rho \in [4, 4 \vee m]$ $\theta \in [6, 6 \vee \mu]$</td><td rowspan="2">existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [4, 4 \vee m]$ $\theta \in [6, \mu]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in [4, 4 \vee m]$ $\theta \in (6, \mu]$</td><td rowspan="2">existence–uniqueness in $H^{1,\rho,2}$ for $\rho \in [4, 4 \vee m]$</td><td rowspan="2">local existence in $H^{1,\rho,\theta}$ for $\rho \in [4, 4 \vee m]$ $\theta \in [6, \mu]$
global existence in $H^{1,\rho,\theta}$ for $\rho \in [4, 4 \vee m]$ $\theta \in [q, \mu]$
existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [4, 4 \vee m]$ $\theta \in \left[\frac{3}{2}(q-2), \mu\right]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in [4, 4 \vee m]$ $\theta \in \left(\frac{3}{2}(q-2), \mu\right]$</td><td rowspan="2">existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [4, 4 \vee m]$ $\theta \in \left[\frac{3}{2}(q-2), \mu\right]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in [4, 4 \vee m]$ $\theta \in \left(\frac{3}{2}(q-2), \mu\right]$</td></tr>
<tr><td>$3 < p < 4$</td></tr>
<tr><td>$4 = p < m$</td><td colspan="2">existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [4, m]$ $\theta \in [6, 6 \vee \mu]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in (4, m]$ $\theta \in [6, 6 \vee \mu]$</td><td>existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [4, m]$ $\theta \in [6, \mu]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in (4, m]$ $\theta \in (6, \mu]$</td><td>existence–uniqueness in $H^{1,\rho,2}$ for $\rho \in [4, m]$</td><td>local existence in $H^{1,\rho,\theta}$ for $\rho \in [4, m]$ $\theta \in [6, \mu]$
global existence in $H^{1,\rho,\theta}$ for $\rho \in [4, m]$ $\theta \in [q, \mu]$
existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [4, m]$ $\theta \in \left[\frac{3}{2}(q-2), \mu\right]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in (4, m]$ $\theta \in \left(\frac{3}{2}(q-2), \mu\right]$</td><td>existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [4, m]$ $\theta \in \left[\frac{3}{2}(q-2), \mu\right]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in (4, m]$ $\theta \in \left(\frac{3}{2}(q-2), \mu\right]$</td></tr>
<tr><td>$4 = p = m$</td><td colspan="2">existence–uniqueness in $H^{1,2,\theta}$ for $\theta \in [6, 6 \vee \mu]$</td><td>existence–uniqueness in $H^{1,2,\theta}$ for $\theta \in [6, \mu]$</td><td>existence–uniqueness in H^1</td><td>local existence in $H^{1,2,\theta}$ for $\theta \in [6, \mu]$
global existence in $H^{1,2,\theta}$ for $\theta \in [q, \mu]$
existence–uniqueness in $H^{1,2,\theta}$ for $\theta \in \left[\frac{3}{2}(q-2), \mu\right]$</td><td>existence–uniqueness in $H^{1,2,\theta}$ for $\theta \in \left[\frac{3}{2}(q-2), \mu\right]$</td></tr>
<tr><td>$4 < p < 1 + 4/m'$</td><td colspan="2">local existence in $H^{1,\rho,\theta}$ for $\rho \in [4, m]$ $\theta \in [6, 6 \vee \mu]$
global existence in $H^{1,\rho,\theta}$ for $\rho \in [p, m]$ $\theta \in [6, 6 \vee \mu]$
existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [2(p-2), m]$ $\theta \in [6, 6 \vee \mu]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in (2(p-2), m]$ $\theta \in [6, 6 \vee \mu]$</td><td>local existence in $H^{1,\rho,\theta}$ for $\rho \in [4, m]$ $\theta \in [6, \mu]$
global existence in $H^{1,\rho,\theta}$ for $\rho \in [p, m]$ $\theta \in [6, \mu]$
existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [2(p-2), m]$ $\theta \in [6, \mu]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in (2(p-2), m]$ $\theta \in (6, \mu]$</td><td>local existence in $H^{1,\rho,2}$ for $\rho \in [4, m]$
global existence in $H^{1,\rho,2}$ for $\rho \in [p, m]$
existence–uniqueness in $H^{1,\rho,2}$ for $\rho \in [2(p-2), m]$</td><td>local existence in $H^{1,\rho,\theta}$ for $\rho \in [4, m]$ $\theta \in [6, \mu]$
global existence in $H^{1,\rho,\theta}$ for $\rho \in [p, m]$ $\theta \in [q, \mu]$
existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [2(p-2), m]$ $\theta \in \left[\frac{3}{2}(q-2), \mu\right]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in (2(p-2), m]$ $\theta \in \left(\frac{3}{2}(q-2), \mu\right]$</td><td>local existence in $H^{1,\rho,\theta}$ for $\rho \in [4, m]$ $\theta \in \left[\frac{3}{2}(q-2), \mu\right]$
global existence in $H^{1,\rho,\theta}$ for $\rho \in [p, m]$ $\theta \in \left[\frac{3}{2}(q-2), \mu\right]$
existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [2(p-2), m]$ $\theta \in \left[\frac{3}{2}(q-2), \mu\right]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in (2(p-2), m]$ $\theta \in \left(\frac{3}{2}(q-2), \mu\right]$</td></tr>
<tr><td>$4 < p = 1 + 4/m'$</td><td colspan="2">existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [2(p-2), m]$ $\theta \in [6, 6 \vee \mu]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in (2(p-2), m]$ $\theta \in [6, 6 \vee \mu]$</td><td>existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [2(p-2), m]$ $\theta \in [6, \mu]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in (2(p-2), m]$ $\theta \in (6, \mu]$</td><td>existence–uniqueness in $H^{1,\rho,2}$ for $\rho \in [2(p-2), m]$</td><td>local existence in $H^{1,\rho,\theta}$ for $\rho \in [2(p-2), m]$ $\theta \in [6, \mu]$
global existence in $H^{1,\rho,\theta}$ for $\rho \in [2(p-2), m]$ $\theta \in [q, \mu]$
existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [2(p-2), m]$ $\theta \in \left[\frac{3}{2}(q-2), \mu\right]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in (2(p-2), m]$ $\theta \in \left(\frac{3}{2}(q-2), \mu\right]$</td><td>existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in [2(p-2), m]$ $\theta \in \left[\frac{3}{2}(q-2), \mu\right]$
well–posedness in $H^{1,\rho,\theta}$ for $\rho \in (2(p-2), m]$ $\theta \in \left(\frac{3}{2}(q-2), \mu\right]$</td></tr>
</table>

TABLE 7. Further results when $N = 5$ and $\operatorname{essinf}_\Omega \alpha, \operatorname{essinf}_{\Gamma_1} \beta > 0$

	$2 \le q < 4$	$2 \le q < 4$	$4 = q < \mu$	$4 = q = \mu$	$4 < q < 1 + 4/\mu'$	$4 < q = 1 + 4/\mu'$
$2 \le p \le \frac{8}{3}$	well–posedness in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in [4, 4 \vee \mu]$		existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in [4, \mu]$ well–posedness in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in (4, \mu]$	existence–uniqueness in $H^{1,\rho,2}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$	local existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in [4, \mu]$ global existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in [q, \mu]$ existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in [2(q-2), \mu]$ well–posedness in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in (2(q-2), \mu]$	existence–uniqueness in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in [2(q-2), \mu]$ well–posedness in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in (2(q-2), \mu]$
$\frac{8}{3} < p < \frac{10}{3}$	existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in [4, 4 \vee \mu]$		existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in [4, \mu]$	existence in $H^{1,\rho,2}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$	local existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in [4, \mu]$ global existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in [q, \mu]$	existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, \frac{10}{3} \vee m\right]$ $\theta \in [2(q-2), \mu]$
$\frac{10}{3} = p \le m$	existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, m\right]$ $\theta \in [4, 4 \vee \mu]$		existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, m\right]$ $\theta \in [4, \mu]$	existence in $H^{1,\rho,2}$ for $\rho \in \left[\frac{10}{3}, m\right]$	local existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, m\right]$ $\theta \in [4, \mu]$ global existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, m\right]$ $\theta \in [q, \mu]$	existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, m\right]$ $\theta \in [2(q-2), \mu]$
$\frac{10}{3} < p < 1 + \frac{10}{3m'}$	local existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, m\right]$ $\theta \in [4, 4 \vee \mu]$ global existence in $H^{1,\rho,\theta}$ for $\rho \in [p, m]$ $\theta \in [4, 4 \vee \mu]$		local existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, m\right]$ $\theta \in [4, \mu]$ global existence in $H^{1,\rho,\theta}$ for $\rho \in [p, m]$ $\theta \in [4, \mu]$	local existence in $H^{1,\rho,2}$ for $\rho \in \left[\frac{10}{3}, m\right]$ global existence in $H^{1,\rho,2}$ for $\rho \in [p, m]$	local existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, m\right]$ $\theta \in [4, \mu]$ global existence in $H^{1,\rho,\theta}$ for $\rho \in [p, m]$ $\theta \in [q, \mu]$	local existence in $H^{1,\rho,\theta}$ for $\rho \in \left[\frac{10}{3}, m\right]$ $\theta \in [2(q-2), \mu]$ global existence in $H^{1,\rho,\theta}$ for $\rho \in [p, m]$ $\theta \in [2(q-2), \mu]$
$\frac{10}{3} < p = 1 + \frac{10}{3m'}$	no results					

TABLE 8. Further results when $N \ge 6$ and $\operatorname{essinf}_\Omega \alpha, \operatorname{essinf}_{\Gamma_1} \beta > 0$

	$2 \le q \le 1 + r_\Gamma/2$	$1 + r_\Gamma/2 < q \le r_\Gamma$	$r_\Gamma < q < 1 + r_\Gamma/\mu'$	$r_\Gamma < q = 1 + r_\Gamma/\mu'$
$2 \le p \le 1 + r_\Omega/2$	well–posedness in $H^{1,\rho,\theta}$ $\rho \in [r_\Omega, r_\Omega \vee m]$ $\theta \in [r_\Gamma, r_\Gamma \vee \mu]$		local existence in $H^{1,\rho,\theta}$ $\rho \in [r_\Omega, r_\Omega \vee m]$ $\theta \in [r_\Gamma, \mu]$	
$1 + r_\Omega/2 < p \le r_\Omega$	existence in $H^{1,\rho,\theta}$ $\rho \in [r_\Omega, r_\Omega \vee m]$ $\theta \in [r_\Gamma, r_\Gamma \vee \mu]$		global existence in $H^{1,\rho,\theta}$ $\rho \in [r_\Omega, r_\Omega \vee m]$ $\theta \in [q, \mu]$	
$r_\Omega < p < 1 + r_\Omega/m'$	local existence in $H^{1,\rho,\theta}$ $\rho \in [r_\Omega, m]$ $\theta \in [r_\Gamma, r_\Gamma \vee \mu]$ global existence in $H^{1,\rho,\theta}$ $\rho \in [p, m]$ $\theta \in [r_\Gamma, r_\Gamma \vee \mu]$		local existence in $H^{1,\rho,\theta}$ $\rho \in [r_\Omega, m]$ $\theta \in [r_\Gamma, \mu]$ global existence in $H^{1,\rho,\theta}$ $\rho \in [p, m]$ $\theta \in [q, \mu]$	
$r_\Omega < p = 1 + r_\Omega/m'$				no results

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